Billanook College

July Exam 2017

VCE Specialist Mathematics
Examination 2

Written Examination

Question and Answer Booklet

Reading time: 15 minutes
Writing time: 1½ hour

Student's Name: ________________________________

Teacher's Name : ________________________________

Structure of Booklet

Section	Number of Questions	Number of marks
1	20	20
2	5	56
total		76

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference book, one approved CAS calculator, and one scientific calculator. Calculator memory DOES NOT need to be cleared.
Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied:
Question and answer booklet
Multiple choice answer sheet

Instructions
Write your name and teacher's name in the space provided above.
Always show your working.
All written responses should be in English

Students are NOT permitted to bring mobile phones and/or any other electronic communications equipment into the examination room.

VCE Specialist Mathematics

Written Examination 2

Multiple-choice Answer Sheet

Student's Name: ANSWERS

Teacher's Name: ______________________

Instructions
Use a pencil for all entries. If you make a mistake, erase the incorrect answer – do not cross it out.
Marks will **not** be deducted for incorrect answers.
No mark will be given if more than **one** answer is completed for any question.

All answers must be completed like **this** example: A **B** C D E

Use pencil only

1	A	B	C	**D**	E
2	A	B	**C**	D	E
3	A	B	C	D	**E**
4	**A**	B	C	D	E
5	A	B	C	D	**E**
– 6	**A**	**B**	C	D	E
7	A	B	C	D	**E**
8	A	B	**C**	D	E
9	A	**B**	C	D	E
10	A	**B**	C	D	E

11	A	B	C	D	**E**
12	A	B	**C**	D	E
13	**A**	B	C	D	E
14	A	B	C	**D**	E
15	A	B	C	D	**E**
16	A	**B**	C	D	E
17	**A**	B	C	D	E
18	A	**B**	C	D	E
19	**A**	B	C	D	E
– 20	A	B	**C**	D	E

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1 2012 Q1

The graph with equation $y = \dfrac{1}{2x^2 - x - 6}$ has asymptotes given by

A. $x = -\dfrac{3}{2}$, $x = 2$ and $y = 1$

B. $x = -\dfrac{3}{2}$ and $x = 2$ only

C. $x = \dfrac{3}{2}$, $x = -2$ and $y = 0$

D. $x = -\dfrac{3}{2}$, $x = 2$ and $y = 0$

E. $x = \dfrac{3}{2}$ and $x = -2$ only

Question 2 2012 Q2

A rectangle is drawn so that its sides lie on the lines with equations $x = -2$, $x = 4$, $y = -1$ and $y = 7$.
An ellipse is drawn inside the rectangle so that it just touches each side of the rectangle.
The equation of the ellipse could be

A. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

B. $\dfrac{(x+1)^2}{9} + \dfrac{(y+3)^2}{16} = 1$

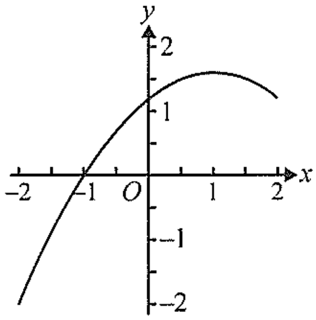
C. $\dfrac{(x-1)^2}{9} + \dfrac{(y-3)^2}{16} = 1$

D. $\dfrac{(x+1)^2}{36} + \dfrac{(y+3)^2}{64} = 1$

E. $\dfrac{(x-1)^2}{36} + \dfrac{(y-3)^2}{64} = 1$

Question 3 2012 Q3

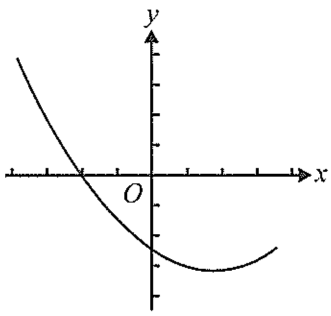
The graph of $y = f(x)$ is shown below.



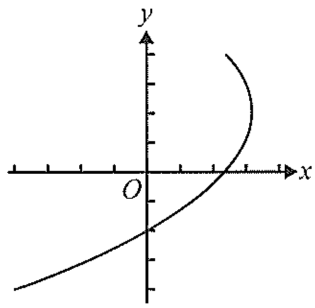
All of the axes below have the same scale as the axes in the diagram above.

The graph of $y = \dfrac{1}{f(x)}$ is best represented by

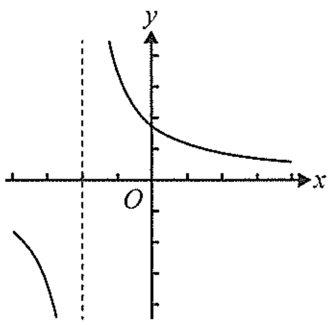
A.



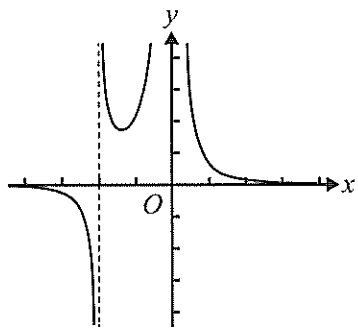
B.



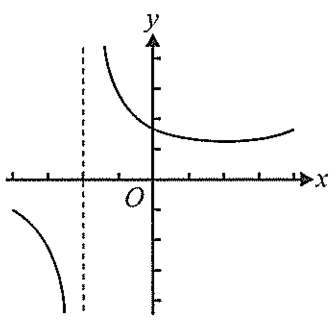
C.



D.



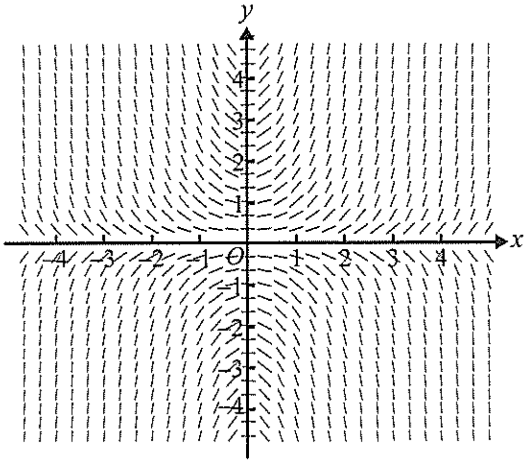
E.



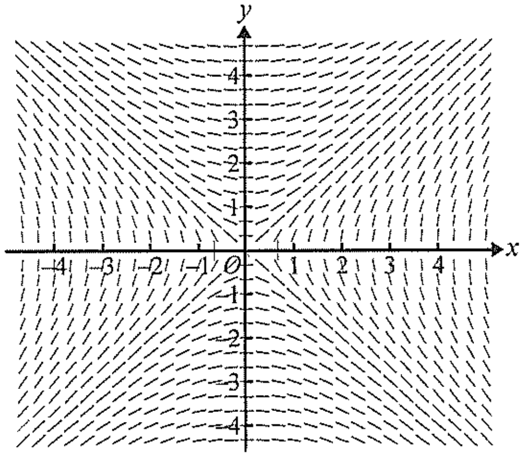
Question 4 2012 Q10

The diagram that best represents the direction field of the differential equation $\frac{dy}{dx} = xy$ is

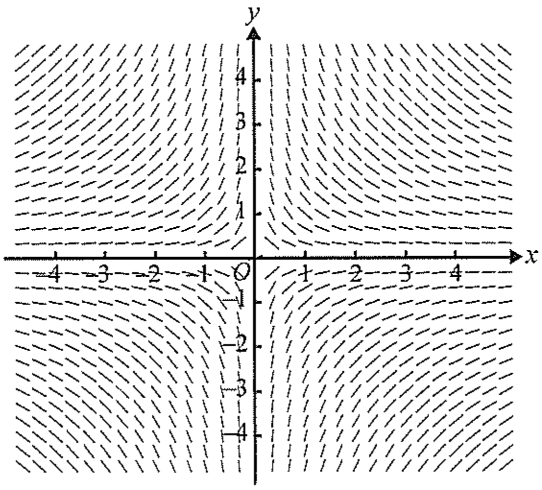
A.



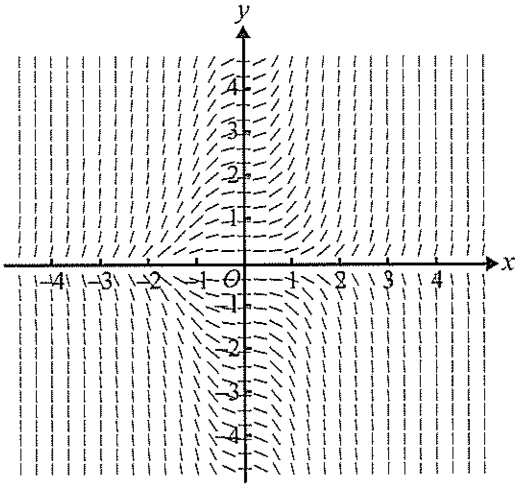
B.



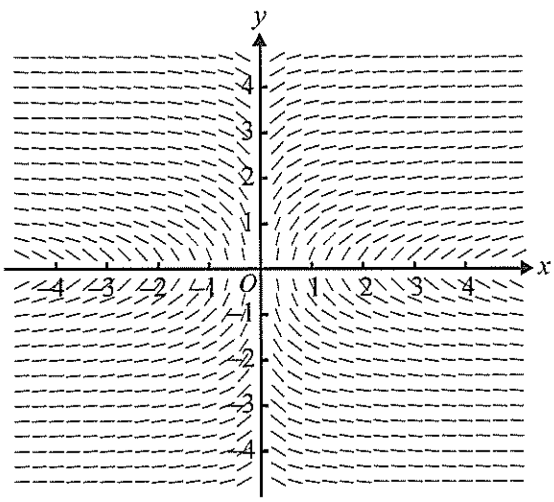
C.



D.



E.



Question 5 2012 Q11

If $\dfrac{d^2y}{dx^2} = x^2 - x$ and $\dfrac{dy}{dx} = 0$ at $x = 0$, then the graph of y will have

A. a local minimum at $x = \dfrac{1}{2}$

B. a local maximum at $x = 0$ and a local minimum at $x = 1$

C. stationary points of inflection at $x = 0$ and $x = 1$, and a local minimum at $x = \dfrac{3}{2}$

D. a stationary point of inflection at $x = 0$, no other points of inflection and a local minimum at $x = \dfrac{3}{2}$

E. a stationary point of inflection at $x = 0$, a non-stationary point of inflection at $x = 1$ and a local minimum at $x = \dfrac{3}{2}$

Question 6 2012 Q12

The volume of the solid of revolution formed by rotating the graph of $y = \sqrt{9-(x-1)^2}$ about the x-axis is given by

A. $4\pi(3)^2$

B. $\pi\int_{-3}^{3}\left(9-(x-1)^2\right)dx$

C. $\pi\int_{-2}^{4}\left(\sqrt{9-(x-1)^2}\right)dx$

D. $\pi\int_{-2}^{4}\left(9-(x-1)^2\right)^2dx$

E. $\pi\int_{-4}^{2}\left(9-(x-1)^2\right)dx$

Question 7 2013 Q7

If $z = r\,\text{cis}(\theta)$, then $\dfrac{z^2}{\bar{z}}$ is equivalent to

A. $r^3\text{cis}(3\theta)$

B. $r^3\text{cis}(-\theta)$

C. $2\,\text{cis}(3\theta)$

D. $r^3\text{cis}(\theta)$

E. $r\,\text{cis}(3\theta)$

Question 8 2013 Q8

The principal arguments of the solutions to the equation $z^2 = 1 + i$ are

A. $\dfrac{\pi}{8}$ and $\dfrac{9\pi}{8}$

B. $-\dfrac{\pi}{8}$ and $\dfrac{7\pi}{8}$

C. $-\dfrac{7\pi}{8}$ and $\dfrac{\pi}{8}$

D. $\dfrac{7\pi}{8}$ and $\dfrac{15\pi}{8}$

E. $-\dfrac{3\pi}{4}$ and $\dfrac{\pi}{4}$

Question 9 2013 Q9

The definite integral $\displaystyle\int_{e^3}^{e^4} \frac{1}{x\log_e(x)}\,dx$ can be written in the form $\displaystyle\int_a^b \frac{1}{u}\,du$ where

A. $u = \log_e(x)$, $a = \log_e(3)$, $b = \log_e(4)$

B. $u = \log_e(x)$, $a = 3$, $b = 4$

C. $u = \log_e(x)$, $a = e^3$, $b = e^4$

D. $u = \dfrac{1}{x}$, $a = e^{-3}$, $b = e^{-4}$

E. $u = \dfrac{1}{x}$, $a = e^3$, $b = e^4$

Question 10 2013 Q14

The distance from the origin to the point $P\left(7,\ -1,\ 5\sqrt{2}\right)$ is

A. $7\sqrt{2}$

B. 10

C. $6+5\sqrt{2}$

D. 100

E. $5\sqrt{6}$

Question 11 2013 Q15

Let $\underset{\sim}{u}=4\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$, $\underset{\sim}{v}=3\underset{\sim}{j}+3\underset{\sim}{k}$ and $\underset{\sim}{w}=-4\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$.

Which one of the following statements is **not** true?

A. $|\underset{\sim}{u}|=|\underset{\sim}{v}|$

B. $|\underset{\sim}{u}|=|-\underset{\sim}{w}|$

C. $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are linearly independent

D. $\underset{\sim}{u}.\underset{\sim}{v}=0$

E. $(\underset{\sim}{u}+\underset{\sim}{w}).\underset{\sim}{v}=12$

Question 12 2013 Q.11

Consider the differential equation $\dfrac{dy}{dx}=\dfrac{1}{3+3x+x^2}$, with $y_0=1$ when $x_0=0$.

Using Euler's method with a step size of 0.1, the value of y_2, correct to three decimal places, is

A. 1.033

B. 1.063

C. 1.064

D. 1.065

E. 1.066

Question 13 2014 Q5

If the complex number z has modulus $2\sqrt{2}$ and argument $\frac{3\pi}{4}$, then z^2 is equal to

A. $-8i$

B. $4i$

C. $-2\sqrt{2}i$

D. $2\sqrt{2}i$

E. $-4i$

Question 14 2014 Q6

Given that $i^n = p$ and $i^2 = -1$, then i^{2n+3} in terms of p is equal to

A. $p^2 - i$

B. $p^2 + i$

C. $-p^2$

D. $-ip^2$

E. ip^2

Question 15 2014 Q7

The sum of the roots of $z^3 - 5z^2 + 11z - 7 = 0$, where $z \in C$, is

A. $1 + 2\sqrt{3}i$

B. $5i$

C. $4 - 2\sqrt{3}i$

D. $2\sqrt{3}i$

E. 5

Question 16 2014 Q 9

The circle $|z-3-2i|=2$ is intersected exactly twice by the line given by

A. $|z-i|=|z+1|$

B. $|z-3-2i|=|z-5|$

C. $|z-3-2i|=|z-10i|$

D. $\text{Im}(z)=0$

E. $\text{Re}(z)=5$

Question 17 2014 Q10

A large tank initially holds 1500 L of water in which 100 kg of salt is dissolved. A solution containing 2 kg of salt per litre flows into the tank at a rate of 8 L per minute. The mixture is stirred continuously and flows out of the tank through a hole at a rate of 10 L per minute.

The differential equation for Q, the number of kilograms of salt in the tank after t minutes, is given by

A. $\frac{dQ}{dt}=16-\frac{5Q}{750-t}$

B. $\frac{dQ}{dt}=16-\frac{5Q}{750+t}$

C. $\frac{dQ}{dt}=16+\frac{5Q}{750-t}$

D. $\frac{dQ}{dt}=\frac{100Q}{750-t}$

E. $\frac{dQ}{dt}=8-\frac{Q}{1500-2t}$

Question 18 2014 Q15

If θ is the angle between $\underset{\sim}{a} = \sqrt{3}\,\underset{\sim}{i} + 4\,\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\,\underset{\sim}{j} + \sqrt{3}\,\underset{\sim}{k}$, then $\cos(2\theta)$ is

A. $-\frac{4}{5}$

B. $\frac{7}{25}$

C. $-\frac{7}{25}$

D. $\frac{14}{25}$

E. $-\frac{24}{25}$

Question 19 2014 Q16

Two vectors are given by $\underset{\sim}{a} = 4\,\underset{\sim}{i} + m\,\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\,\underset{\sim}{i} + n\,\underset{\sim}{j} - \underset{\sim}{k}$, where $m,\ n \in R^+$.

If $|\underset{\sim}{a}| = 10$ and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$, then m and n respectively are

A. $5\sqrt{3},\ \frac{\sqrt{3}}{3}$

B. $5\sqrt{3},\ \sqrt{3}$

C. $-5\sqrt{3},\ \sqrt{3}$

D. $\sqrt{93},\ \frac{5\sqrt{93}}{93}$

E. 5, 1

Question 20 2014 Q12

If $\frac{dy}{dx} = \sqrt{(2x^6 + 1)}$ and $y = 5$ when $x = 1$, then the value of y when $x = 4$ is given by

A. $\int_1^4 \left(\sqrt{(2x^6 + 1)} + 5\right) dx$

B. $\int_1^4 \sqrt{(2x^6 + 1)}\, dx$

C. $\int_1^4 \sqrt{(2x^6 + 1)}\, dx + 5$

D. $\int_1^4 \sqrt{(2x^6 + 1)}\, dx - 5$

E. $\int_1^4 \left(\sqrt{(2x^6 + 1)} - 5\right) dx$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
Unless otherwise specified an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1 2011 Q1

Consider the graph with rule $|z - i| = 1$ where $z \in C$.

a. Write this rule in cartesian form.

$|z - i| = 1$

$|x + iy - i| = 1$

$|x + i(y-1)| = 1$

$x^2 + (y-1)^2 = 1^2$

$x^2 + (y-1)^2 = 1$

2 marks

b. Find the points of intersection of the graphs with rules $|z - i| = 1$ and $|z - 1| = 1$ in cartesian form.

$x^2 + (y-1)^2 = (x-1)^2 + y^2$

$x^2 + y^2 - 2y + 1 = x^2 - 2x + 1 + y^2$

$-2y = -2x$

$y = x$ is where the circles intersect

from $x^2 + (y-1)^2 = 1$ $x = y$

2 marks

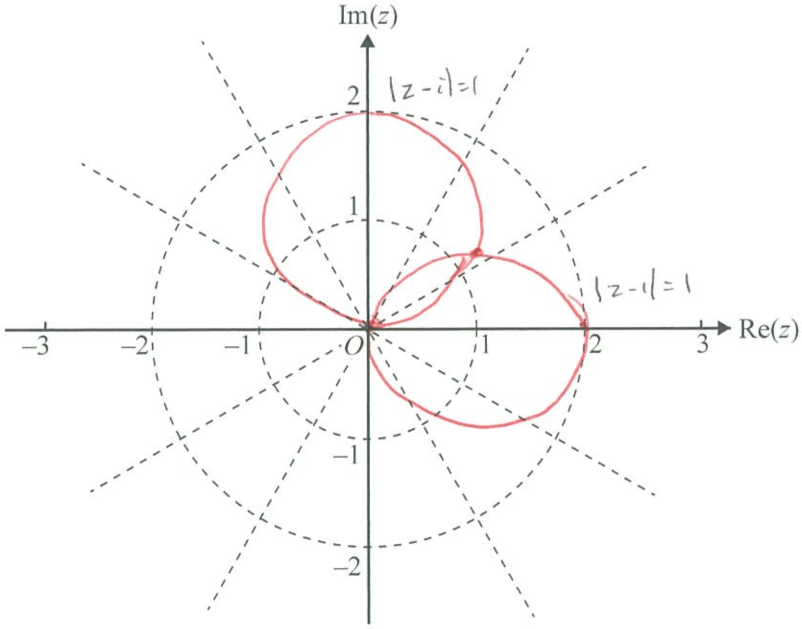
$x^2 + x^2 - 2x + 1 = 0$

$2x^2 - 2x = 0$

$x(x-1) = 0$

so points are $(0,0)$ and $(1,1)$

c. Sketch **and label** the graphs with rules $|z - i| = 1$ and $|z - 1| = 1$ on the argand diagram below.



$|z - i| = 1$

$|z|$ is a circle, r=1 centre origin

$-i$ shifts up

-1 " right

2 marks

d. **i.** Find the equation of the straight line which passes through the points of intersection of the graphs with rules $|z - i| = 1$ and $|z - 1| = 1$.

Express your answer in cartesian form.

already found this $y = x$

ii. The straight line found in **part d. i.** can be expressed in the form $z = a\bar{z}$ where $a \in C$.
Find the value of a.

$x + iy = a(x - iy)$ know $(1,1)$

$1 + i = a(1 - i)$

$a = \frac{1+i}{1-i}$

$\frac{1+i}{1-i} \times \frac{1+i}{1+i}$ rationalise

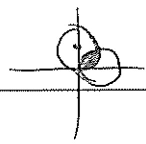
$= \frac{2i}{2}$

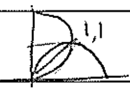
$= i$

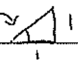
1 + 2 = 3 marks

e. i. Shade the region $\{z : |z - 1| \leq 1, z \in C\} \cap \{z : |z - i| \leq 1, z \in C\}$ on the argand diagram in **part c.**

ii. Find the area of the shaded region in **part e. i.**

Intersection is the section shown 

Area of a segment  

$\therefore \frac{\pi}{4}$ 

Sector $A = \frac{1}{2}r^2\theta$

Segment $A = \frac{1}{2}r^2(\theta - \sin\theta)$ $r = \cancel{\sqrt{2}}$

for the circle 1 + 2 = 3 marks

Total 12 marks

$\cancel{= \frac{1}{2} \times (\frac{\pi}{4} - \sin\frac{\pi}{4})}$

$\cancel{= \frac{1}{2}(\frac{\pi}{4} - \frac{1}{\sqrt{2}})}$

$= \frac{1}{2} \times 1^2\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$= \frac{1}{2}\left(\frac{\pi}{2} - 1\right)$

$= \frac{\pi}{4} - \frac{1}{2}$

area of shaded region is double this segment.

$\frac{\pi}{2} - 1$

The region in the first quadrant bounded by the ellipse, the coordinate axes and the line $y = 20$ is rotated about the y-axis to form a volume of revolution, which is to model a fish bowl. Values on the coordinate axes represent centimetres.

b. **i.** Write down a definite integral **in terms of y** which will give the volume of the bowl.

$$\frac{x^2}{400} + \frac{(y-10)^2}{900} = 1$$

$$x^2 + \frac{\cancel{400}}{\cancel{900}}(y-10)^2 = 400$$

$$x^2 = 400 - \frac{4}{9}(y-10)^2$$

$$V = \pi \int_0^{20} x^2 \, dy$$

$$= \pi \int_0^{20} \left(400 - \frac{4}{9}(y-10)^2\right) dy$$

ii. Evaluate the integral in **part b. i.** to find the volume of the bowl, correct to the nearest cubic centimetre.

use CAS

$V \approx 24202 \text{ cm}^3$

2 + 1 = 3 marks

Now consider a **different** fish bowl for which the volume V cubic centimetres of water contained in the bowl is related to the depth h centimetres by

$$\frac{dV}{dh} = \frac{25\pi}{36}\left(800 + 20h - h^2\right).$$

Water flows in at a rate of 500 cubic centimetres per minute.

c. At what rate is the depth rising, in centimetres per minute, when the depth is 15 centimetres? Give your answer correct to two decimal places.

have $\frac{dV}{dh}$ $\quad \frac{dV}{dt} = +500$ cm/min $\quad$ all V cm^3/min, height cm.

need dh/dt at $h = 15$ cm

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{1}{\frac{25\pi}{36}(800 + 20h - h^2)} \times 500$$

when $h = 15$ $\quad \frac{dh}{dt} = \frac{1}{\frac{25\pi}{36}(800 + 20 \times 15 - 15^2)} \times 500$

≈ 0.26 cm/min.

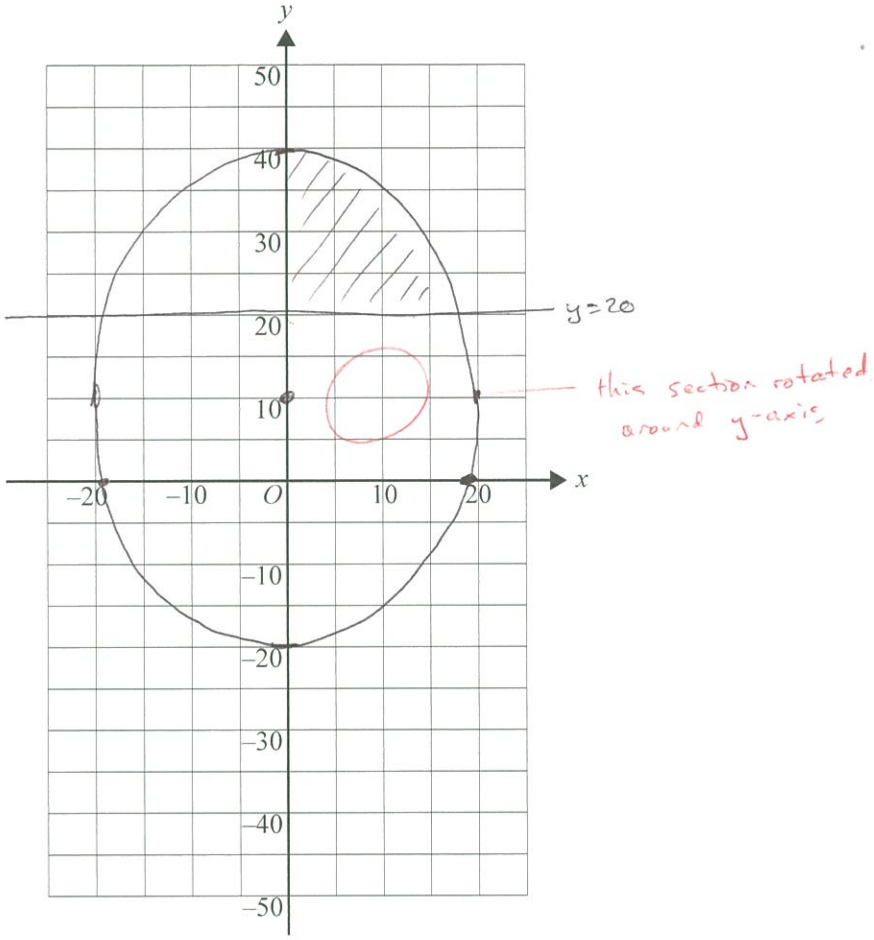
3 marks

Total 10 marks

Question 2 2011 Q3

a. Sketch the ellipse with equation $\dfrac{x^2}{400} + \dfrac{(y-10)^2}{900} = 1$ on the axes below.

Write down the intercepts with the x-axis.



Intercepts with x-axis - where $y = 0$

$\dfrac{x^2}{400} + \dfrac{100}{900} = 1$

$\dfrac{x^2}{400} = \dfrac{8}{9}$

$x^2 = \dfrac{3200}{9}$

$x = \pm\sqrt{\dfrac{3200}{9}} = \pm\dfrac{40\sqrt{2}}{3}$

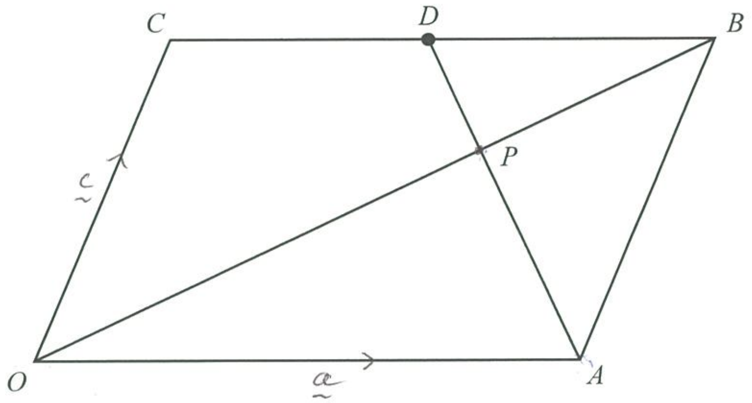
$\left(\dfrac{-40\sqrt{2}}{3}, 0\right)$ & $\left(\dfrac{40\sqrt{2}}{3}, 0\right)$

4 marks

$OABC$ is a parallelogram where D is the midpoint of $\overline{CB}$.

$\overline{OB}$ and $\overline{AD}$ intersect at point P.

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.



b. **i.** Given that $\overrightarrow{AP} = \alpha\,\overrightarrow{AD}$, write an expression for $\overrightarrow{AP}$ in terms of α, $\underset{\sim}{a}$ and $\underset{\sim}{c}$. 2 marks

$AD = AB - \frac{1}{2}BC$

$= \underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$

$\therefore AP = \alpha(\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a})$

ii. Given that $\overrightarrow{OP} = \beta\,\overrightarrow{OB}$, write another expression for $\overrightarrow{AP}$ in terms of β, $\underset{\sim}{a}$ and $\underset{\sim}{c}$. 1 mark

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ $\quad$ $\overrightarrow{AP} = \overrightarrow{PO} + \overrightarrow{OA}$

$= \overrightarrow{AO} + \overrightarrow{OP}$

$\overrightarrow{AP} = -\overrightarrow{OA} + \beta\,\overrightarrow{OB}$ $\quad$ $AP = -\overrightarrow{OA} + \overrightarrow{OP}$

$= -\hat{a} + \beta(a+c) = -a + \beta a + \beta c = a(\beta-1) + \beta c$

iii. **Hence** deduce the values of α and β. 2 marks

$\alpha(c - \frac{1}{2}a) = a(\beta-1) + \beta c$

$\alpha c - \frac{1}{2}\alpha a = a\beta - a + \beta c$

$\alpha c - \frac{1}{2}\alpha a = a(\beta-1) + \beta c$

for $\hat{a}$ $\quad -\frac{1}{2}\alpha = \beta - 1$

$\underset{\sim}{c}$ $\quad \alpha = \beta$ $\quad \therefore \alpha = \beta$

$-\frac{1}{2}\alpha = \beta - 1$

$-\frac{1}{2}\alpha = \alpha - 1$

$1 = 1\frac{1}{2}\alpha$

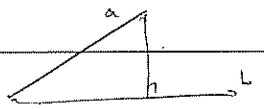
$1 = \frac{3}{2}\alpha$

$\alpha = \frac{2}{3}$ & $\beta = \frac{2}{3}$

Question 3 (10 marks) 2014 Q3

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$.

a. Express $\underset{\sim}{a}$ as the **sum** of two vector resolutes, one of which is parallel to $\underset{\sim}{b}$ and the other of which is perpendicular to $\underset{\sim}{b}$. Identify clearly the parallel vector resolute and the perpendicular vector resolute. 5 marks



parallel $(a.\hat{b})\hat{b}$

perp. $\underset{\sim}{a} - (a.\hat{b})\hat{b}$

$a.\hat{b} = (3i+2j+k) \cdot \frac{1}{\sqrt{9}}(2i-2j-k)$

$= \frac{1}{3}(3\times 2 - 2\times 2 - 1)$

$= \frac{1}{3}$

$(a.\hat{b})\hat{b} = \frac{1}{9}(2i-2j-k)$ ← parallel (resolute)

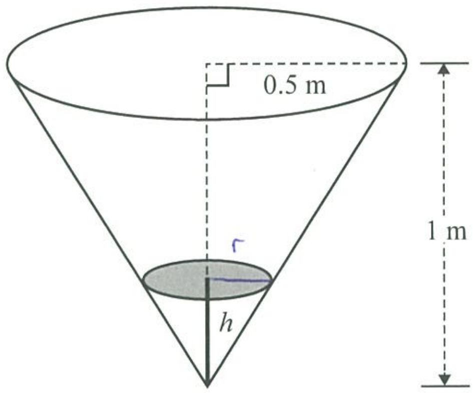
$3i+2j+k - \frac{1}{9}(2i-2j-k)$

$= \frac{25}{9}i + \frac{20}{9}j + \frac{10}{9}k$ ← perpendicular

$\therefore \underset{\sim}{a} = \frac{1}{9}(2i-2j-k) + \frac{5}{9}(5i+4j+2k)$

Question 4 (12 marks) 2014 Q4

At a water fun park, a conical tank of radius 0.5 m and height 1 m is filling with water. At the same time, some water flows out from the vertex, wetting those underneath. When the tank eventually fills, it tips over and the water falls out, drenching all those underneath. The tank then returns to its original position and begins to refill.



$\frac{r}{h} = \frac{0.5}{1} = \frac{1}{2}$

$r = \frac{1}{2}h$

Water flows in at a constant rate of 0.02π m^3/min and flows out at a variable rate of $0.01\pi\sqrt{h}$ m^3/min, where h metres is the depth of the water at any instant.

a. Show that the volume, V cubic metres, of water in the cone when it is filled to a depth of h metres is given by $V = \frac{\pi}{12}h^3$. 1 mark

$V = \frac{1}{3}\pi r^2 \times h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 \times h = \frac{1}{3}\pi \frac{1}{4}h^3$

$= \frac{1}{12}\pi h^3$

b. Find the rate, in m/min, at which the depth of the water in the tank is increasing when the depth is 0.25 m. 4 marks

need $\frac{dh}{dt}$ $\quad \frac{dV}{dh} = \frac{\pi}{4}h^2$ $\quad \frac{dV}{dt} = 0.02\pi - 0.01\pi\sqrt{h}$

$= 0.01\pi(2-\sqrt{h})$

$\frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = 0.01\pi(2-\sqrt{h}) \times \frac{4}{\pi h^2}$

$= 0.04\frac{2-\sqrt{h}}{h^2}$

when $h = 0.25$ $\quad \frac{dh}{dt} = 0.04\frac{2-0.5}{1/16} = 0.96$ m/min

The tank is empty at time $t = 0$ minutes.

c. By using an appropriate definite integral, find the time it takes for the tank to fill. Give your answer in minutes, correct to one decimal place. 2 marks

$\frac{dh}{dt} = \frac{0.04(2-\sqrt{h})}{h^2}$ $\qquad \frac{dt}{dh} = \frac{h^2}{0.04(2-\sqrt{h})}$

$t = \int_0^1 \frac{h^2}{0.04(2-\sqrt{h})}\,dh$

$\simeq 7.4$ minutes (CAS)

Question 5 (12 marks) 2015 Q1

Consider $y=\sqrt{2-\sin^2(x)}$.

a. Use the relation $y^2 = 2 - \sin^2(x)$ to find $\frac{dy}{dx}$ in terms of x and y. 1 mark

$y\frac{dy}{dx} = -2 \sin x \cos x$

$= -\sin(2x)$ $\frac{dy}{dx} = \frac{-\sin(2x)}{y}$

b. **i.** Write down the values of y where $x = 0$ and where $x = \frac{\pi}{2}$. 1 mark

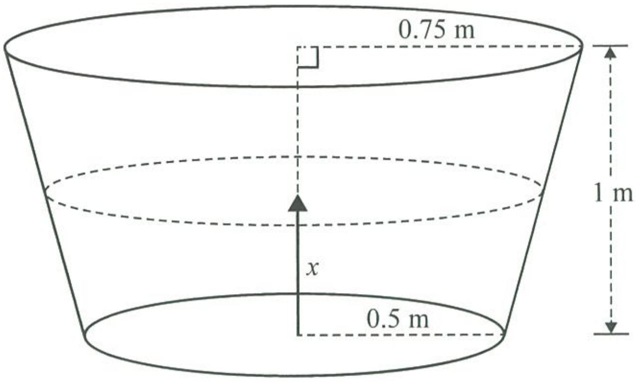
$x=0$ $y = \sqrt{2-\sin^2 0} = \sqrt{2}$

$x = \frac{\pi}{2}$ $y = \sqrt{2 - \sin^2 \frac{\pi}{2}} = 1$

ii. Write down the values of $\frac{dy}{dx}$ where $x = 0$ and where $x = \frac{\pi}{2}$. 1 mark

$x=0$ $\frac{dy}{dx} = \frac{-\sin 0}{y} = 0$ $x = \frac{\pi}{2}$ $\frac{dy}{dx} = -\frac{\sin \pi}{y} = 0$

Another water tank, shown below, has the shape of a large bucket (part of a cone) with the dimensions given. Water fills the tank at a rate of 0.05π m^3/min, but no water leaks out.



When filled to a depth of x metres, the volume of water, V cubic metres, in the tank is given by

$$V = \frac{\pi}{48}\left(x^3 + 6x^2 + 12x\right)$$

d. Given that the tank is initially empty, find the depth, x metres, as a function of time t. 5 marks

$\frac{dV}{dx} = \frac{\pi}{48}\left(3x^2 + 12x + 12\right) = \frac{\pi}{16}\left(x^2 + 4x + 4\right)$

$= \frac{\pi}{16}(x+2)^2$

need dx/dt

$\frac{dV}{dt} = 0.05\pi$

$\frac{dV}{dx} = \frac{dV}{dt}\frac{dt}{dx}$ $\quad \frac{dx}{dt} = \frac{dV}{dt}\cdot\frac{dx}{dV}$

$= \frac{1}{\frac{\pi}{16}(x+2)^2}\cdot 0.05\pi$

$= \frac{16 \times 0.05\cancel{\pi}}{\cancel{\pi}(x+2)^2}$

$= \frac{0.8}{(x+2)^2}$

$\frac{dt}{dx} = \frac{(x+2)^2}{0.8}$

$t = \int_0^x \frac{(x+2)^2}{0.8}\,dx$

$t = \left[5\frac{(x+2)^3}{12}\right]_0^x = \frac{5(x+2)^3}{12} - \frac{10}{3}$

$\therefore x = 2(0.3t+1)^{1/3} - 2$

Now consider the function f with rule $f(x) = \sqrt{2 - \sin^2(x)}$ for $0 \le x \le \frac{\pi}{2}$.

c. Find the rule for the inverse function f^{-1}, and state the domain and range of f^{-1}. 3 marks

$y = \sqrt{2 - \sin^2 x}$

$y^2 = 2 - \sin^2 x$ dom $[0, \frac{\pi}{2}]$ range $[1, \sqrt{2}]$

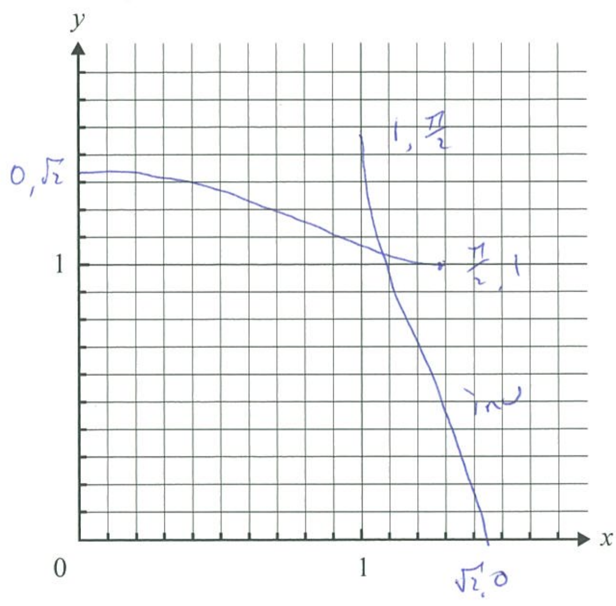
inverse $x^2 = 2 - \sin^2 y$

$\sin^2 y = 2 - x^2$

$\therefore 1 - 2\sin^2 y = 2x^2 - 3$

$y = \frac{1}{2}\cos^{-1}(2x^2 - 3)$ dom $[1, \sqrt{2}]$ range $[0, \frac{\pi}{2}]$

d. Sketch and label the graphs of f and f^{-1} on the axes below. 2 marks



e. The graphs of f and f^{-1} intersect at the point $P(a, a)$.

Find a, correct to three decimal places. 1 mark

$\sqrt{2-\sin^2 x} = x$ CAS $a = x \approx 1.099$

The region bounded by the graph of f, the coordinate axes and the line $x = 1$ is rotated about the x-axis to form a solid of revolution.

f. **i.** Write down a definite integral in terms of x that gives the volume of this solid of revolution. 2 marks

$V = \int_0^1 \pi y^2 \, dx = \int_0^1 \pi (2 - \sin^2 x) \, dx$

ii. Find the volume of this solid, correct to one decimal place. 1 mark

CAS $V = 5.4$ cubic units.