# Billanook College

## July Exam 2017

## VCE Specialist Mathematics
## Examination 1

Written Examination

**Question and Answer Booklet**

Reading time: 15 minutes
Writing time: 1 hour

Student's Name: ANSWERS

Teacher's Name : ______________________

Structure of Booklet

| Section | Number of Questions | Number of marks |
|---|---|---|
| Exam 1 | 9 | 38 |

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
No calculator is permitted in this examination.

**Materials supplied:**
Question and answer booklet
Formula sheet.

**Instructions**
Write your name and teacher's name in the space provided above.
Always show your working.
All written responses should be in English

**Students are NOT permitted to bring mobile phones and/or any other electronic communications equipment into the examination room.**

**Question** 1 (4 marks) Q.4 2015

**a.** Find all solutions of $z^3 = 8i,\ z \in C$ in cartesian form. 3 marks

$\sqrt{-8z}$ $\quad -2i$ is a solution (by inspection)

$z^3 - 8i = 0 \quad (z+2i)(z^2 - 2iz - 4) = 0$ by long division etc.

$$z+2i \,\overline{\big)\, z^3 - 0z^2 - 0z - 8i}$$ quotient $z^2 - 2iz + 4$

$z^2 - 2iz - 4 = 0 \quad \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$z = \pm\sqrt{3} + i$

solutions $-2i,\ \pm\sqrt{3} + i$

OR. $r^3 \operatorname{cis} 3\theta = 8 \operatorname{cis} \frac{\pi}{2}$

$2 \operatorname{cis} \frac{\pi}{6}$ $\quad \pi/6$

$x: \frac{\sqrt{3}}{2} \times \sqrt{?} = \sqrt{3}$

$y: \frac{1}{2} \times 2 = 1\ i$

$\pm\sqrt{3} + i,\ -2i$

**b.** Find all solutions of $(z - 2i)^3 = 8i,\ z \in C$ in cartesian form. 1 mark

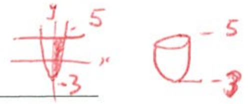
$z - 2i = -2i \quad \therefore z = 0$

$z - 2i = \pm\sqrt{3} + i \quad \therefore z = \pm\sqrt{3} + 3i$

translated 2 units up

**Question** 2 (3 marks) Q5. 2015

Find the volume generated when the region bounded by the graph of $y = 2x^2 - 3$, the line $y = 5$ and the $y$-axis is rotated about the $y$-axis.



$2x^2 - 3 = 5$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$V = \int \pi r^2\, dy$ $\quad$ y-axis

$r$ is $x$ $\quad y = 2x^2 - 3$

$\dfrac{y+3}{2} = x^2$

$= \int_{-3}^{5} \pi \left(\dfrac{y+3}{2}\right) dy$

$= \frac{\pi}{2} \int_{-3}^{5} (y+3)\, dy$

$= \frac{\pi}{2} \left[\frac{y^2}{2} + 3y\right]_{-3}^{5}$

$= \frac{\pi}{2} \times \left(\frac{25}{2} + 15 - \frac{9}{4} + 9\right)$

$= \frac{\pi}{2} \times (8 + 15 + 9)$

$= \frac{\pi}{2} \times 32$

$= 16\pi$

**Question 3** (6 marks) Q9 2015

Consider the curve represented by $x^2 - xy + \frac{3}{2}y^2 = 9$.

**a.** Find the gradient of the curve at any point $(x, y)$. 2 marks

$$2x - \left(y + x\frac{dy}{dx}\right) + \frac{3}{2}\times 2y\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 3y\frac{dy}{dx} = 0$$

$$2x - y = (x - 3y)\frac{dy}{dx}$$

$$dy/dx = \frac{2x-y}{x-3y}$$

**b.** Find the equation of the tangent to the curve at the point $(3, 0)$ **and** find the equation of the tangent to the curve at the point $\left(0, \sqrt{6}\right)$.

Write each equation in the form $y = ax + b$. 2 marks

at $(3,0)$ $\quad \frac{dy}{dx} = \frac{6}{3} = 2$ $\quad y - y_1 = m(x - x_1)$

$y = 2(x-3)$

$y = 2x - 6$ ①

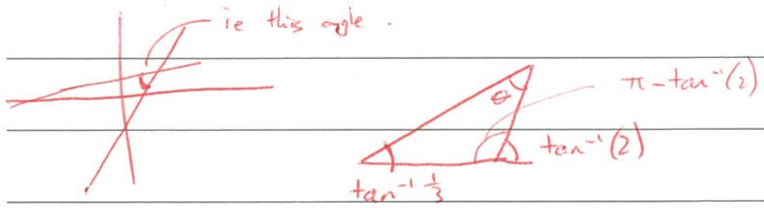
at $(0,\sqrt{6})$ $\quad \frac{dy}{dx} = \frac{-\sqrt{6}}{-3\sqrt{6}} = \frac{1}{3}$ $\quad y - y_1 = m(x - x_1)$

$y - \sqrt{6} = \frac{1}{3}(x)$

$y = \frac{1}{3}x + \sqrt{6}$ ①

**c.** Find the acute angle between the tangent to the curve at the point $(3, 0)$ and the tangent to the curve at the point $\left(0, \sqrt{6}\right)$.

Give your answer in the form $k\pi$, where $k$ is a real constant. 2 marks




$\theta = \pi - (\tan^{-1}\frac{1}{3}) - (\pi - \tan^{-1}(2))$

$= \tan^{-1}(2) - \tan^{-1}(\frac{1}{3})$

$\tan\theta = \tan\left((\tan^{-1}(2)) - (\tan^{-1}(\frac{1}{3}))\right)$

$= \cancel{\tan\left(\tan^{-1}2 - \tan\right.}$

$= \frac{\tan(\tan^{-1}(2)) - \tan(\tan^{-1}(\frac{1}{3}))}{1 + \tan(\tan^{-1}(2))\tan(\tan^{-1}(\frac{1}{3}))} = \frac{2 - \frac{1}{3}}{1 + 2/3} = \frac{1^{2/3}}{1^{2/3}} = 1$

$\tan\theta = 1$

$\therefore \theta = \pi/4$

**Question 4** (4 marks) 2016 Q4

Chemicals are added to a container so that a particular crystal will grow in the shape of a cube. The side length of the crystal, $x$ millimetres, $t$ days after the chemicals were added to the container, is given by $x = \arctan(t)$.

Find the rate at which the surface area, $A$ square millimetres, of the crystal is growing one day after the chemicals were added. Give your answer in square millimetres per day.

$x = \tan^{-1}(t)$ Surface area $= 6x^2$

$A(t) = 6\left(\tan^{-1}(t)\right)^2$ mm$^2$

need $\frac{dA}{dt} = 6 \times \left|2 \tan^{-1}(t)\right| \times \frac{1}{1+t^2}$ chain rule

$= \frac{12 \tan^{-1} t}{1+t^2}$

when $t=1$ $\frac{dA}{dt} = \frac{12 \tan^{-1} 1}{1+1^2} = \frac{12 \times \pi/4}{2} = \frac{3\pi}{2}$ mm$^2$/day

4

**Question 5** (4 marks) 2016 Q5

Consider the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + d\underset{\sim}{k}$, where $d$ is a real constant.

**a.** Find the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. 2 marks

vector resolute $(\underset{\sim}{a}.\hat{b})\hat{b}$ $\hat{b} = \frac{1}{\sqrt{14}}(i - 2j + 3k)$

$\underset{\sim}{a}\hat{b} = \frac{1}{\sqrt{14}}(3 - 10 - 6) = \frac{-13}{\sqrt{14}}$

$\frac{-13}{14} \times (i - 2j + 3k)$

**b.** Find the value of $d$ if the vectors are **linearly dependent**. 2 marks

find d. let. $\hat{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$

i) $3m + 1n = 1$ ①

j) $5m + -2n = 0$ ② k) $-2m + 3n = d$

①-② $-2m + 3n = 1$ $\therefore$ $d = 1$

nice coincidence!

**Question 6** (3 marks) 2016 Q9

Given that $\cos(x-y)=\frac{3}{5}$ and $\tan(x)\tan(y)=2$, find $\cos(x+y)$.

$\cos(x)\cos(y)-\sin(x)\sin(y)$

need this.

$\cos(x)\cos(y)+\sin(x)\sin(y)=\frac{3}{5}$

$\tan(x)\tan(y)=2=\frac{\sin x\cos y}{\cos x\cos y}$

$\therefore \sin(x)\sin(y)=2\cos(x)\cos(y)$

$\left(\frac{3}{5}-\cos(x)\cos(y)\right)=2\cos(x)\cos(y)$

$\frac{3}{5}=3\cos(x)\cos(y)$

$\cos(x)\cos(y)=\frac{1}{5}$

$\cos(x+y)=\frac{1}{5}-2\cos(x)\cos(y)$

$=\frac{1}{5}-2\times\frac{1}{5}$

$=-\frac{1}{5}$

**Question 7** (5 marks) 2016 Q10

Solve the differential equation $\sqrt{2-x^2}\,\frac{dy}{dx}=\frac{1}{2-y}$, given that $y(1)=0$. Express $y$ as a function of $x$.

group $\quad (2-y)\frac{dy}{dx}=\frac{1}{\sqrt{2-x^2}}$

int. w.r.t. $x$ $\quad \int(2-y)\,dy=\int\frac{1}{\sqrt{2-x^2}}\,dx$

$2y-\frac{y^2}{2}+c=\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)+k$

$y(1)=0$ $\quad (c-k)=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$c-k=\frac{\pi}{4}$

$2y-\frac{y^2}{2}+\frac{\pi}{4}=\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$\frac{y^2}{2}-2y=-\sin^{-1}()+\frac{\pi}{4}$

$(y-1)^2-1=\frac{\pi}{4}-\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$(y-1)^2=1+\frac{\pi}{4}-\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$y-1=\sqrt{1+\frac{\pi}{4}-\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)}$

$y=1+\sqrt{1+\frac{\pi}{4}-\sin^{-1}\frac{x}{\sqrt{2}}}$

$2y-\frac{y^2}{2}+\frac{\pi}{4}=\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$4y-y^2+\frac{\pi}{2}=2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$y^2-4y=\frac{\pi}{2}-2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$(y-2)^2-4=$

$(y-2)^2=4+\frac{\pi}{2}-2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)$

$y=2\pm\sqrt{4+\frac{\pi}{2}-2\sin^{-1}\frac{x}{\sqrt{2}}}$

given $y(1)=0$, needs to be $2-\sqrt{4}$ to get 0.

$y=2-\sqrt{4+\frac{\pi}{2}-2\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)}$

**Question** 8 (5 marks) 2014 Q3

Let $f$ be a function of a complex variable, defined by the rule $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$.

**a.** Given that $z = i$ is a solution of $f(z) = 0$, write down a quadratic factor of $f(z)$. 2 marks

$(z-i)$ factor $\therefore$ $(z+i)$ is a factor (real coefficients)

$(z-i)(z+i)$ gives a quadratic factor

$z^2+1$

**b.** Given that the other quadratic factor of $f(z)$ has the form $z^2 + bz + c$, find all solutions of $z^4 - 4z^3 + 7z^2 - 4z + 6 = 0$ in cartesian form. 3 marks

$(z^2+1)(z^2+bz+c) = z^4 + bz^3 + (c+1)z^2 + bz + c$

$\therefore$ $b = -4$ $c = 6$

as $z^2 + bz + c = 0$

$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{4 \pm \sqrt{16-24}}{2}$

$= 2 \pm \sqrt{2}\, i$

all solutions: $\pm i$, $2 \pm \sqrt{2}\, i$

**Question** 9 (4 marks) 2013 Q2

Evaluate $\int_0^1 \frac{x-5}{x^2-5x+6}\,dx$.

$$\frac{x-5}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$

$B = -2 \qquad A = 3$

$$\int_0^1 \frac{3}{x-2} - \frac{2}{x-3}\,dx$$

$$= \left[3\ln|x-2| - 2\ln|x-3|\right]_0^1$$

$$= (3\ln|-1| - 2\ln|-2|) - (3\ln|-2| - 2\ln|-3|)$$

$$= -2\ln(2) - 3\ln(2) + 2\ln(3)$$

$$= -\ln 2^5 + \ln 3^2$$

$$= \ln\frac{9}{32}$$