Sample exam 1 2016

Sunday, 21 August 2016 4:41 PM

Sample exam 1 20...

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
Year

answers

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER

SPECIALIST MATHEMATICS
Written examination 1

Day Date

Reading time: *. to *.** (15 minutes)**
Writing time: *. to *.** (1 hour)**

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 11 pages.
- Formula sheet.
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Version 2 – April 2016

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Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (3 marks)

a. Show that $\sqrt{5} - i$ is a solution of the equation $z^3 - (\sqrt{5} - i)z^2 + 4z - 4\sqrt{5} + 4i = 0$. 1 mark

Only 1 mark, so simplify $\cancel{z^3} - \cancel{(z)z^2} + 4z - 4\sqrt{5} + 4i = 0$

$4(\sqrt{5} - i) - 4\sqrt{5} + 4i$

$4\sqrt{5} - 4i - 4\sqrt{5} + 4i = 0$ $\therefore$ a solution

b. Find all other solutions of the equation $z^3 - (\sqrt{5} - i)z^2 + 4z - 4\sqrt{5} + 4i = 0$. 2 marks

* complex coeff, so can't use conjugate

short division gives

$z^2(z - \sqrt{5} + i) + 4(z - \sqrt{5} + i) = 0$

$(z^2 + 4)(z - \sqrt{5} + i) = 0$

$z = \sqrt{5} - i, \pm 2i$

TURN OVER

Question 2 (4 marks)

Given the relation $3x^2 + 2xy + y^2 = 11$, find the gradient of the **normal** to the graph of the relation at the point in the first quadrant where $x = 1$.

gradient first: $\frac{d}{dx} 3x^2 + \frac{d}{dx} 2xy + \frac{d}{dx} y^2 = 0$

$$6x + 2\left(x \frac{dy}{dx} + y\right) + 2y \frac{dy}{dx} = 0$$

$$6x + 2y + \frac{dy}{dx}(2x + 2y) = 0$$

$$\frac{dy}{dx} = \frac{-6x - 2y}{2x + 2y} = \frac{-3x - y}{x + y}$$

normal : $\frac{x+y}{3x+y}$

when $x = 1$ $\quad 3 + 2y + y^2 = 11 \quad y^2 + 2y - 8 = 0$

$$(y+1)^2 - 3^2 = 0$$

$$(y + 1 + 3)(y + 1 - 3) = 0$$

$$y = -4 \text{ or } 2$$

in Q1, so $x = 1$, $y = 2$

gradient of normal $\frac{dy}{dx} = \frac{x+y}{3x+y} = \frac{1+2}{3+2} = \frac{3}{5}$

Question 3 (5 marks)

A coffee machine dispenses volumes of coffee that are normally distributed with mean 240 mL and standard deviation 8 mL. The machine also has the option of adding milk to a cup of coffee, where the volume of milk dispensed is also normally distributed with mean 10 mL and standard deviation 2 mL.

Let the random variable X represent the volume of coffee the machine dispenses and let the random variable Y represent the volume of milk the machine dispenses. X and Y are independent random variables.

a. Find the mean and variance of the volume of the combined drink, that is, a cup of coffee with milk. 2 marks

$E(x+y) = E(x) + E(y)$

mean of combined : $240 + 10 = 250$ mL

$Var(x+y) = Var(x) + Var(y)$

variance of combined : $64 + 4 = 68$

A second coffee machine also dispenses volumes of coffee that are normally distributed. The owner has been told that the mean volume is again 240 mL. The owner is concerned that the second coffee machine is, on average, dispensing less coffee than the first. A sample of 16 cups of coffee (with no milk) is dispensed and it is found that the mean volume of all coffees served in this sample is 235 mL. Assume that the population standard deviation of 8 mL is unchanged.

b. **i.** State appropriate null and alternative hypotheses for the volume V in this situation. 1 mark

H_0 : null hypothesis is that there is no difference

H_1 : alternative hypothesis is that there is a difference and new mean < original machine mean.

ii. The p value for this test is given by the expression $\Pr(Z \le a)$, where Z has the standard normal distribution.

Find the value of a and **hence** determine whether the null hypothesis should be rejected at the 0.05 level of significance. 2 marks

$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{235 - 240}{8/\sqrt{16}} = \frac{-5}{8/4} = -2.5$

$p(-3\sigma) \Rightarrow 0.0013$

$p(-2\sigma) \Rightarrow 0.0227$

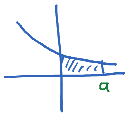
p is less than 0.05, so reject H_0

TURN OVER

Question 4 (4 marks)

The region in the first quadrant enclosed by the coordinate axes, the graph with equation $y = e^{-x}$ and the straight line $x = a$ where $a > 0$, is rotated about the x-axis to form a solid of revolution.

a. Express the volume of the solid of revolution as a definite integral. 1 mark



$$V = \pi \int_0^a \left(e^{-x}\right)^2 \, dx$$

$$= \pi \int_0^a e^{-2x} \, dx$$

b. Calculate the volume of the solid of revolution in terms of a. 1 mark

$$V = \pi \int_0^a e^{-2x} \, dx$$

$$= \pi \left[-\tfrac{1}{2} e^{-2x} \right]_0^a$$

$$= \pi \left[\left(-\tfrac{1}{2} e^{-2a} \right) - \left(-\tfrac{1}{2} \right) \right]$$

$$= \frac{\pi}{2} - \frac{\pi}{2e^{2a}}$$

c. Find the exact value of a if the volume is $\frac{5\pi}{18}$ cubic units. 2 marks

If $V = \frac{5\pi}{18}$

$$\frac{5\pi}{18} = \frac{\pi}{2} - \frac{\pi}{2} e^{-2a} \qquad \div \pi$$

$$\frac{5}{18} - \frac{9}{18} = -\frac{1}{2} e^{-2a}$$

$$-\frac{4}{18} = -\frac{1}{2} e^{-2a}$$

$$\frac{4}{9} = e^{-2a}$$

$$\ln \frac{4}{9} = -2a$$

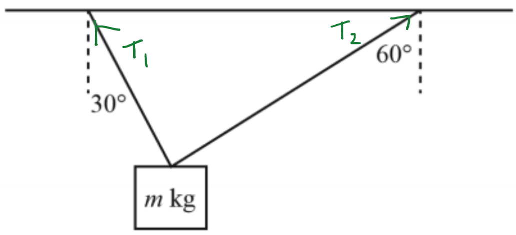
$$a = -\frac{1}{2} \ln \frac{4}{9}$$

$$= \ln \left(\frac{4}{9} \right)^{-1/2}$$

$$= \ln \frac{3}{2}$$

Question 5 (4 marks)

A flowerpot of mass m kilograms is held in equilibrium by two light ropes, both of which are connected to a ceiling. The first rope makes an angle of 30° to the vertical and has tension T_1 newtons. The second rope makes an angle of 60° to the vertical and has tension T_2 newtons.



a. Show that $T_2 = \frac{T_1}{\sqrt{3}}$. 1 mark

Horizontal forces: $T_1 \sin 30 = T_2 \sin 60$

$T_1 \times \frac{1}{2} = T_2 \times \frac{\sqrt{3}}{2}$

$T_2 = \frac{T_1}{\sqrt{3}}$

b. The first rope is strong, but the second rope will break if the tension in it exceeds 98 N.

Find the maximum value of m for which the flowerpot will remain in equilibrium. 3 marks

Vertical forces $T_1 \cos 30 + T_2 \cos 60 = 9.8\,m$ $T_1 = \sqrt{3}\,T_2$

$\sqrt{3}\,T_2 \times \frac{\sqrt{3}}{2} + T_2 \times \frac{1}{2} = 9.8\,m$

$\frac{3}{2} T_2 + \frac{1}{2} T_2 = 9.8\,m$

$T_2 = 4.9\,m$

T_2 max is 98 N $\frac{98}{4.9} = \max m$

$m = 20$ kg

TURN OVER

Question 6 (3 marks)

Evaluate $\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cos^2(2x)\sin(2x)\,dx$.

let $u = 2x$ $\quad \frac{du}{dx} = 2 \quad \frac{dx}{du} = \frac{1}{2}$

let $v = \cos u \quad \frac{dv}{du} = -\sin u$

$\sin u = -\frac{dv}{du}$

$\int \sin 2x \cos^2(2x)\, dx$

$= \frac{1}{2}\int \sin u \cos^2 u \, du$

$= \frac{1}{2}\int -\frac{dv}{du} v^2 \, du$

$= -\frac{1}{2}\int v^2 \, dv$

$= -\frac{1}{2}\frac{v^3}{3} + c$

$= -\frac{1}{6}v^3 + c = -\frac{1}{6}\cos^3 2x + c$

definite integral

$-\frac{1}{6}\left[\cos^3 2x\right]_{\pi/2}^{3\pi/4}$

$= -\frac{1}{6}\left(\cos^3 \frac{3\pi}{2} - \cos^3 \pi\right)$

$= -\frac{1}{6}(0 - -1)$

$= -\frac{1}{6}$

OR $\quad u = \cos 2x \quad \frac{du}{dx} = -2\sin 2x$

$\int_{\pi/2}^{3\pi/4} \cos^2 2x \sin 2x \, dx \qquad x = \frac{3\pi}{4} \quad u = 0$

$= -\frac{1}{2}\int_{-1}^{0} u^2 \, du \qquad x = \frac{\pi}{2} \quad u = -1$

$= -\frac{1}{2}\cdot\frac{1}{3}\left[u^3\right]_{-1}^{0} = -\frac{1}{6}(0 - -1) = -\frac{1}{6}$

Question 7 (4 marks)

Solve the following differential equation $\frac{dy}{dx} = \frac{y}{x^2}$ for y, given that when $x = 1$, $y = -1$.

$\frac{dy}{dx} = \frac{y}{x^2}$

$\frac{dy}{dx \cdot y} = \frac{1}{x^2}$ $\quad$ integrate both sides

$\int \frac{dy}{dx}\cdot\frac{1}{y}\, dx = \int \frac{1}{x^2}\, dx$

$\int \frac{1}{y}\, dy = \int \frac{1}{x^2}\, dx$

$\log y = -\frac{1}{x} + c$

$y = e^{-\frac{1}{x} + c}$ $\quad$ when $x = 1$, $y = 1$

$1 = e^0$ $\quad$ so $\quad -\frac{1}{x} + c = 0 \quad c = \frac{1}{x} = 1$

$y = e^{-\frac{1}{x} + 1}$

Question 8 (4 marks)

a. Write down a definite integral in terms of θ that gives the arc length from $\theta = 0$ to $\theta = \pi$ for the curve defined parametrically by 2 marks

$$x = \cos(2\theta) - 3$$

$$y = \sin(2\theta) + 1$$

$\cos(2\theta) = x+3$

$\sin(2\theta) = y-1$

$\sin^2(2\theta) + \cos^2(2\theta) = (x+3)^2 + (y-1)^2$

$= 1$

$(x+3)^2 + (y-1)^2 = 1$ circle

centre $-3, 1$

$\int_0^{\pi} \sqrt{dx^2 + dy^2}$

b. **Hence** find the length of this arc. 2 marks

$\int_0^{\pi} 2 \, d\theta = 2\pi$

TURN OVER

Question 9 (5 marks)

Consider the three vectors $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + m\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, where $m \in R$.

a. Find the value(s) of m for which $|\underset{\sim}{b}| = 2\sqrt{3}$. 2 marks

|b| means mag.

$2\sqrt{3} = \sqrt{1^2 + 2^2 + m^2}$ so $m = \pm\sqrt{7}$

$4 \times 3 = 1 + 4 + m^2$

$12 = 5 + m^2$

b. Find the value of m such that $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$. 1 mark

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |a||b| \cos\theta$ if perp. $\cos 90 = 0$, so $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$1 - 2 + 2m = 0$ $2m = 1$ $m = \frac{1}{2}$

c. i. Calculate $3\underset{\sim}{c} - \underset{\sim}{a}$. 1 mark

$3(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) - (\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k})$

$3\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k} - \underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} = 2\underset{\sim}{i} + 4\underset{\sim}{j} - 5\underset{\sim}{k}$

ii. Hence find a value of m such that $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are **linearly dependent**. 1 mark

$\underset{\sim}{c} = \underset{\sim}{\alpha}\,\underset{\sim}{a} + \beta\underset{\sim}{b}$

$i + j - k = \alpha(i - j + 2k) + \beta(i + 2j + mk)$

$i + j - k = \alpha i - \alpha j + 2\alpha k + \beta i + 2\beta j + m\beta k$

$i: \quad 1 = \alpha + \beta$ $\quad\quad \beta = \frac{2}{3}$

$j: \quad 1 = -\alpha + 2\beta$

$k: \quad -1 = 2\alpha + m\beta$ $\quad\quad \alpha = \frac{1}{3}$

$m = \frac{-5}{2}$

Question 10 (4 marks)

a. Verify that $\dfrac{5x^3+12x+4}{x^2(x^2+4)}$ can be written as $\dfrac{1}{x^2}+\dfrac{3}{x}+\dfrac{2x-1}{x^2+4}$. 1 mark

show $5x^3+12x+4 = 1(x^2+4)+3(x)(x^2+4)+(2x-1)(x^2)$

$= \cancel{x^2}+4+3x^3+12x+2x^3-\cancel{x^2}$

$= 5x^3+12x+4$

$= LHS$

b. Find an antiderivative of $\dfrac{5x^3+12x+4}{x^2(x^2+4)}$. 3 marks

$\int\left(\frac{1}{x^2}+\frac{3}{x}+\frac{2x-1}{x^2+4}\right)dx$

$= \int\left(\frac{1}{x^2}+\frac{3}{x}+\frac{2x}{x^2+4}-\frac{1}{x^2+4}\right)dx$

$= -\frac{1}{x}+3\ln(|x|)+\ln(x^2+4)-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

END OF QUESTION AND ANSWER BOOK