

Billanook College

September Exam 2016

VCE Specialist Mathematics Examination 2

Written Examination

Question and Answer Booklet

Reading time: 15 minutes

Writing time: 2 hour

Student's Name: _____

Instructions to students

This exam consists of Section A and Section B.

Section A consists of 20 multiple-choice questions and should be answered on the separate answer sheet.

Section B consists of 6 extended-answer questions.

Section A begins on page 2 of this exam and is worth 20 marks.

Section B begins on page 11 of this exam and is worth 60 marks.

There is a total of 80 marks available.

All questions in Section A and B should be answered.

In Section B, where more than one mark is allocated to a question, appropriate working must be shown.

An exact value is required to a question unless otherwise directed.

Unless otherwise stated, diagrams in this exam are not drawn to scale.

The acceleration due to gravity should be taken to have magnitude $g \text{ m/s}^2$ where $g = 9.8$

Students may bring one bound reference into the exam.

Students may bring into the exam one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory does not need to be cleared. For approved computer-based CAS, full functionality may be used.

Students are NOT permitted to bring mobile phones and/or any other electronic communications equipment into the examination room.

SECTION A – Multiple-choice questions

Question 1

The graph of $y = \frac{x^2 - 3x - 4}{x^2 - x - 12}$ will include

- A. asymptotes at $x = 1$ and $x = -3$
- B. asymptotes at $x = 4$ and $x = -3$
- C. an asymptote at $x = 1$ and a point of discontinuity at $x = 2$
- D. an asymptote at $x = 4$ and a point of discontinuity at $x = -3$
- E. an asymptote at $x = -3$ and a point of discontinuity at $x = 4$

Question 2

The domain of the function $f(x) = \arccos(3 - x)$ is

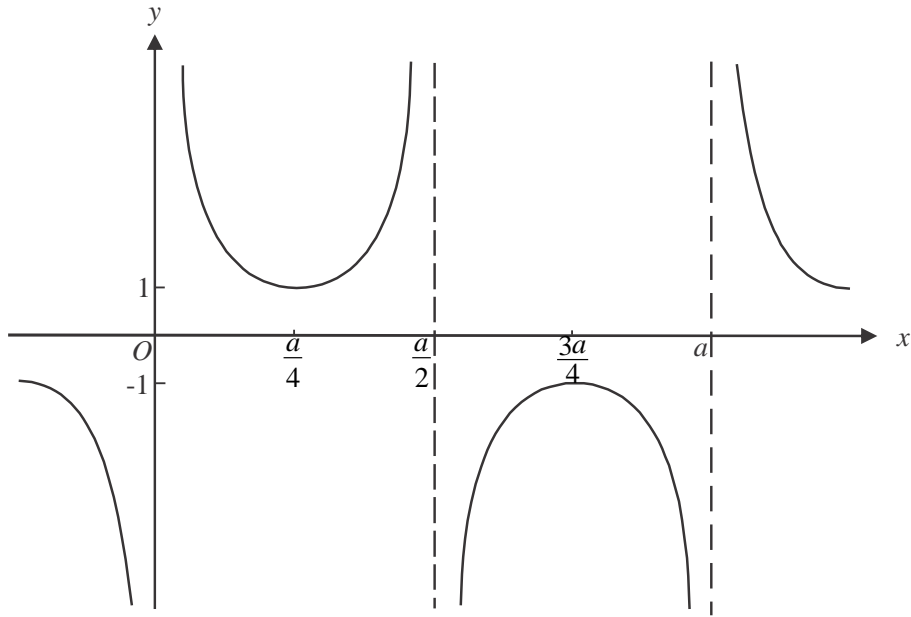
- A. $[-1, 1]$
- B. $[-4, 2]$
- C. $[-2, 4]$
- D. $[2, 4]$
- E. $(-\infty, 2] \cup [4, \infty)$

Question 3

The algebraic function $\frac{2x^2 + 1}{(x^2 - 1)(x^2 + 4)}$ could be expressed as the partial fractions

- A. $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x-2}$
- B. $\frac{A}{x-1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4}$
- C. $\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+4}$
- D. $\frac{A}{x-1} + \frac{B}{x^2-1} + \frac{Cx+D}{x^2+4}$
- E. $\frac{Ax^2+B}{x^2-1} + \frac{Cx+D}{x^2+4}$

Question 4



A rule for the function shown above where a is a positive constant, could be

- A. $y = \sec\left(\frac{2\rho}{a}x - \frac{a}{4}\right)$
- B. $y = \sec\left(\frac{\rho}{a}x - \frac{a}{4}\right)$
- C. $y = \sec\left(\frac{2\rho}{a}x + \frac{a}{4}\right)$
- D. $y = \sec\left(\frac{\rho}{a}x + \frac{a}{4}\right)$
- E. $y = \sec\left(\frac{\rho}{a}x - \frac{a}{2}\right)$

Question 5

On an Argand diagram, the graphs of the relations $|z - 2 - 2i| = 2$ and $\text{Arg}(z) = \theta$ intersect. The maximum possible interval for values of θ is

- A. $-\rho < \theta < \rho$
- B. $0 < \theta < \rho$
- C. $0 < \theta < \frac{\rho}{2}$
- D. $0 \leq \theta \leq \frac{\pi}{2}$
- E. $-\frac{\rho}{2} < \theta < \frac{\rho}{2}$

Question 6

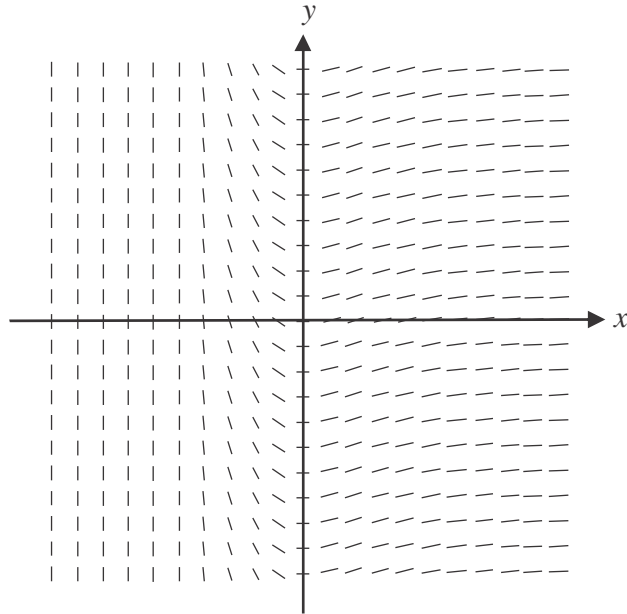
If $z_1 = 1 - \sqrt{3}i$ and $z_2 = (z_1)^{12}$ then

- A. $\text{Arg}(z_2) < 0$
- B. $\text{Im}(z_2) + \text{Re}(z_2) < 0$
- C. $|z_2| < 0$
- D. $\text{Re}(z_2) = 0$
- E. $\text{Im}(z_2) = 0$

Question 7

Using a suitable substitution the definite integral $\int_1^2 (x-3)(2x+1)^5 dx$ is equivalent to

- A. $\int_1^2 (u^6 - 7u^5) du$
- B. $\frac{1}{4} \int_3^5 (u^6 - 7u^5) du$
- C. $\frac{1}{4} \int_1^2 (u^6 - 7u^5) du$
- D. $\frac{1}{2} \int_3^5 (u^6 - 7u^5) du$
- E. $\frac{1}{2} \int_1^2 (u^6 - 7u^5) du$

Question 8

The differential equation that best represents the direction field shown above is

- A. $\frac{dy}{dx} = e^x$
 B. $\frac{dy}{dx} = xe^x$
 C. $\frac{dy}{dx} = xe^{-x}$
 D. $\frac{dy}{dx} = xye^x$
 E. $\frac{dy}{dx} = ye^{-x}$

Question 9

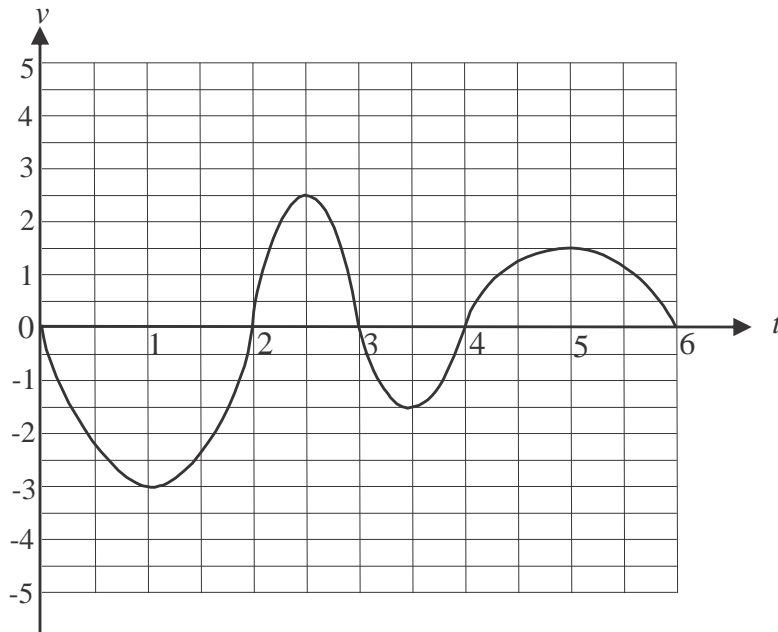
Consider the differential equation $\frac{dy}{dx} = x^{\frac{3}{2}} + 2x$, with $x_0 = 1$ and $y_0 = 2$.

Using Euler's method with a step size of 0.1, the value of y_2 correct to three decimal places is

- A. 1.508
 B. 1.635
 C. 2.300
 D. 2.635
 E. 3.109

Question 10

The velocity-time graph for a particle travelling in a straight line is shown below.



The particle is furthest from its initial position during the time interval

- A. (0.5, 1.5)
- B. (1.5, 2.5)
- C. (3, 4)
- D. (4.5, 5.5)
- E. (5, 6)

Question 11

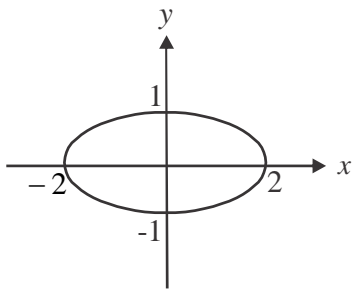
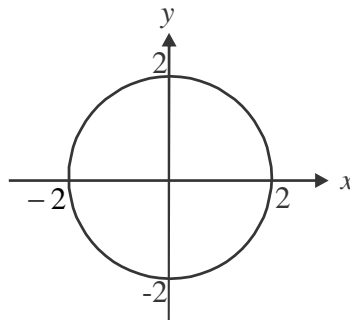
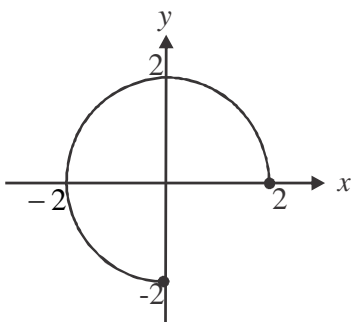
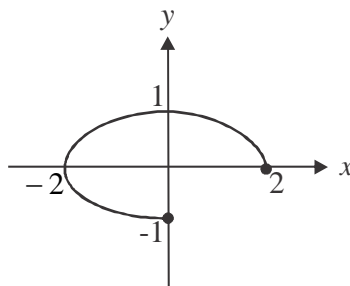
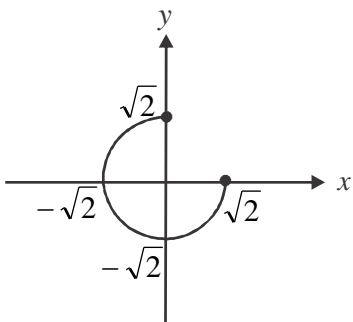
The vector resolute of $\underline{a} = 2\underline{i} + \underline{j} + \underline{k}$ in the direction of $\underline{b} = \underline{i} - \underline{k}$ is

- A. $\frac{1}{\sqrt{2}}$
- B. $\frac{1}{2}$
- C. $\frac{1}{2}(\underline{i} - \underline{k})$
- D. $\underline{i} - \underline{k}$
- E. $\frac{1}{2}(3\underline{i} + 2\underline{j} + 3\underline{k})$

Question 12

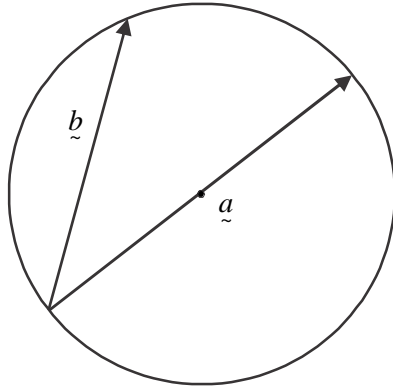
The position vector of a particle at time t seconds is given by $\underline{r}(t) = 2\cos(t)\underline{i} + \sin(t)\underline{j}$, $0 \leq t \leq \frac{3\pi}{2}$.

The path of the particle is shown by

A.**B.****C.****D.****E.**

Question 13

The diagram below shows a circle. The vector \underline{a} passes through the centre of the circle and vectors \underline{a} and \underline{b} meet at the circumference of the circle.



It can definitely be deduced from this diagram that

- A. $\underline{a} \cdot \underline{b} = 0$
- B. $\underline{a} \cdot \underline{b} = 1$
- C. $\underline{a} \cdot \underline{a} = \underline{b} \cdot \underline{b}$
- D. $|\underline{a}| |\underline{b}| = 1$
- E. $(\underline{a} - \underline{b}) \cdot \underline{b} = 0$

Question 14

The position vectors of the points P , Q and R are given respectively by $\underline{p} = \underline{i} + \underline{k}$, $\underline{q} = 2\underline{j} - \underline{k}$ and $\underline{r} = \underline{i} - \underline{j} - 2\underline{k}$.

The angle PQR is closest to

- A. 6.2°
- B. 59.8°
- C. 72.5°
- D. 123.6°
- E. 134.7°

Question 15

A particle moving in a straight line has acceleration, $a \text{ ms}^{-2}$, given by $a = \sqrt{v+4}$, where v is the velocity of the particle in ms^{-1} at time t seconds. Initially the velocity of the particle is -3 ms^{-1} .

The velocity of the particle at time t seconds is given by

- A. $v = \frac{3t+2}{2} - 4$
- B. $v = \left(\frac{3t+2}{2}\right)^{\frac{2}{3}} - 4$
- C. $v = e^2 - 4$
- D. $v = \frac{t^2 - 8}{2}$
- E. $v = \frac{t^2 + 4t - 12}{4}$

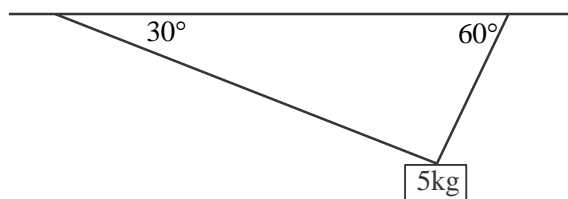
Question 16

An 85 kg man stands in a lift which is accelerating downwards at 1.8 ms^{-2} . The reaction force, in newtons, of the lift floor on the man is

- A. 17
- B. 102
- C. 680
- D. 831.8
- E. 935

Question 17

A mass of 5 kg is held in equilibrium by two light strings which are each connected to a horizontal ceiling. One of the strings makes an angle of 30° with the ceiling and has a tension T_1 newtons. The other string makes an angle of 60° with the ceiling and has a tension of T_2 newtons.



The ratio of T_2 to T_1 is

- A. $1 : \frac{1}{\sqrt{3}}$
- B. $1 : \sqrt{3}$
- C. $1 : 2$
- D. $1 : \sqrt{5}$
- E. $1 : 2\sqrt{5}$

Question 18

A popular brand of pasta sells its penne in packets labelled as 750 grams.

A random sample of 40 packets of penne have a mean weight of 748.6 grams and a standard deviation of 2.7 grams.

An approximate 95% confidence interval for the population mean μ is closest to

- A. $747.8 < \mu < 749.4$
- B. $747.9 < \mu < 748.1$
- C. $748.2 < \mu < 749.0$
- D. $748.5 < \mu < 751.5$
- E. $749.05 < \mu < 750.95$

Question 19

A census found that the mean weekly earnings for adult full time workers in a particular region was \$1554 with a standard deviation of \$162.

A random sample of 50 adult full time workers in this region was taken. The probability that the mean weekly earnings for this group was less than \$1500 is closest to

- A. 0.0092
- B. 0.0522
- C. 0.2389
- D. 0.3694
- E. 0.4986

Question 20

A statistical test was conducted involving a null hypothesis H_0 and an alternative hypothesis H_1 . The results of the test indicated that H_0 should be rejected.

Which one of the following statements **could not** be correct?

- A. A correct decision has been made.
- B. A type 1 error has been made.
- C. An incorrect decision has been made.
- D. A type 2 error has been made.
- E. H_1 is true.

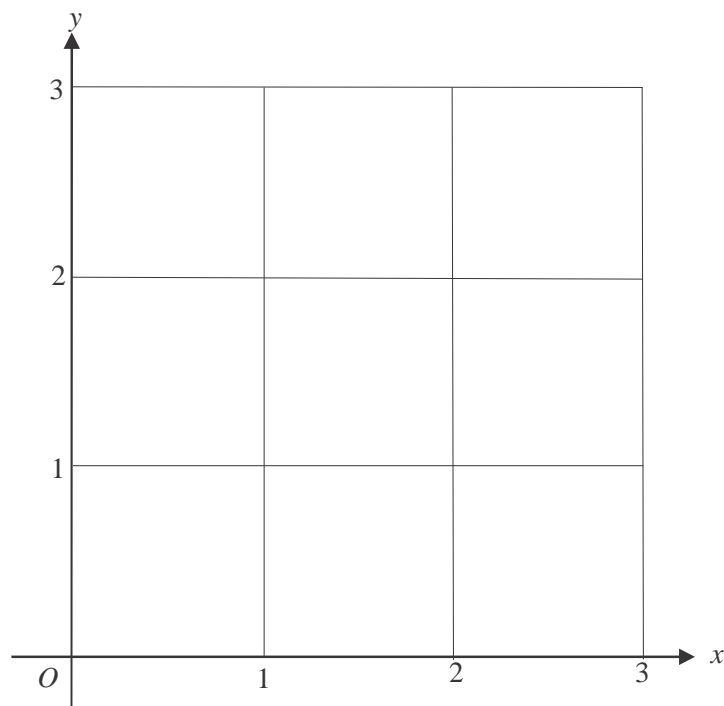
SECTION B

Question 1 (11 marks)

Consider $f : [0, 2] \rightarrow \mathbb{R}$, $f(x) = 1 + \frac{4}{\pi} \arctan(x-1)$.

- a. Given that $f''(x) = \frac{-8(x-1)}{\pi(x^2 - 2x + 2)^2}$, show that the graph of f has a point of inflection at $x = 1$. 2 marks

- b. The inverse function of f is f^{-1} . Sketch the graphs of f and f^{-1} on the set of axes below. Indicate clearly endpoints and points of inflection. 2 marks



- c. Given that $f^{-1}(x) = a + \tan\left(\frac{b}{c}(x-a)\right)$, find the values of a , b , and c . 2 marks

Let A represent the area, in square units, enclosed by the graphs of f and f^{-1} .

- d. Given that $\int_0^1 (f^{-1}(x) - g(x)) dx = \frac{A}{4}$, find the rule for the function g . 1 mark

- e. Hence or otherwise find A . Give your answer correct to one decimal place. 2 marks

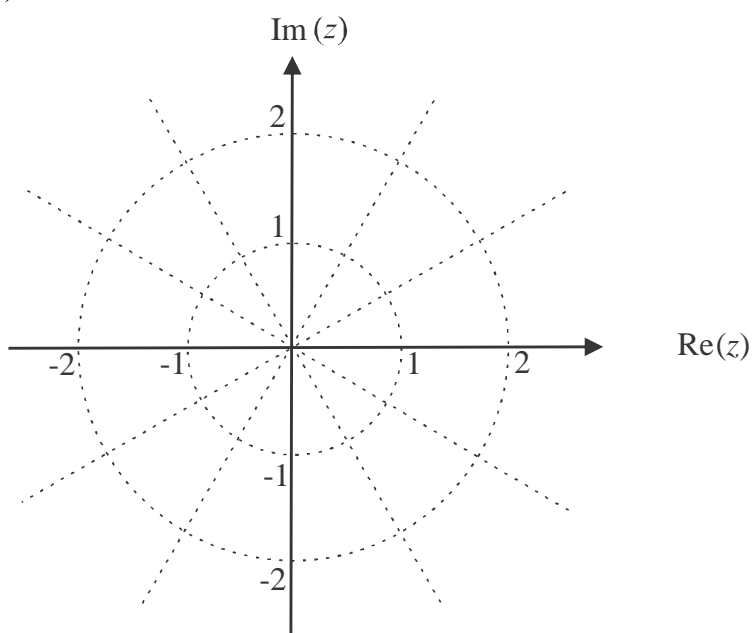
- f. The region enclosed by the graph of f , the y -axis, and the line $y = 1$ is rotated about the y -axis to form a solid of revolution. Find the volume of this solid. Give your answer correct to one decimal place. 2 marks

Question 2 (10 marks)

Let $u = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

- a. i.** Express u in polar form. 2 marks

- ii.** The complex number u is one root of the equation $z^3 - i = 0$.
Plot all the roots of this equation on the Argand diagram below labelling them u , v and w ,
where $\text{Im}(w) < 0$. 2 marks



- iii.** Let $S = \{z : |z - u| = |z - w|\}$.

Sketch S on the Argand diagram above. 1 mark

- iv.** Show algebraically that v lies on S . 2 marks

- b.** Consider the equation $z^2 + 6i \sin(\alpha)z - 9 = 0$, where $z \in C$, α is a real constant and $0 < \alpha < \frac{\pi}{3}$.

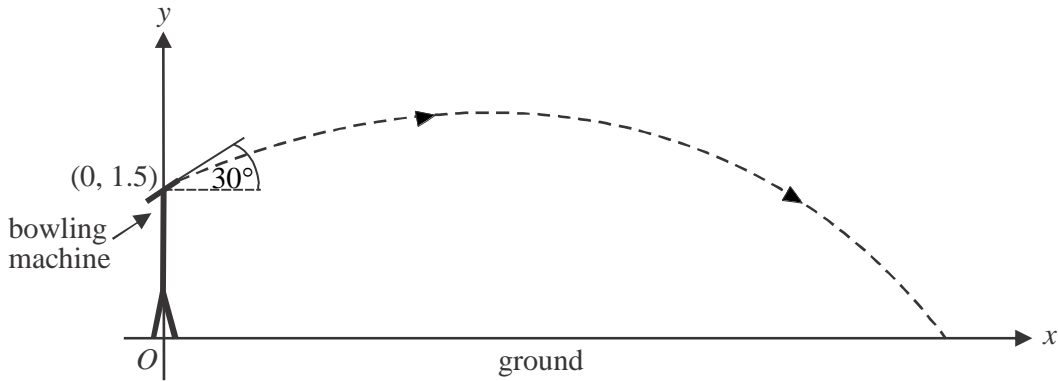
Find the roots z_1 and z_2 of this equation. Express them in polar form in terms of α .

3 marks

Question 3 (11 marks)

A bowling machine is programmed to project a ball from a height of 1.5 m above the ground at an angle of 30° to the horizontal as shown below. The ball's initial speed is 8 ms^{-1} and it travels in a vertical plane, landing on the ground T seconds later.

On a Cartesian graph, the origin O is located 1.5 m vertically below the point where the ball is projected. Let \hat{i} represent a unit vector in the positive x direction and let \hat{j} represent a unit vector in the positive y direction where displacement is measured in metres and time is in seconds.



- a. Show that the ball has an initial velocity when projected of $4\sqrt{3}\hat{i} + 4\hat{j}$ 1 mark

- b.** The acceleration of the ball t seconds after being projected is given by

$$\ddot{\mathbf{r}}(t) = -\frac{t}{50}\mathbf{i} + \left(\frac{t}{20} - g\right)\mathbf{j}, \quad 0 \leq t \leq T.$$

Show that the position vector of the ball t seconds after being projected by the machine is given by

$$\mathbf{r}(t) = 4\sqrt{3}t - \frac{t^3}{300}\mathbf{i} + \left(\frac{t^3}{120} - \frac{gt^2}{2} + 4t + 1.5\right)\mathbf{j}, \quad 0 \leq t \leq T \quad 3 \text{ marks}$$

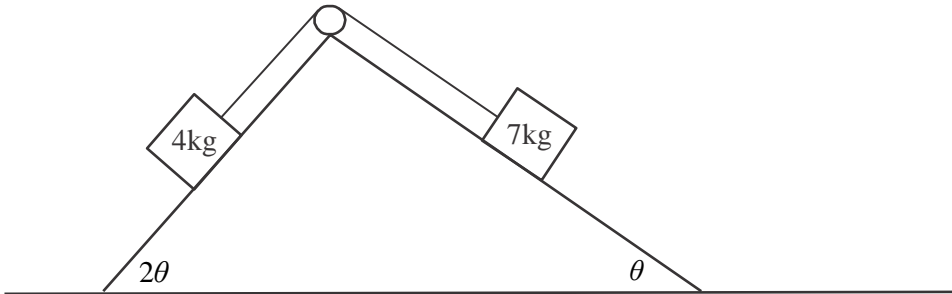
- c.** How far does the ball travel horizontally before it hits the ground? Give your answer in metres correct to 1 decimal place. 2 marks

- d.** What is the distance, in metres, that the ball travels from when it is projected until it hits the ground?
Give your answer correct to one decimal place. 3 marks

- e.** Find the speed of the ball, in metres per second, when it hits the ground. Give your answer correct to one decimal place. 2 marks

Question 4 (9 marks)

Particles of mass 4 kg and 7 kg are shown on adjoining smooth planes in the diagram below. The planes are inclined to the horizontal at angles of 2θ and θ respectively where $0^\circ < \theta < 45^\circ$. A light inextensible string passes over a smooth pulley and connects the two particles. The tension in the string is T newtons. The 7 kg mass accelerates down the plane at $a \text{ ms}^{-2}$.



- a. Find an equation for the motion of the 4 kg particle up the plane. 1 mark

- b. Hence show that the acceleration of the 7 kg mass down the other plane is given by

$$a = \frac{g \sin(\theta)}{11} (7 - 8 \cos(\theta)).$$

3 marks

- c. Find the angle θ when the system is in equilibrium. Give your answer correct to one decimal place. 2 marks

Suppose θ is fixed at 30° and a mass of m kg is added to the 4 kg mass. The 7 kg mass now accelerates up the plane with an acceleration of $\frac{g(11\sqrt{3}-14)}{50}\text{ms}^{-2}$.

d. Find the value of m .

3 marks

Question 5 (10 marks)

A tyre manufacturer produces thirteen inch car tyres using rubber and other components. The random variable R represents the weight, in kg, of rubber in one of these tyres.

The random variable C represents the weight, in kg, of the other components in one of these tyres.

Both variables are independent and are normally distributed with mean and variance as shown in the table below.

random variable	mean (μ)	variance (σ^2)
R	2.2	0.18
C	4.9	0.31

Let the random variable W represent **four** times the weight of **one** of these thirteen inch car tyres.

- a. Find the mean and standard deviation of W .

2 marks

The tyre manufacturer also produces fifteen inch car tyres. The distribution of the weights of these tyres is normal and it is claimed by the manufacturer that $\mu = 8.6$ kg .

A motoring organization tests these claims by taking a random sample of 36 tyres. It finds the mean weight for the sample to be $\bar{x} = 8.4$ kg and the standard deviation to be $s = 0.54$ kg. Using this sample data it checks the manufacturer's claims that $\mu = 8.6$ kg by conducting a statistical test.

It is assumed that the population standard deviation (σ) of these tyre weights can be estimated with sufficient accuracy by s .

- b. Write down two hypotheses, H_0 and H_1 , that could be used to test whether the mean weight of the tyres is less than that being claimed by the manufacturer.

2 marks

- c. Find the p value for this statistical test. Give your answer correct to three decimal places. 2 marks

- d. State whether H_0 should be rejected or not at the 5% level of significance. Give a reason to support your answer. 1 mark

The motoring organization decides to repeat its test of the manufacturers claim that $\mu = 8.6$ kg but this time it will be at the 1% level of significance.
Using the same hypotheses, it again takes a random sample of 36 tyres. The standard deviation of this second sample is found again to be 0.54 kg.

The motoring organization concludes that according to this second sample, the null hypothesis should not be rejected at the 1% level of significance.

- e. Find the minimum value of the sample mean \bar{x} that the motoring organization could have found in its second sample. Give your answer correct to three decimal places. 3 marks

Question 6 (9 marks)

A tank initially contains 500 litres of brine which is a solution of water and salt. There is 100 grams of salt in the tank initially.

Water is pumped into the tank at the rate of 60 litres per minute and brine is pumped out of the tank at the rate of 20 litres per minute.

The amount of salt in the tank t minutes after the pumping begins is x grams. The solution in the tank is kept uniform by constant stirring.

- a. i.** Write down an expression for the concentration of salt in the tank, in grams per litre, in terms of x and t . 1 mark

- ii.** Show that the differential equation which represents the rate of change of x with respect to t is given by $\frac{dx}{dt} + \frac{x}{25+2t} = 0$. 2 marks

- b.** Show that $x(t) = \frac{500}{\sqrt{25+2t}}$. 2 marks

- c. Verify by substitution that $x(t)$ satisfies the differential equation $\frac{dx}{dt} + \frac{x}{25+2t} = 0$ and the initial conditions. 2 marks

- d. Find the amount of salt that flowed out of the tank in the first twenty minutes after the pumping began. Give your answer in grams correct to one decimal place. 2 marks

SPECIALIST MATHS
TRIAL EXAMINATION 2
MULTIPLE - CHOICE ANSWER SHEET

STUDENT NAME:.....

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A B C D E

The answer selected is B. Only one answer should be selected.

1. A B C D E
2. A B C D E
3. A B C D E
4. A B C D E
5. A B C D E
6. A B C D E
7. A B C D E
8. A B C D E
9. A B C D E
10. A B C D E

11. A B C D E
12. A B C D E
13. A B C D E
14. A B C D E
15. A B C D E
16. A B C D E
17. A B C D E
18. A B C D E
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