SPECIALIST MATHEMATICS

Written examination 1

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2024 Trial Examination

SOLUTIONS

Question 1 (4 marks)

Answer:

$S(n) = 3^n + 3, n \in N$
$S(1) = 3^1 + 3 = 6$
So, S is divisible for the smallest Natural number.

1 W

Assume the proposition is true for $x = k,\ k \in N$.
So, $S(k) = 3^k + 3, n \in N$ is divisible by 6.
$S(k) = 3^k + 3 = 6m, m \in N$

1 W

Test the proposition for $n = k + 1,\ k \in N$.
$S(k+1) = 3^{k+1} + 3$
$= 3 \times 3^k + 3$
$= 3(3^k + 3) - 6$
$= 3(6m) - 6$
$= 6(3m - 1)$

1 W

Since $S(k+1)$ has a common factor of 6, $S(k+1)$ must be divisible by 6.
Since $S(1), S(K)$and $S(k+1)$ are all divisible by 6, S must be divisible by 6.

1 A

Question 2 (5 marks)

a. (1 mark)

$z_1 = 2cis\left(\frac{\pi}{6}\right)$

$z_2 = \overline{z_1} = 2cis\left(-\frac{\pi}{6}\right)$

1 A

b. (2 marks)

$z^2 + bz + c = 0$

$z_1 = 2cis\left(\frac{\pi}{6}\right) = \sqrt{3} + i$

$z_2 = 2cis\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$

$z^2 + bz + c = (z - \sqrt{3} - i)(z - \sqrt{3} + i)$

1 W

$= z^2 - 2\sqrt{3}\,z + 4$

$b = -2\sqrt{3}\,, c = 4$

1 A

c. (2 marks)

$P(z) = z^2 + 2iz - 4 = 0$

$P(z_2) = (\sqrt{3} - i)^2 + 2i(\sqrt{3} - i) - 4$

$= 3 - 2\sqrt{3}\,i - 1 + 2\sqrt{3}\,i + 2 - 4 = 0$

So, $z = \sqrt{3} - i$ is a root of $P(z) = z^2 + 2iz - 4 = 0$

$P(z) = z^2 + 2iz - 4 = (z - \sqrt{3} + i)(z + \sqrt{3} + i) = 0$

1 W

Roots are $z_2 = \sqrt{3} - i, z_3 = -\sqrt{3} - i$

$z_2 \times\ z_3 = (\sqrt{3} - i)(-\sqrt{3} - i)$

$= -3 - i\sqrt{3} + i\sqrt{3} - 1 = -4 \in R$

1 A

Question 3 (4 marks)

$3x^2y - 2xy^2 + y = 1$

$\frac{d}{dx}(3x^2y - 2xy^2 + y) = \frac{d}{dx}(1)$

$6xy + 3x^2\frac{dy}{dx} - 2y^2 - 4xy\frac{dy}{dx} + \frac{dy}{dx} = 0$

$\frac{dy}{dx}(3x^2 - 4xy + 1) = 2y^2 - 6xy$

$\frac{dy}{dx} = \frac{2y^2 - 6xy}{3x^2 - 4xy + 1}$

1 W

$3x^2y - 2xy^2 + y = 1$

When $y = 1$

$3x^2 - 2x + 1 = 1$

$x(3x - 2) = 0$

$x = 0 \text{ or } x = \frac{2}{3}$

Point on curve is $(\frac{2}{3}, 1)$

1 W

$\frac{dy}{dx} = \frac{2 - 6\left(\frac{2}{3}\right)}{3(\frac{2}{3})^2 - 4\left(\frac{2}{3}\right) + 1} = 6$

Gradient of normal is: $-\frac{1}{6}$

1 W

$y = -\frac{1}{6}x + c$

Sub. in $(\frac{2}{3}, 1)$

$y = -\frac{1}{6}x + \frac{10}{9}$

1 A

Question 4 (3 marks)

Answer:
$\bar{x} = 99$

$sd(\bar{x}) = \dfrac{3}{\sqrt{n}}$

$\Pr(Z > 2) \approx 0.025 = 2.5\%$

1 W

$$z = 2 = \frac{100 - 99}{\frac{3}{\sqrt{n}}}$$

1 W

$\dfrac{6}{\sqrt{n}} = 1$

$\sqrt{n} = 6$

$n = 36$

36 strips of turf were sampled.

1 A

Question 5 (4 marks)

a.

$y = \dfrac{x^3}{x^2 - 9}$

$y = \dfrac{x(x^2 - 9) + 9x}{x^2 - 9}$

$y = x + \dfrac{9x}{x^2 - 9}$

$y = x + \dfrac{9x}{(x+3)(x-3)}$

1 W

Vertical asymptotes: $x = -3, x = 3$

Oblique asymptote: $y = x$

1 A

b.

The only intercept on $y = \frac{x^3}{x^2-9}$ is at (0,0)

1 A

c.

The graph crosses the asymptote $y = x$ at (0,0). Tammy is correct.

1 A

Question 6 (4 marks)

a.

$\frac{dP}{dt} = \frac{1}{8}P(4-P)$

$\int \left(\frac{8}{P(4-P)}\right) dP = \int 1\, dt$

$$\frac{8}{P(4-P)} = \frac{2}{P} + \frac{2}{4-P}$$

1 W

$\int \left(\frac{2}{P} + \frac{2}{4-P}\right) dP = \int 1\, dt$

$2log_e P - 2log_e(4-P) + c = t$

$t = 0, P = 2$

$2log_e 2 - 2log_e(2) + c = 0$

$c = 0$

$2log_e P - 2log_e(4-P) = t$

$$2log_e\left(\frac{P}{4-P}\right) = t$$

1 W

$$\frac{P}{4-P} = e^{\frac{t}{2}}$$

$P = (4-P)e^{\frac{t}{2}}$

$P = 4e^{\frac{t}{2}} - Pe^{\frac{t}{2}}$

$P + Pe^{\frac{t}{2}} = 4e^{\frac{t}{2}}$

$P(1 + e^{\frac{t}{2}}) = 4e^{\frac{t}{2}}$

$$P = \frac{4e^{\frac{t}{2}}}{1 + e^{\frac{t}{2}}} = 4 - \frac{4}{1 + e^{\frac{t}{2}}}$$

1 A

b.

$P = 4 - \dfrac{4}{1 + e^{\frac{t}{2}}}$

As $t \to \infty, P \to 4$

$P \in [2,4)$

1 A

Question 7 (8 marks)

a.

For a suitable domain, $y = \tan^{-1} x$ and $\boldsymbol{x = \tan y}$ are equivalent.

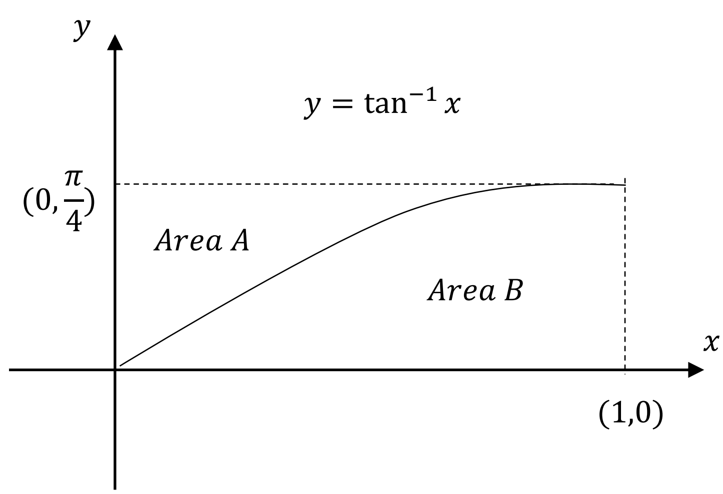
1 A

b.

$a = f(0) = \tan^{-1} 0 = 0$, $b = f(1) = \tan^{-1} 1 = \dfrac{\pi}{4}$

1 A

c.



Area A

$\int_0^{\frac{\pi}{4}} \tan y \; dy$

$= [-log_e(\cos y)]_0^{\frac{\pi}{4}}$

1 W

$= -log_e(\cos\frac{\pi}{4}) + log_e(\cos 0)$

$= log_e(\sqrt{2})$

1 W

Area B

$= \frac{\pi}{4} \times 1 - log_e(\sqrt{2})$

$= \frac{\pi}{4} - \frac{1}{2} log_e(2)$

1 A

d.

$V_y = \pi \int_0^{\frac{\pi}{4}} (x^2)\, dy$

$V_y = \pi \int_0^{\frac{\pi}{4}} (\tan^2 y)\, dy$

1 W

$V_y = \pi \int_0^{\frac{\pi}{4}} (\sec^2 y - 1)\, dy$

$= \pi[\tan y - y]_0^{\frac{\pi}{4}}$

1 W

$= \pi\left(\tan\frac{\pi}{4} - \frac{\pi}{4} - \tan 0 + 0\right)$

$= \pi(1 - \frac{\pi}{4})$

1 A

Question 8 (8 marks)
$A(1,1,-2), B(0,2,-1), C(2,-1,-5)$.

a.
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

1 A

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$
$= \underset{\sim}{i} - 2\underset{\sim}{j} - 3\underset{\sim}{k}$

1 A

b.
$\overrightarrow{AB} \times \overrightarrow{AC} = (-3 - -2)\underset{\sim}{i} - (3-1)\underset{\sim}{j} + (2-1)\underset{\sim}{k} = -\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{n} = -\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ is normal to the plane.

1 W, 1 A

c.
$\underset{\sim}{\hat{n}} = \frac{1}{\sqrt{6}}\left(-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

1 A

d.
Use any point on the plane, say A, and $\underset{\sim}{n}$.

$\overrightarrow{OA} = \underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

Now consider a general point on the plane, say D.

$\overrightarrow{OD} = \underset{\sim}{d} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$\overrightarrow{AD}.\underset{\sim}{n} = 0$

1 W

$(\underset{\sim}{d} - \underset{\sim}{a}).\underset{\sim}{n} = 0$

$\underset{\sim}{d}\underset{\sim}{n} = \underset{\sim}{a}.\underset{\sim}{n}$

$\left(x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}\right).\left(-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) = \left(\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right).\left(-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

1 W

$-x - 2y + z = -1 - 2 - 2$
$-x - 2y + z = -5$
$x + 2y - z = 5$

1 A

