Neap

VCE Specialist Mathematics Units 3&4

Suggested Solutions

2024 Trial Examination 1

Question 1 (4 marks)

Applying implicit differentiation gives:

$$1\times\tan^{-1}y + x\times\frac{1}{1+y^2}\frac{dy}{dx}+2\frac{dy}{dx}=2x$$ A2

Note: Deduct 1 mark for a single error.

Substituting point (0, 1) gives:

$$\frac{\pi}{4}+0+2\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\pi}{8}$$ A1

$$m_{\perp}=\frac{8}{\pi}$$

$$\therefore y=\frac{8}{\pi}x+1$$ A1

Question 2 (3 marks)

$u=x$ $du=dx$

$dv=\cos(x)dx$ $v=\sin(x)$

M1

$$\int_0^{\pi}x\cos(x)dx=[x\sin(x)]_0^{\pi}-\int_0^{\pi}\sin(x)dx$$ A1

$$=0+[\cos(x)]_0^{\pi}$$

$$=-2$$ A1

Question 3 (5 marks)

a. $$\left|2-\frac{2}{\sqrt{3}}i\right|=\sqrt{4+\frac{4}{3}}$$

$$=\frac{4}{\sqrt{3}}$$ A1

$$\text{Arg}\left(2-\frac{2}{\sqrt{3}}i\right)=\tan^{-1}\left(\frac{-\frac{2}{\sqrt{3}}}{2}\right)$$

$$=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$=-\frac{\pi}{6}$$ A1

$$2-\frac{2}{\sqrt{3}}i=\frac{4}{\sqrt{3}}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\left(2-\frac{2}{\sqrt{3}}i\right)^4=\frac{256}{9}\text{cis}\left(-\frac{2\pi}{3}\right)$$ M1

$$=\frac{256}{9}\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$=-\frac{128}{9}-\frac{128\sqrt{3}}{9}i$$ A1

b. $\text{Arg}\left(\left(2-\dfrac{2}{\sqrt{3}}i\right)^n\right)=\pi k,\ k\in Z$

$-\dfrac{n\pi}{6}=\pi k$

$n=6k$ **OR** $-6k,\ k\in Z$ A1

*Note: Consequential on answer to **Question 3a.***

Question 4 (4 marks)

a. The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by $\left(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$.

$\hat{\underset{\sim}{b}}=\dfrac{1}{\sqrt{1+9+25}}\left(-\underset{\sim}{i}+3\underset{\sim}{j}-5\underset{\sim}{k}\right)$

$=\dfrac{1}{\sqrt{35}}\left(-\underset{\sim}{i}+3\underset{\sim}{j}-5\underset{\sim}{k}\right)$ A1

$\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}=\dfrac{1}{\sqrt{35}}((2\times-1)+(-4\times3)+(1\times-5))$

$=\dfrac{1}{\sqrt{35}}(-2-12-5)$

$=-\dfrac{19}{\sqrt{35}}$

$\left(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}=-\dfrac{19}{35}\left(-\underset{\sim}{i}+3\underset{\sim}{j}-5\underset{\sim}{k}\right)$

$=\dfrac{19}{35}\underset{\sim}{i}-\dfrac{57}{35}\underset{\sim}{j}+\dfrac{19}{7}\underset{\sim}{k}$ A1

b. The Cartesian equation is given by $\underset{\sim}{n}\cdot\overrightarrow{P_0P}$, where $\underset{\sim}{n}=\underset{\sim}{a}\times\underset{\sim}{b}$, $P_0=(3, 1, -1)$ and $P=(x, y, z)$.

$\underset{\sim}{a}\times\underset{\sim}{b}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & -4 & 1\\ -1 & 3 & -5\end{vmatrix}$

$=(20-3)\underset{\sim}{i}-(-10+1)\underset{\sim}{j}+(6-4)\underset{\sim}{k}$ M1

$=17\underset{\sim}{i}+9\underset{\sim}{j}+2\underset{\sim}{k}$

$17(x-3)+9(y-1)+2(z+1)=0$

$17x+9y+2z=58$ A1

Question 5 (3 marks)

$W \sim N\left(25, 1.8^2\right)$

$\bar{W} \sim N\left(25, \left(\frac{1.8}{\sqrt{36}}\right)^2\right)$ A1

$\Pr(\bar{W} < a) = 0.84$

$\Pr(Z < 1) = 0.84$ A1

$1 = \frac{a - 25}{\frac{1.8}{6}}$

$a = 25.3$ kg A1

Question 6 (4 marks)

$f'(x) = 4x^3e^{2x} + 2x^4e^{2x}$

$= 2e^{2x}\left(2x^3 + x^4\right)$ A1

$f''(x) = 4e^{2x}\left(2x^3 + x^4\right) + 2e^{2x}\left(6x^2 + 4x^3\right)$

$= 4x^2e^{2x}\left(x^2 + 4x + 3\right)$ A1

$f''(x) = 0$

$4x^2e^{2x}(x+1)(x+3) = 0$

$x = -3, -1, 0$

$f''\left(-3^-\right) > 0 \Rightarrow$ concave up for $x < -3$ M1

$f''\left(-3^+\right) < 0$

$f''\left(-1^+\right) > 0 \Rightarrow$ concave up for $x > -1$, $x \neq 0$ (stationary point of inflection at 0)

$x \in (-\infty, -3) \cup (-1, \infty) \setminus \{0\}$ A1

Question 7 (4 marks)

$S = 2\pi\int_0^1 x^3\sqrt{1 + \left(3x^2\right)^2}\,dx$

$= 2\pi\int_0^1 x^3\sqrt{1 + 9x^4}\,dx$ M1

$u = 1 + 9x^4 \Rightarrow du = 36x^3dx$ M1

$S = 2\pi\int_0^{10} \frac{1}{36}u^{\frac{1}{2}}\,du$ A1

$= \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{10}$

$= \frac{\pi}{27}\left(10\sqrt{10} - 1\right)$ A1

Question 8 (6 marks)

a. $\dot{\underset{\sim}{r}}(t) = -e^t \underset{\sim}{i} + \left(2e^{2t} - 2e^t\right)\underset{\sim}{j}$ A1

$\dot{\underset{\sim}{r}}(0) = -e^0 \underset{\sim}{i} + \left(2e^0 - 2e^0\right)\underset{\sim}{j}$

$= -\underset{\sim}{i}$

$\left|\dot{\underset{\sim}{r}}(0)\right| = \sqrt{(-1)^2 + 0^2}$

$= 1$ A1

b. $x = -e^t$

$x^2 = \left(-e^t\right)^2$

$x^2 = e^{2t}$

$y = e^{2t} - 2e^t - 1$

$= x^2 + 2x - 1$ A1

Given that $t \geq 0$, $e^t \geq 1 \Rightarrow -e^t \leq -1 \Rightarrow x \leq -1$. A1

c. Let P be a point on the curve $y = x^2 + 2x - 1$ (from **part b.**).

The coordinates of P are given by $(x, x^2 + 2x - 1)$.

Finding the distance from P to the origin gives:

$$d_{OP} = \sqrt{x^2 + \left(x^2 + 2x - 1\right)^2}$$

$$\frac{d}{dx}\left(d_{OP}\right) = \frac{1}{2}\left(x^2 + (x^2 + 2x - 1)^2\right)^{-\frac{1}{2}} \times \left(2x + 2\left(x^2 + 2x - 1\right)(2x + 2)\right)$$ M1

Minimising the distance gives:

$$\frac{d}{dx}\left(d_{OP}\right) = 0$$

Considering the numerator only gives:

$$\left(2x + 2\left(x^2 + 2x - 1\right)(2x + 2)\right) = 0$$

$$2x + 4x^3 + 4x^2 + 8x^2 + 8x - 4x - 4 = 0$$

$$4x^3 + 12x^2 + 6x - 4 = 0$$

$$2x^3 + 6x^2 + 3x - 2 = 0$$

When $x = a$:

$2a^3 + 6a^2 + 3a - 2 = 0$ M1

Note: Consequential on answer to ***Question 8b.***

Question 9 (4 marks)

$$\frac{-2x^2+13x-1}{(x+5)(x^2+4)}=\frac{A}{x+5}+\frac{Bx+C}{x^2+4}$$ M1

$$-2x^2+13x-1=A(x^2+4)+(Bx+C)(x+5)$$

Substituting $x=-5$ into the equation gives:

$$A((-5)^2+4)+(B(-5)+C)(-5+5)=-2\times(-5)^2+13\times-5-1$$
$$29A=-116$$
$$A=-4$$

Substituting $x=0$ and then $A=-4$ gives:

$$A((0)^2+4)+(B(0)+C)(0+5)=-2\times(0)^2+13\times0-1$$
$$4\times-4+5C=-1$$
$$C=3$$

Substituting $x=1$ and then $A=-4$, $C=3$ gives:

$$A((1)^2+4)+(B(1)+C)(1+5)=-2\times(1)^2+13\times1-1$$
$$5\times-4+6B+6\times3=10$$
$$B=2$$ A1

$$\int\left(\frac{-4}{x+5}+\frac{2x+3}{x^2+4}\right)dx$$ A1

$$=\int\left(\frac{-4}{x+5}+\frac{2x}{x^2+4}+\frac{3}{x^2+4}\right)dx$$

$$=-4\log_e|x+5|+\log_e(x^2+4)+\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$$ A1

Question 10 (3 marks)

Assuming $\sin(x)+\cos(x)<1$ for some $x\in\left[0,\frac{\pi}{2}\right]$: A1

$$(\sin(x)+\cos(x))^2<1^2$$
$$\sin^2(x)+2\sin(x)\cos(x)+\cos^2(x)<1$$
$$2\sin(x)\cos(x)+1<1$$
$$2\sin(x)\cos(x)<0$$ M1

This is false, since $\sin(x)\geq0$ and $\cos(x)\geq0$ for $x\in\left[0,\frac{\pi}{2}\right]$. M1

Therefore, by contradiction, proof is complete.