The Mathematical Association of Victoria

Trial Examination 2024

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of examination

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 22 pages.
- Formula sheet.
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination

- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A- Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple – choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

For the curve $2xy - y^2 = -3$, the gradient to the curve at $x = 1$ for $y > 0$ is

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. $\frac{-2y}{2x-2y}$

D. $\frac{y}{y-1}$

Question 2

The graph of $y = \frac{ax^3}{x^2 + bx - 2}$ has two vertical asymptotes at $x = 1$ and $x = -2$ and an oblique asymptote at $y = 3x - 3$. The values of a and b could be

A. $a = 1, b = 1$

B. $a = 3, b = 1$

C. $a = \frac{1}{3}, b = 1$

D. $a = 1, b = -2$

Question 3

The graph of $y = a\,\text{cosec}\left(\frac{\pi}{2}x + \pi\right)$ where $a \in R$, will have vertical asymptotes in the interval $-3 < x < 3$ when

A. $x = -\pi, x = -\frac{\pi}{2}, x = 0, x = \frac{\pi}{2}, x = \pi$

B. $x = -a, x = -\frac{a}{3}, x = \frac{a}{3}, x = a$

C. $x = -2, x = 0, x = 2$

D. $x = -2a, x = 0, x = 2a$

Question 4

Let z be a complex number where $\frac{z\bar{z}}{|z|} = \sqrt{13}$, then iz could be equivalent to

A. 13

B. $\sqrt{13}$

C. $3 + 3i$

D. $-3 + 2i$

Question 5

The position of a particle moving in the Cartesian plane, at time t, is given by the parametric equations

$$x(t) = (t+1)^3 \text{ and } y(t) = \frac{2}{t+1} \text{ where } t \geq 0.$$

The slope of the line perpendicular to the tangent to the path at $t = 1$ is

A. $-\frac{1}{24}$

B. 24

C. $\frac{-2}{3(t+4)^4}$

D. $\frac{3(t+4)^4}{2}$

Question 6

Consider the following pseudocode.

```
a ← 0
t ← 1
while t < 90
    a ← a + sin²t
    t ← t + 1
end while
print a
```

Assume t is in degrees. What would be the output of the above pseudocode?

A. 45.5

B. 44.5

C. 45

D. 44

Question 7

The length of the curve defined by the parametric equations $x = 2t$ and $y = (t-1) - (1-2t)^2$ for $-1 \leq t \leq 2$ is closest to

A. 20

B. 18

C. 16

D. 6

Question 8

The acceleration, a ms^{-2}, of a particle moving in a straight line, starting with an initial velocity of 1 ms^{-1} at 1 metre in the positive direction of its displacement, is described by $a = v(1+v)^2$, where v ms^{-1} is its velocity after t seconds. The displacement of the particle when the velocity is 10 ms^{-1} equals

A. $\frac{21}{11}$

B. $-\frac{1}{11}$

C. $\frac{1033}{33}$

D. $\frac{31}{22}$

Question 9

Consider the vectors $\underset{\sim}{a} = m\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} - n\underset{\sim}{j}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + p\underset{\sim}{k}$, where m, n and p are real numbers. If these vectors are linearly **dependent**, then

A. $m = -n + p$

B. $m = \frac{4}{n} + \frac{10}{p}$

C. $m = n + p$

D. $m = -4n + 5p$

Question 10

The velocity of a body moving in a plane is given by $\dot{\underset{\sim}{r}}(t) = \sin(t)\cos(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$ where $t \geq 0$. Given that $\underset{\sim}{r}(\pi) = 2\underset{\sim}{i} - 3\underset{\sim}{j}$, the displacement of the body at time t, $\underset{\sim}{r}(t)$ is given by

A. $\left(\frac{1}{4}\cos(2t)\right)\underset{\sim}{i} + \left(-2\sin(2t)\right)\underset{\sim}{j}$

B. $\left(-\frac{1}{4}\cos(2t)\right)\underset{\sim}{i} + \left(\frac{1}{2}\sin(2t)\right)\underset{\sim}{j}$

C. $\left(-\frac{1}{4}\cos(2t) + \frac{7}{4}\right)\underset{\sim}{i} + \left(\frac{1}{2}\sin(2t) + 3t\right)\underset{\sim}{j}$

D. $\left(-\frac{1}{4}\cos(2t) + \frac{9}{4}\right)\underset{\sim}{i} + \left(\frac{1}{2}\sin(2t) - 3\right)\underset{\sim}{j}$

Question 11

The differential equation $\frac{dP}{dt} = P\left(6 - \frac{P}{8000}\right)$ where t hours has an initial population, P, of 4000 bacteria. The graph of $y = P(t)$ has an asymptote at

A. $P = 48000$

B. $P = 4800$

C. $P = 6$

D. $P = \frac{1}{11}$

Question 12

Which of the following statements is true?

A. A number l is irrational if and only if l^2 is irrational

B. An integer n is odd if and only if $n^2 + 2$ is odd

C. At least one of two numbers x and y is irrational if and only if the product xy is irrational.

D. A number m is odd if and only if $m(m+1)$ is even.

Question 13

Let there be two non-zero vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, such that $\underset{\sim}{a} \cdot \underset{\sim}{b} = 3$ and $\tan(\theta) = \frac{1}{3}$ where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Let $\underset{\sim}{c} = \sqrt{3}\underset{\sim}{a} + \sqrt{2}\underset{\sim}{b}$ then the area of the parallelogram spanned by the vectors $\underset{\sim}{a}$ and $\underset{\sim}{c}$ is

A. 0

B. 1

C. $\sqrt{2}$

D. $\sqrt{3}$

Question 14

Let vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$ such that $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$ then $\underset{\sim}{a} \times \underset{\sim}{b} =$

A. $\underset{\sim}{c} \times \underset{\sim}{b}$

B. $\underset{\sim}{b} \times \underset{\sim}{c}$

C. $\underset{\sim}{a} \times \underset{\sim}{c}$

D. $\underset{\sim}{b} \times \underset{\sim}{a}$

Question 15

Let the acceleration of a particle be $a(t)=e^{-2t}$ at any time $t \geq 0$. It is known that when $t=0$, $v=\frac{5}{2}$ and $x=\frac{17}{4}$. Then at any time $t>0$,

A. $v=3-\frac{a}{2}$

B. $x=\frac{3}{2}\log_e\left(\frac{-1}{2(v-3)}\right)$

C. $v=\frac{-1}{2}e^{-2t}$

D. $x=\frac{a}{4}$

Question 16

If the planes $2x+2y+\lambda z+4=0$ and $x+y+z-8=0$ are perpendicular to each other, then λ is equal to

A. 2

B. −4

C. 1

D. 0

Question 17

Let A_x and A_y be the surface areas generated by the curve $y=\frac{x}{5}$ bounded by $x=a$ and the x-axis rotated around the x and y axis respectively. $\frac{A_x}{A_y}=$

A. 25

B. $\frac{1}{5}$

C. $\frac{1}{25}$

D. $\frac{1}{125}$

Question 18

Which of the following differential equations will only have negative slopes in the fourth quadrant if a slope field is plotted?

A. $\frac{dy}{dx} = -\frac{x}{y}$

B. $\frac{dy}{dx} = y\log_e|x|$

C. $\frac{dy}{dx} = 1 - e^{y(1-y)}$

D. $\frac{dy}{dx} = \frac{y}{x^2} - 3$

Question 19

Let X and Y be two independent normal distributions, where both X and Y have the mean of μ and standard deviation of σ. Therefore, the $\Pr(-1 < X - Y < 1)$ is

A. dependent on the value of σ and independent of the value of μ.

B. dependent on the value of μ and independent of the value of σ.

C. dependent on both of the values of μ and σ.

D. independent of both of the values of μ and σ.

Question 20

A random sample of lengths of 36 croissants in a large batch gives a sample mean of 15 cm. Given the variance of the batch is 2, the 99% confidence interval for the mean of the lengths of all croissants, round to the nearest integer would be

A. $(14,16)$

B. $(12,18)$

C. $(-4,34)$

D. $(11,19)$

END OF SECTION A
TURN OVER

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (10 marks)

Consider the family of functions f with rule $f(x)=\dfrac{ax^2+1}{x^2-3x+2}$ where $a\in R\setminus\{0\}$.

a. Write down the equations of all asymptotes of the graph of f. 1 mark

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b. State the coordinates, for $a=1$, of the local maximum stationary point on the graph of f 1 mark

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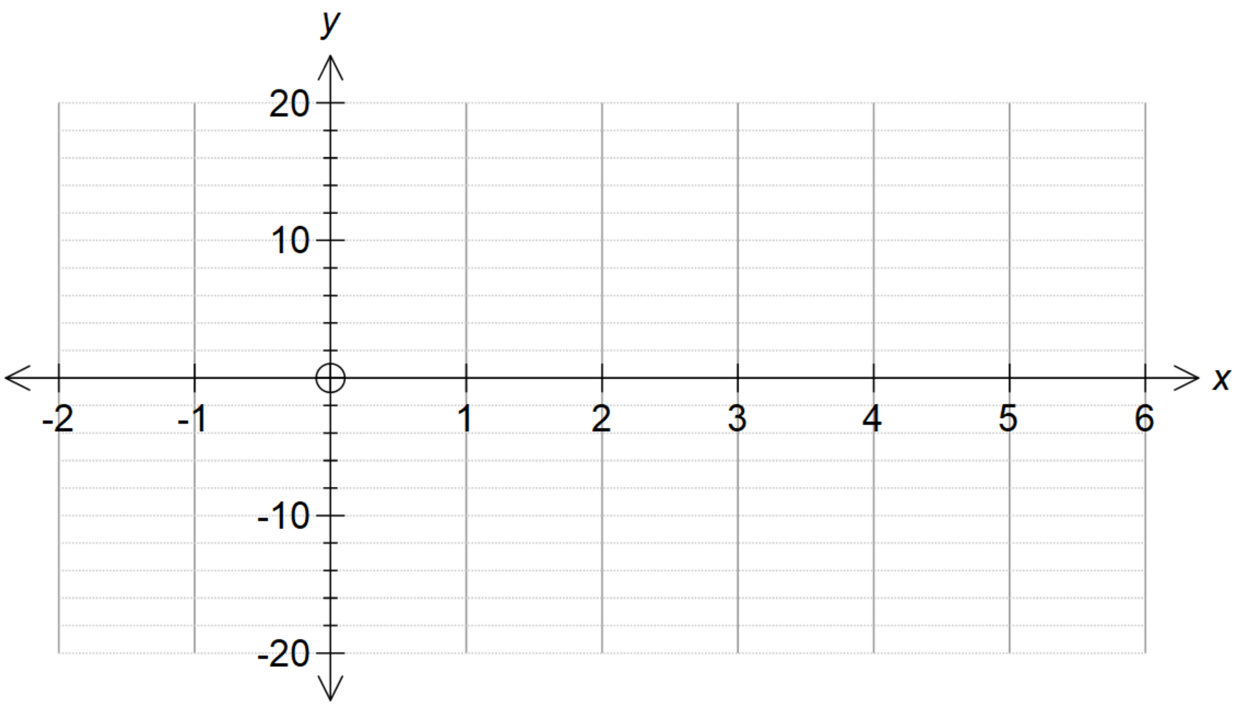
c. Sketch the graph of $y=f(x)$ for $a=1$ on the set of axes below. Clearly label any turning points with coordinates, expressed correct to two decimal places, and all asymptotes with their equations. 3 marks



d. **i.** Write down, in terms of a, the x-value of any stationary points on the graph of f. 1 mark

ii. **Hence**, state the values of a for which the graph of f has two stationary points. 1 mark

Now consider the functions h and g where $h(x) = x$ and $g(x) = |f(x)|$ where $a = 1$.

e. How many times do the graphs of h and g intersect? 1 mark

f. The region bounded by the curves of h and g and the line $x = 5$ is rotated around the x-axis. Find the volume of this solid. Give your answer correct to two decimal places 2 marks

Question 2 (10 marks)

A particle moves in a straight line so that its distance, x metres, from a fixed origin O after time t seconds, $t \geq 0$, is given by the diffferential equation

$$\frac{dx}{dt} = 2\operatorname{cosec}(x)\sin^2(2t) \text{ where } x = \pi \text{ when } t = \frac{\pi}{4}$$

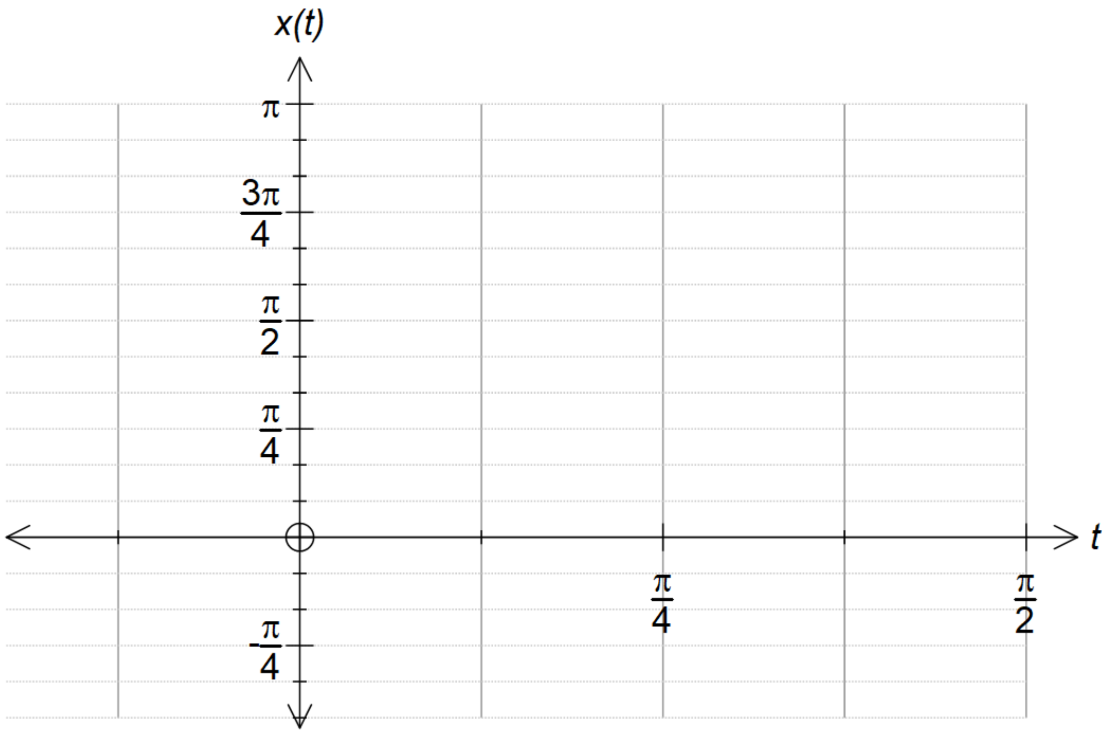
a. i. Use the fact that $\sin^2(2t) = \frac{1}{2}\left(1-\cos(4t)\right)$, rewrite the given diffferential equation in the form $\int f(x)dx = \int g(t)dt$. 1 mark

ii. Solve the diffferential equation of the form $\int f(x)dx = \int g(t)dt$ giving your answer as x in terms of t in the form $x = \cos^{-1}\left(at + b\sin(4t) + c\right)$ where a, b and c are real constants. 3 marks

iii. Write down the domain and range of the graph of $x(t)$. 2 marks

b. The particle moves along the graph of $x(t)$, beginning at $t = 0$. Find the speed of the particle after $\frac{\pi}{8}$ seconds. Give your answer in decimals correct to two decimal places. 1 mark

c. Sketch the graph of $x(t)$ on the grid below. Label non-exact endpoints with coordinates correct to two decimal places. 3 marks

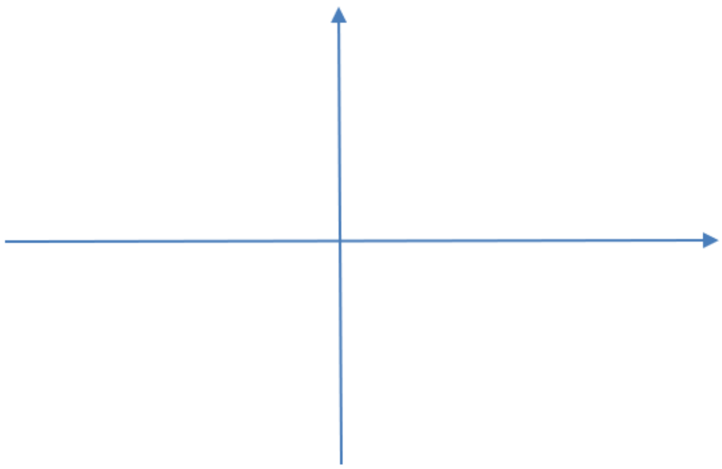


Question 3 (11 marks)

a. Use appropriate trigonometric formulas to show that $\text{cis}\left(\frac{5\pi}{12}\right)$ can be expressed in the form $A + Bi$, where $A, B \in R$ as $\text{cis}\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{6}+\sqrt{2}}{4}i$. 2 marks

b. Express $\left\{z : \left|z - 3\,\text{cis}\left(\frac{5\pi}{12}\right)\right| = \left|z + 3\,\text{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$ in the form $y = ax + b$, where $a, b \in R$. 3 marks

c. Sketch and label, on the axes below, the graph of $\left\{z : \left|z - 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| = \left|z + 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$ in the form $y = ax + b$, where $a, b \in R$, 1 mark



d. **i.** On the Argand diagram below sketch and label $A = \left\{z : z\bar{z} = 4, z \in C\right\}$ and 2 marks

$B = \left\{z : \left|z - 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| = \left|z + 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}$.

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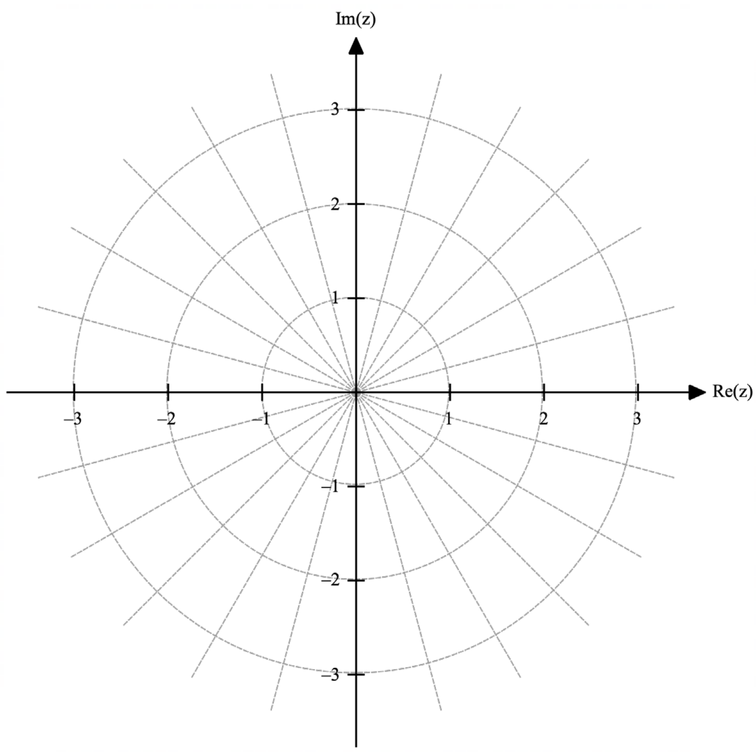
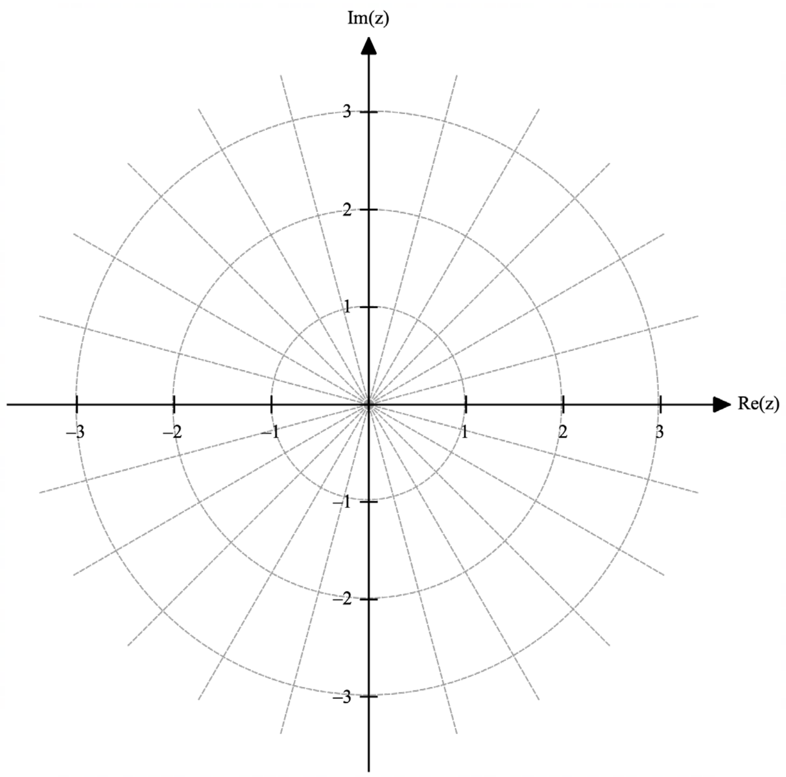


ii. State the points of intersection found in the graph in part i) above 1 mark

e. Shade the region defined by

$$\left\{z : z\bar{z} \leq 4, z \in C\right\} \cap \left\{z : \left|z - 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right| \leq \left|z + 3\operatorname{cis}\left(\frac{5\pi}{12}\right)\right|, z \in C\right\}.$$ 1 mark



f. Find the area of the shaded region in **part e**. 1 mark

Question 4 (11 marks)

Consider the plane, Π_1, with cartesian equation $2x+4y-7z=5$.

a. Verify that the $P(5,0,1)$ does not lie on the plane. 1 mark

b. Write down a vector perpendicular to the plane. 1 mark

c. Find the shortest distance between the point $P(5,0,1)$ to the plane Π_1 2 marks

Consider a second plane Π_2 given by $\underset{\sim}{r}\cdot\left(3\underset{\sim}{i}+6\underset{\sim}{j}+7\underset{\sim}{k}\right)=28$ and the plane Π_2 contains the points $A(\alpha,\beta,1)$, $B(2,-1,4)$, and $C(5,1,1)$ where $\alpha>\beta>1$.

d. Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of α and β. 1 mark

e. Hence find the coordinates of A. 2 marks

f. Find the acute angle between the planes, Π_1 and Π_2. Express your answer in degrees, correct to 2 decimal places. 2 marks

g. Show that a vector equation of the line of intersection of the planes Π_1 and Π_2 could be expressed as $\underset{\sim}{\ell} = \left(\frac{33}{5}\underset{\sim}{i} + 0\underset{\sim}{j} + \frac{41}{35}\underset{\sim}{k}\right) + t\left(-2\underset{\sim}{i} + \underset{\sim}{j} + 0\underset{\sim}{k}\right)$. 2 marks

Question 5 (10 marks)

Two dragons are flying around and the patterns can be described by some parametric curves.

The first dragon's flying pattern is given by the parametric curves defined by

$$\underset{\sim}{r_1}(t) = 13\sin(t)\underset{\sim}{i} + 5\cos(t)\underset{\sim}{j} + 12\cos(t)\underset{\sim}{k}$$

where $t \in [0, \infty)$ in seconds.

a. Find the velocity of the dragon at time t. 1 mark

b. Show that the speed of this dragon is constant. 2 marks

The second dragon's flying pattern is given by the parametric curves defined by

$\underset{\sim}{r_2}(t) = \left(\frac{t^2}{16}\right)\underset{\sim}{i} + \left(\log_e(t)\right)\underset{\sim}{j} + \left(\cos\left(\frac{t}{2}\right)\right)\underset{\sim}{k}$ and $t \in (0, \infty)$.

c. Find the coordinates of the point where the second dragon first passes through the xy plane. 2 marks

d. Find the time and distance when the two dragons are the closest.
Correct your answer to 3 decimal places. 3 marks

e. How far along the curve has the second dragon travelled from $t = 1$ to $t = 3$?
Correct your answer to 3 decimal places. 2 marks

Question 6 (8 marks)

Let S be a normally distributed random variable representing the daily sales figure of a café in a shopping mall.

Caitlin wanted to purchase this café. The current owner claims that the sales figures are related to the maximum temperature on the day and the number of people visiting the shopping mall on the day (in thousands).

daily sales = 3000 + k_1 × maximum temperature + k_2 × number of people visited

where $k_1 \sim N(-10,25)$ and $k_2 \sim N(200,15^2)$.
Assume that maximum temperature and number of people visiting the shopping mall on a day are independent.
On a particular day, the maximum temperature is 25 and 5 thousands people visited the mall.

a. Find the expected sales figure and the standard deviation on that particular day. 2 marks

b. Find the probability that the sales figure on that particular day is more than 3800.
Correct your answer to 4 decimal places. 1 mark

Caitlin visited the café for five days and found that all five days have the maximum temperature of 25 degrees and 5 thousands people visited the mall.

c. Write down a 95% confidence interval for the average sales figure of these five claimed by the owner, correct your answer to 3 decimal places. 1 mark

d. Find the probability that the average sales of these five days exceed 3800.

Give your answer correct to 4 decimal places. 1 mark

Caitlin found that the average sales of these five days are 3700. She wants to use a one-tailed statistical test using a 5% level of significance to determine that if the owner's claim is true.

e. Write down suitable hypotheses H_0 and H_1 for this test. 1 mark

f. Find the p value for the test, correct to two decimal places. Thus, state if the mean sales that Caitlin found support the owner's claim at the 5% level of significance for a one-tailed test.

2 marks

END OF QUESTION AND ANSWER BOOK