The Mathematical Association of Victoria

Trial Exam 2024

SPECIALIST MATHEMATICS

WRITTEN EXAMINATION 1

SOLUTIONS

Question 1

a. $g(x)=\dfrac{3}{2x^2+1}$

$g'(x)=\dfrac{-12x}{\left(2x^2+1\right)^2}$

$g''(x)=\dfrac{\left(2x^2+1\right)^2\times -12+12x\times 2\left(2x^2+1\right)^1 4x}{\left(2x^2+1\right)^4}$

$g''(x)=\dfrac{-12\left(2x^2+1\right)+96x^2}{\left(2x^2+1\right)^3}$ where $2x^2+1\neq 0$

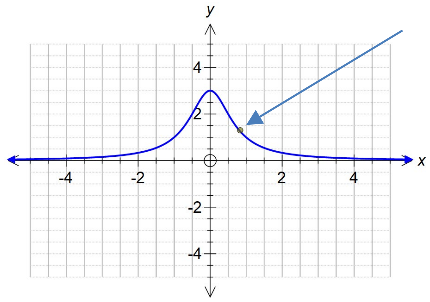
$g''(x)=\dfrac{-12\left(2x^2+1-8x^2\right)}{\left(2x^2+1\right)^3}=\dfrac{12\left(6x^2-1\right)}{\left(2x^2+1\right)^3}$ **1M**

solve $g''(x)=0$

$g''(x)=6x^2-1=0$

Points of inflection occur at $x=\pm\dfrac{\sqrt{6}}{6}$

Concavity on graph of $g(x)$ changes either side of $x=\pm\dfrac{\sqrt{6}}{6}$



1M

b. Area $= 2\int_0^{\frac{\sqrt{6}}{6}} \frac{3}{2x^2+1}\,dx$ by symmetry

$$= 2\int_0^{\frac{\sqrt{6}}{6}} \frac{3}{2\left(x^2+\frac{1}{2}\right)}\,dx$$ **1M**

$$= 3\sqrt{2}\int_0^{\frac{\sqrt{6}}{6}} \frac{\frac{1}{\sqrt{2}}}{\left(x^2+\frac{1}{2}\right)}\,dx$$

$$= 3\sqrt{2}\left[\tan^{-1}\left(\sqrt{2}x\right)\right]_0^{\frac{\sqrt{6}}{6}}$$

$$= 3\sqrt{2}\left[\tan^{-1}\left(\sqrt{2}\,\frac{\sqrt{6}}{6}\right) - \tan^{-1}(0)\right]$$

$$= 3\sqrt{2}\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$= 3\sqrt{2}\,\frac{\pi}{6}$$

$$2\int_0^{\frac{\sqrt{6}}{6}} \frac{3}{2x^2+1}\,dx = \frac{\pi\sqrt{2}}{2}$$ **1A**

Question 2

$z_1 = 2 - 2\sqrt{3}i$ and $z_2 = 5\,\mathrm{cis}\left(-\frac{5\pi}{6}\right)$

a. $z_1 = 2 - 2\sqrt{3}i = 4\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

So $z_1 z_2 = \left(4\,\mathrm{cis}\left(-\frac{\pi}{3}\right)\right)\left(5\,\mathrm{cis}\left(-\frac{5\pi}{6}\right)\right)$ **1M**

$$= 20\,\mathrm{cis}\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right)$$

$$= 20\,\mathrm{cis}\left(-\frac{7\pi}{6}\right)$$

In Principal Argument form

$$= 20\,\mathrm{cis}\left(\frac{5\pi}{6}\right)$$ **1A**

b. $\overline{z_2} = 5\,\mathrm{cis}\left(\frac{5\pi}{6}\right) = 5\left(-\frac{\sqrt{3}}{2}\right) + 5\left(\frac{1}{2}\right)i$

$\overline{z_2} = 5\,\mathrm{cis}\left(\frac{5\pi}{6}\right) = -\frac{5\sqrt{3}}{2} + \frac{5}{2}i$ **1M**

So $z_1 + \overline{z_2} = 2 - 2\sqrt{3}i - \frac{5\sqrt{3}}{2} + \frac{5}{2}i$

$z_1 + \overline{z_2} = 2 - \frac{5\sqrt{3}}{2} + \left(\frac{5}{2} - 2\sqrt{3}\right)i$ in Cartesian Form **1A**

Question 3

Find k using the point $(1,1)$

$-2(1)(1)^2 + k(1) = 3(1)$

giving $k = 5$ **1M**

$-2xy^2 + ky = 3x$ where $k = 5$

$-2xy^2 + 5y = 3x$

Using implicit differentiation $-\left(y^2(2) + 2x\left(2y\frac{dy}{dx}\right)\right) + 5\frac{dy}{dx} = 3$

$-2y^2 - 4xy\frac{dy}{dx} + 5\frac{dy}{dx} = 3$

$-4xy\frac{dy}{dx} + 5\frac{dy}{dx} = 3 + 2y^2$

$\frac{dy}{dx}(-4xy + 5) = 3 + 2y^2$ **1M**

At point $(1,1)$

$\frac{dy}{dx}(1) = 5$

$\frac{dy}{dx} = 5$

gradient of normal $= -\frac{1}{5}$ **1A**

Question 4

$\int_1^2 \left(\frac{2x}{2x^2 - 5x - 3}\right) dx$

Partial Fractions

$\frac{2x}{2x^2 - 5x - 3} = \frac{A}{2x+1} + \frac{B}{x-3}$

$\frac{A}{2x+1} + \frac{B}{x-3} = \frac{A(x-3) + B(2x+1)}{(2x+1)(x-3)}$

$2x = A(x-3) + B(2x+1)$

Let $x = 3 \therefore 6 = B(7)$

Let $x = -\frac{1}{2} \therefore -1 = A\left(-\frac{7}{2}\right)$

Giving $A = \frac{2}{7}, B = \frac{6}{7}$ **1M**

$$\int_1^2 \left(\frac{2x}{2x^2-5x-3}\right)dx = \frac{1}{7}\int_1^2 \left(\frac{2}{2x+1}+\frac{6}{x-3}\right)dx$$

$$= \frac{1}{7}\left[\log_e|2x+1| + 6\log_e|x-3|\right]_1^2$$

$$= \frac{1}{7}\left[\log_e\left|(2x+1)(x-3)^6\right|\right]_1^2$$ **1M**

$$= \frac{1}{7}\left[\log_e\left|(5)(-1)^6\right| - \log_e\left|(3)(-2)^6\right|\right]$$

$$= \frac{1}{7}\left[\log_e(5) - \log_e(192)\right]$$

In required form $\frac{\log_e(a)}{b}$ where $a, b \in Q$

$$= \frac{\log_e\left(\frac{5}{192}\right)}{7}$$ **1A**

Question 5

$2y\frac{dy}{dx} = \frac{1}{\sqrt{12-x^2-4x}}$ where $y(0)=2$.

$$\int (2y)dy = \int \left(\frac{1}{\sqrt{12-x^2-4x}}\right)dx$$ **1M**

$$\frac{2y^2}{2} = \int \left(\frac{1}{\sqrt{-x^2-4x+12}}\right)dx$$

$$y^2 = \int \left(\frac{1}{\sqrt{-\left(x^2+4x-12\right)}}\right)dx$$

$$y^2 = \int \left(\frac{1}{\sqrt{-\left(x^2+4x+4-4-12\right)}}\right)dx$$ complete the square

$$y^2 = \int \left(\frac{1}{\sqrt{-\left((x+2)^2-16\right)}}\right)dx$$ complete the square

$$y^2 = \int \left(\frac{1}{\sqrt{16-(x+2)^2}}\right)dx$$

$$y^2 = \sin^{-1}\left(\frac{x+2}{4}\right)+c$$ **1M**

gven $y(0)=2$

$$2^2 = \sin^{-1}\left(\frac{0+2}{4}\right)+c$$

$$4 = \sin^{-1}\left(\frac{1}{2}\right)+c \Rightarrow c = 4-\frac{\pi}{6}$$

$\therefore y^2 = \sin^{-1}\left(\frac{x+2}{4}\right) + 4 - \frac{\pi}{6}$

$y = \pm\sqrt{\sin^{-1}\left(\frac{x+2}{4}\right) + 4 - \frac{\pi}{6}}$

Because $y(0) = 2$ take +ve solution

In required form y in terms of x

$y = \sqrt{\sin^{-1}\left(\frac{x+2}{4}\right) + 4 - \frac{\pi}{6}}$ **1A**

Question 6

$\int x\sin(x+1)dx$

Integration by parts $\int\left(u\frac{dv}{dx}\right)dx = uv - \int\left(v\frac{du}{dx}\right)dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \sin(x+1) \Rightarrow v = -\cos(x+1)$ **1M**

$\int\left(u\frac{dv}{dx}\right)dx = uv - \int\left(v\frac{du}{dx}\right)dx$

gives $\int\left(x\sin(x+1)\right)dx = -x\cos(x+1) - \int\left(-\cos(x+1)\times 1\right)dx$

So $\int x\sin(x+1)dx = -x\cos(x+1) + \sin(x+1) + c$ **1A**

Question 7

a. Solve $f(x) = \frac{x^2}{2} - \frac{2}{x^2} = 0$

$f(x) = \frac{x^4 - 4}{2x^2} = 0$

Gives $x^4 - 4 = 0$

$x^4 = 4$

$\therefore x = \pm\sqrt{2}$

$(-\sqrt{2}, 0), (\sqrt{2}, 0)$ **1A**

b. $f'(x) = x + \frac{4}{x^3} = \frac{x^4 + 4}{x^3}$

No real solutions for $x^4 + 4 = 0$

No stationary points on the graph of $f(x)$. **1A**

c. Volume $= \pi\int_{\sqrt{2}}^{2\sqrt{2}}\left(\frac{x^2}{2} - \frac{2}{x^2}\right)^2 dx$ **1M**

$= \pi\int_{\sqrt{2}}^{2\sqrt{2}}\left(\frac{x^4}{4} - 2 + \frac{4}{x^4}\right)dx$

$= \pi \left[\frac{x^5}{20} - 2x - \frac{4}{3x^3} \right]_{\sqrt{2}}^{2\sqrt{2}}$ **1M**

$= \pi \left[\left(\frac{(2\sqrt{2})^5}{20} - 2(2\sqrt{2}) - \frac{4}{3(2\sqrt{2})^3} \right) - \left(\frac{(\sqrt{2})^5}{20} - 2(\sqrt{2}) - \frac{4}{3(\sqrt{2})^3} \right) \right]$

$= \pi \left[\left(\frac{128\sqrt{2}}{20} - 4\sqrt{2} - \frac{4}{48\sqrt{2}} \right) - \left(\frac{4\sqrt{2}}{20} - 2\sqrt{2} - \frac{4}{6\sqrt{2}} \right) \right]$

$= \pi \left(\frac{124\sqrt{2}}{20} - 2\sqrt{2} + \frac{7\sqrt{2}}{24} \right)$

$= \pi \left(\frac{31\sqrt{2}}{5} - 2\sqrt{2} + \frac{7\sqrt{2}}{24} \right)$

in the form of $\pi a \sqrt{b}$ where $a, b \in Q$

Volume $= \pi \frac{539}{120} \sqrt{2}$ **1A**

Question 8

a i. $\mathrm{E}(M) = \mathrm{E}(T_1) + \mathrm{E}(T_2) + \mathrm{E}(A) = 65 + 65 + 125 = 255$ **1A**

a ii. $\mathrm{sd}(M) = \sqrt{\mathrm{Var}(M)} = \sqrt{\mathrm{Var}(T_1) + \mathrm{Var}(T_2) + \mathrm{Var}(A)} = \sqrt{5^2 + 5^2 + 10^2} = \sqrt{150} = 5\sqrt{6}$ **1A**

b. $\mathrm{E}(T_1 + T_2 - A) = \mathrm{E}(T_1) + \mathrm{E}(T_2) - \mathrm{E}(A) = 65 + 65 - 125 = 5$

$\mathrm{Var}(M) = \mathrm{Var}(T_1) + \mathrm{Var}(T_2) + \mathrm{Var}(A)$

$\mathrm{sd}(M) = \sqrt{\mathrm{Var}(T_1 + T_2 - A)} = \sqrt{\mathrm{Var}(T_1) + \mathrm{Var}(T_2) + \mathrm{Var}(A)} = \sqrt{5^2 + 5^2 + 10^2} = \sqrt{150} = 5\sqrt{6}$

$\Pr(T_1 + T_2 < A) = \Pr(T_1 + T_2 - A < 0) = \Pr(z < k)$

$(T_1 + T_2 - A) \sim N(5, 150)$ **1M**

$\Pr(T_1 + T_2 - A < 0) = \Pr\left(z < \frac{0 - 5}{5\sqrt{6}} \right) = \Pr\left(z < \frac{-1}{\sqrt{6}} \right)$

$k = -\frac{1}{\sqrt{6}}$ **1A**

Question 9

We proceed by induction on n. Let $p(n)$ be the following proposition over $[2,\infty)$ where n is an integer

$M^n = M$ where $M = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$.

$p(2)$ is the statement that

$M^2 = M$

Since

$$M^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}^2 = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} \times \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 3+(-6)\times 1 & 3\times(-6)+(-6)\times(-2) \\ 1\times 3+(-2)\times 1 & 1\times(-6)+(-2)\times(-2) \end{bmatrix} = \begin{bmatrix} 9-6 & -18+12 \\ 3-2 & -6+4 \end{bmatrix}$$ **1M**

$$= \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = M$$

then $p(2)$ is true.

Consider some integer $k \ge 2$ so that $p(k)$ is true. Since $p(k)$ is true, we know that $M^k = M$.

So that $M^k = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}^k = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix} = M$

Consider the statement $p(k+1)$: $M^{k+1} = M$

Notice that the LHS of $p(k+1)$ can be written as $M^k \times M^1$.

Since $p(k)$ is true, we have $M^k \times M^1 = M \times M = M$ **1A**

Therefore, $M^{k+1} = M$ and so it follows that $p(k+1)$ is true.

Since both of the proposition in the Principle of Mathematical Induction hold, necessarily the conclusion holds. That is, $p(n)$ is true for each integer $n \ge 2$. Therefore for every $n \ge 2$, where n is an integer, we have $M^n = M$ where $M = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$. **1A**

Question 10

a. $\cos(\theta) = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}| \cdot |\underset{\sim}{v}|}$

$$= \frac{(-\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}) \cdot (4\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})}{\sqrt{1^2+2^2+4^2} \times \sqrt{4^2+2^2+1^2}} = \frac{-4+4+-4}{\sqrt{21} \cdot \sqrt{21}}$$

$$\cos(\theta) = \frac{-4}{21}$$ **1A**

b. $\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -1 & 2 & 4 \\ 4 & 2 & -1 \end{vmatrix} = -10\underset{\sim}{i} + 15\underset{\sim}{j} - 10\underset{\sim}{k}$

$\underset{\sim}{w} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = (x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) \cdot (-10\underset{\sim}{i} + 15\underset{\sim}{j} - 10\underset{\sim}{k})$ **1M**

$= -10x + 15y - 10z = -85$

in required form $ax + by + cz = -85$

$-10x + 15y - 10z = -85$ **1A**

c. As $\underset{\sim}{w}$ is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{v}$. It implies that $\underset{\sim}{w} \cdot \underset{\sim}{v} = 0$ and $\underset{\sim}{w} \cdot \underset{\sim}{u} = 0$.

Hence $-x + 2y + 4z = 0$ and $4x + 2y - z = 0$. **1M**

From part b, $-10x + 15y - 10z = -85 \Rightarrow -2x + 3y - 2z = -17$.

Solve the three equations simultaneously gives $x = 2$, $y = -3$ and $z = 2$.

Therefore, $\underset{\sim}{w} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$. **1A**

Question 11

a. $\underset{\sim}{r}(t) = \left(\sin^2(t)\right)\underset{\sim}{i} + \left(\cos(t) - \cos^3(t)\right)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \left(\sin(2t)\right)\underset{\sim}{i} + \left(2\sin(t)\cos^2(t) - \sin^3(t)\right)\underset{\sim}{j}$

$\underset{\sim}{r}\left(\frac{\pi}{4}\right) = \frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{2}}{4}\underset{\sim}{j}$ **1A**

$\dot{\underset{\sim}{r}}\left(\frac{\pi}{4}\right) = \underset{\sim}{i} + \frac{\sqrt{2}}{4}\underset{\sim}{j}$ **1A**

b. Let $\underset{\sim}{r}(t) = (x, y)$.
Thus $x = \sin^2(t)$ and $y = \cos(t) - \cos^3(t)$. **1M**

$\text{LHS} = x^3 + y^2 = \left(\sin^2(t)\right)^3 + \left(\cos(t) - \cos^3(t)\right)^2$

$= \left(\sin^2(t)\right)^3 + \left(\cos(t)\left(1 - \cos^2(t)\right)\right)^2$

$= \sin^6(t) + \left(\cos(t)\sin^2(t)\right)^2$

$= \sin^6(t) + \cos^2(t) \times \sin^4(t)$

$= \sin^4(t)\left(\sin^2(t) + \cos^2(t)\right)$

$= \sin^4(t)$

$= \left(\sin^2(t)\right)^2 = x^2 = \text{RHS}$

OR

$$\text{LHS} = x^3 + y^2 = \left(\sin^2(t)\right)^3 + \left(\cos(t) - \cos^3(t)\right)^2$$

$$= \left(\sin^2(t)\right)^3 + \left(\cos^2(t) + \left(\cos^3(t)\right)^2 - 2\cos(t)\cos^3(t)\right)$$

$$= \sin^6(t) + \cos^2(t) + \left(\cos^2(t)\right)^3 - 2\left(\cos^2(t)\right)^2$$

$$= \sin^6(t) + \left(1 - \sin^2(t)\right) + \left(1 - \sin^2(t)\right)^3 - 2\left(1 - \sin^2(t)\right)^2$$

$$= \sin^6(t) + 1 - \sin^2(t) - \sin^6(t) + 3\sin^4(t) - 3\sin^2(t) + 1 - 2\left(1 + \sin^4(t) - 2\sin^2(t)\right)$$

$$= \cancel{\sin^6(t)} + 1 - \sin^2(t) \cancel{- \sin^6(t)} + 3\sin^4(t) - 3\sin^2(t) + 1 - 2 - 2\sin^4(t) + 4\sin^2(t)$$

$$= 1 \cancel{- \sin^2(t)} + 3\sin^4(t) \cancel{- 3\sin^2(t)} + 1 - 2 - 2\sin^4(t) \cancel{+ 4\sin^2(t)}$$

$$= \cancel{1} + 3\sin^4(t) \cancel{+1} \cancel{-2} - 2\sin^4(t)$$

$$= 3\sin^4(t) - 2\sin^4(t)$$

$$= \sin^4(t)$$

$$= \left(\sin^2(t)\right)^2 = x^2 = \text{RHS}$$

Therefore, $(x, y) \in C$ **1A**

END OF SOLUTIONS