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Specialist Mathematics

Written Examination 1

2024 Insight Publications Trial Examination

- **Reading time:** 15 minutes
- **Writing time:** 1 hour

Approved materials

- Question and Answer book
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and /or correction fluid/tape.

Materials supplied

- Question and Answer book of 15 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions

- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All responses must be in English.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents

Instructions

- Answer **all** questions in the spaces provided.
- Unless otherwise specified, an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (3 marks)

Evaluate $\int_1^4 \sqrt{x}\log_e(2x)\,dx$.

Give your answer in the form $a\log_e(2)+b$, where $a, b \in R$.

Question 2 (4 marks)

a. Evaluate $\left(1+\sqrt{3}i\right)^5+\left(1-\sqrt{3}i\right)^5$.

2 marks

b. Find the values of k for which $\left(1+\sqrt{3}i\right)^k+\left(1-\sqrt{3}i\right)^k=128$, where k is a positive integer.

2 marks

Question 3 (3 marks)

Consider the curve defined parametrically by

$$x = 2\cos(t) + \cos(2t),\ y = 2\sin(t) - \sin(2t), \text{ where } 0 \le t \le 2\pi.$$

Find the arc length of the curve from $t = 0$ to $t = \frac{\pi}{3}$.

Question 4 (3 marks)

Consider the relation $x\cot(y) - 2y^2 = 8$.

Find the gradient of the curve at the point $\left(\frac{\pi^2}{8} + 8, \frac{\pi}{4}\right)$.

Give your answer in the form $\frac{a}{\pi^2 + a\pi + b}$, where $a, b \in R$.

Question 5 (4 marks)

Use mathematical induction to prove that $\left(\cos\left(\theta\right)-\sin\left(\theta\right)i\right)^n=\cos\left(n\theta\right)-\sin\left(n\theta\right)i$ for all $n\in Z^+$.

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Question 6 (6 marks)

Consider the Cartesian equations of two planes:

$\prod_1 : x + 2y - z = 15$

$\prod_2 : x + y + pz = 5$

a. Given that $\prod_1$ and $\prod_2$ are perpendicular, show that $p = 3$.

1 mark

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b. Find the Cartesian equation of the plane that is perpendicular to both $\prod_1$ and $\prod_2$, and which also passes through point $(1, 2, 1)$.

2 marks

_____

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_____

Line L, given by the equation $\underset{\sim}{\mathrm{r}}(t) = \underset{\sim}{\mathrm{i}} + \underset{\sim}{\mathrm{j}} + \underset{\sim}{\mathrm{k}} + t\left(\underset{\sim}{\mathrm{i}} + 3\underset{\sim}{\mathrm{j}} - 6\underset{\sim}{\mathrm{k}}\right), t \in R$, intersects $\prod_1$ at point A and $\prod_2$ at point B.

c. Find the distance AB.

3 marks

Question 7 (4 marks)

An ice cube tray with 6 cuboid compartments is filled with water and placed in a freezer to make 6 ice cubes. Each compartment has a 3 cm by 3 cm square base.

The height of each ice cube, H, is normally distributed with a mean of 2 cm and a standard deviation of $\frac{1}{6} \text{ cm}$. Assume that the height of each ice cube is independent.

a. Find the expected total volume of ice in the ice cube tray in cm^3.

1 mark

b. Find the standard deviation of the total volume of ice in the ice cube tray in cm^3.

Give your answer in the form $\frac{a\sqrt{b}}{c}$ where a, b and c are positive integers.

2 marks

c. Find the expected surface area of one ice cube in cm^2.

1 mark

Question 8 (4 marks)

The position vector of a particle moving relative to the origin at time t seconds is given by $\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} + 4\sin(t)\underset{\sim}{j} + 2t\underset{\sim}{k}$ for $t \geq 0$. All distances are measured in metres.

a. Find the maximum and minimum speeds of the particle.

2 marks

b. Find the number of times, between $t = 0$ and $t = 2\pi$, that the velocity is perpendicular to the acceleration.

2 marks

Question 9 (5 marks)

A tank initially holds $200\ \text{L}$ of water in which $16\ \text{kg}$ of salt has been dissolved. Pure water then flows into the tank at a rate of $2\ \text{L}$ per minute. The mixture is stirred continuously and flows out of the tank at a rate of $4\ \text{L}$ per minute.

a. Show that the differential equation for x, the number of kilograms of salt in the tank after t minutes, is given by

$$\frac{dx}{dt} = -\frac{2x}{100 - t}.$$

1 mark

b. Solve the differential equation given in **part a.** to find x as a function of t. Express your answer in the form $x = \left(\frac{a - t}{b}\right)^2$, where a and b are positive integers.

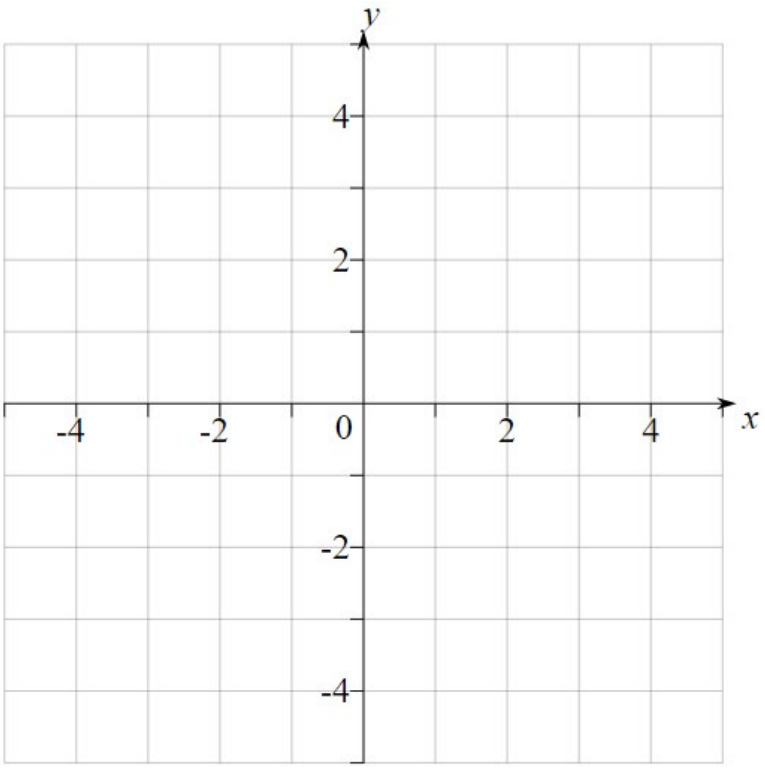
2 marks

c. Hence, find the amount of salt which has flowed out of the tank over the first 25 minutes and the concentration of salt remaining in the tank after 25 minutes.

2 marks

Question 10 (4 marks)

Sketch the graph of $y = \dfrac{x^2 - 1}{x^2 - 4}$ on the axes below. Label any axis intercepts and turning points with their coordinates, and any asymptotes with their equation.



End of Question and Answer Book

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