Specialist Mathematics

Written examination 1

2024 Insight Publications Trial Examination

Worked Solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1

Worked solution

To perform integration by parts we must identify which terms we will assign u and v' respectively.

Let $u = \log_e(2x)$ Let $v' = x^{\frac{1}{2}}$

Then $u' = \frac{1}{x}$ Then $v = \frac{2}{3}x^{\frac{3}{2}}$

Therefore

$$\begin{aligned}\int_1^4 \sqrt{x}\log_e(2x)\,dx &= \left[\frac{2}{3}x^{\frac{3}{2}}\log_e(2x)\right]_1^4 - \frac{2}{3}\int_1^4 x^{\frac{1}{2}}\,dx \\ &= \frac{2}{3}(8)\log_e(8) - \frac{2}{3}\log_e(2) - \frac{2}{3}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 \\ &= 16\log_e(2) - \frac{2}{3}\log_e(2) - \frac{4}{9}(8-1) \\ &= \frac{46}{3}\log_e(2) - \frac{28}{9}\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for evidence of performing integration by parts (i.e. correctly selecting u and v)
- 1 mark for deriving $\left[\frac{2}{3}x^{\frac{3}{2}}\log_e(2x)\right]_1^4 - \frac{2}{3}\int_1^4 x^{\frac{1}{2}}\,dx$ (or equivalent)
- 1 mark for the correct answer: $\frac{46}{3}\log_e(2) - \frac{28}{9}$

Tip

- *If you get stuck when performing integration by parts, look for a term that you know how to differentiate and consider setting that term equal to u.*

Question 2a.

Worked solution

$$\begin{aligned}\left(1+\sqrt{3}i\right)^5+\left(1-\sqrt{3}i\right)^5 &= \left(2cis\left(\frac{\pi}{3}\right)\right)^5+\left(2cis\left(-\frac{\pi}{3}\right)\right)^5 \\ &= 2^5 cis\left(\frac{5\pi}{3}\right)+2^5 cis\left(-\frac{5\pi}{3}\right) \\ &= 32\cos\left(\frac{5\pi}{3}\right)+32\sin\left(\frac{5\pi}{3}\right)i+32\cos\left(-\frac{5\pi}{3}\right)+32\sin\left(-\frac{5\pi}{3}\right)i \\ &= 2\times 32\cos\left(\frac{5\pi}{3}\right) \\ &= 64\times\frac{1}{2} \\ &= 32\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for converting both $1+\sqrt{3}i$ and $1-\sqrt{3}i$ into polar form
- 1 mark for the correct answer: 32

Question 2b.

Worked solution

$$2^k cis\left(\frac{k\pi}{3}\right)+2^k cis\left(-\frac{k\pi}{3}\right)=128$$

$$2^k \cos\left(\frac{k\pi}{3}\right)+2^k \sin\left(\frac{k\pi}{3}\right)i+2^k \cos\left(-\frac{k\pi}{3}\right)+2^k \sin\left(-\frac{k\pi}{3}\right)i=128$$

$$2^k \cos\left(\frac{k\pi}{3}\right)+2^k \sin\left(\frac{k\pi}{3}\right)i+2^k \cos\left(\frac{k\pi}{3}\right)-2^k \sin\left(\frac{k\pi}{3}\right)i=128$$

$$2^{k+1} \cos\left(\frac{k\pi}{3}\right)=128=2^7$$

Case 1:

$$\cos\left(\frac{k\pi}{3}\right)=1$$

Therefore

$$\frac{k\pi}{3}=2\pi$$

$$k=6$$

Alternatively,

if $\cos\left(\frac{k\pi}{3}\right)=1$

$$2^{k+1}=2^7$$

$$k=6$$

Case 2:

$$\cos\left(\frac{k\pi}{3}\right)=\frac{1}{2}$$

Therefore

$$2^{k+1}\times\frac{1}{2}=2^7$$

$$2^k=2^7$$

$$k=7$$

Mark allocation: 2 marks

- 1 mark for $k=6$
- 1 mark for $k=7$

Question 3

Worked solution

As the information provided is in parametric form, the parametric form of arc length must be used.

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Substituting the expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into the equation, and integrating from $t = 0$ to $t = \frac{\pi}{3}$, gives

$$\int_0^{\frac{\pi}{3}} \sqrt{\left(-2\sin(t) - 2\sin(2t)\right)^2 + \left(2\cos(t) - 2\cos(2t)\right)^2}\, dt$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{4\sin^2(t) + 8\sin(t)\sin(2t) + 4\sin^2(2t) + 4\cos^2(t) - 8\cos(t)\cos(2t) + 4\cos^2(2t)}\, dt$$

We have $4\sin^2(t) + 4\cos^2(t) = 4$ and $4\sin^2(2t) + 4\cos^2(2t) = 4$ by Pythagoras's theorem.

$$= \int_0^{\frac{\pi}{3}} \sqrt{8\sin(t)\sin(2t) - 8\cos(t)\cos(2t) + 8}\, dt$$

$8\sin(t)\sin(2t) - 8\cos(t)\cos(2t) = -8\left(\cos(t)\cos(2t) - \sin(t)\sin(2t)\right) = -8\cos(3t)$ using compound angle formulas.

$$t = \int_0^{\frac{\pi}{3}} \sqrt{8 - 8\cos(3t)}\, dt$$

$8 - 8\cos(3t)$ can be simplified into $16\sin^2\left(\frac{3t}{2}\right)$ using the double angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

$$\cos(3t) = 1 - 2\sin^2\left(\frac{3t}{2}\right)$$

Therefore

$$\int_0^{\frac{\pi}{3}} \sqrt{8 - 8\cos(3t)}\, dt = \int_0^{\frac{\pi}{3}} \sqrt{16\sin^2\left(\frac{3t}{2}\right)}\, dt$$

$$= \int_0^{\frac{\pi}{3}} 4\sin\left(\frac{3t}{2}\right) dt$$

$$= \left[-\frac{8}{3}\cos\left(\frac{3t}{2}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{8}{3}$$

Mark allocation: 3 marks

- 1 mark for constructing the expression for arc length:
  $\int_0^{\frac{\pi}{3}} \sqrt{(-2\sin(t)-2\sin(2t))^2+(2\cos(t)-2\cos(2t))^2}\,dt$
- 1 mark for applying Pythagoras's theorem and the compound angle formula to simplify the expression for arc length to $\int_0^{\frac{\pi}{3}} \sqrt{8-8\cos(3t)}\,dt$
- 1 mark for the correct answer: $\frac{8}{3}$

Tips

- *Refer to the formula sheet when working with double angle or compound angle formulas. While you may not remember them, you may be expected to use them within a question.*
- *When calculating the arc length of a curve, aim to make the expression inside the square root a perfect square, either by factorising or using trigonometric identities.*

Question 4

Worked solution

To find the gradient of the curve we need to perform implicit differentiation. Implicitly differentiating the relation $x\cot(y)-2y^2=8$ with respect to x gives

$$\cot(y)-x\operatorname{cosec}^2(y)\frac{dy}{dx}-4y\frac{dy}{dx}=0.$$

From here we can substitute the point $\left(\frac{\pi^2}{8}+8,\frac{\pi}{4}\right)$ into the equation and solve for $\frac{dy}{dx}$

$$1-\left(\frac{\pi^2}{8}+8\right)(2)\frac{dy}{dx}-\pi\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{4}{\pi^2+4\pi+64}$$

Mark allocation: 3 marks

- 1 mark for applying the product rule to $x\cot(y)$
- 1 mark for performing implicit differentiation and obtaining $\cot(y)-x\operatorname{cosec}^2(y)\frac{dy}{dx}-4y\frac{dy}{dx}=0$
- 1 mark for the correct answer: $\frac{4}{\pi^2+4\pi+64}$

Tip

- *When finding the gradient at a point, you may not be required to find an expression for $\frac{dy}{dx}$. If the question does not require an expression for $\frac{dy}{dx}$, then consider substituting the given point into the equation and then making $\frac{dy}{dx}$ the subject.*

Question 5

Worked solution

Let $P(n)$ be the statement $\left(\cos(\theta)-\sin(\theta)i\right)^n=\cos(n\theta)-\sin(n\theta)i$ for all $n\in Z^+$.

Base case

$P(1):\left(\cos(\theta)-\sin(\theta)i\right)^1=\cos(1\theta)-\sin(1\theta)i=\cos(\theta)-\sin(\theta)i$. Therefore $P(1)$ is true.

Assume $P(k)$ is true for some $k\in Z^+$.

$P(k):\left(\cos(\theta)-\sin(\theta)i\right)^k=\cos(k\theta)-\sin(k\theta)i$

Show that $P(k)\Rightarrow P(k+1)$

$P(k+1):\left(\cos(\theta)-\sin(\theta)i\right)^{k+1}=\left(\cos(\theta)-\sin(\theta)i\right)^k\times\left(\cos(\theta)-\sin(\theta)i\right)$

$=\left(\cos(k\theta)-\sin(k\theta)i\right)\times\left(\cos(\theta)-\sin(\theta)i\right)$, since we assume $P(k)$ is true.

$=\cos(\theta)\cos(k\theta)-\sin(\theta)\sin(k\theta)-\left(\sin(\theta)\cos(k\theta)+\cos(\theta)\sin(k\theta)\right)i$

$=\cos\left((k+1)\theta\right)-\sin\left((k+1)\theta\right)i$

Therefore $P(k+1)$ is true.

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$, $P(n)$ is true by mathematical induction.

Mark allocation: 4 marks

- 1 mark for showing that the base case is true
- 1 mark for stating the assumption that $P(k):\left(\cos(\theta)-\sin(\theta)i\right)^k=\cos(k\theta)-\sin(k\theta)i$ is true
- 1 mark for applying the inductive step:
  $\left(\cos(\theta)-\sin(\theta)i\right)^{k+1}=\left(\cos(\theta)-\sin(\theta)i\right)^k\times\left(\cos(\theta)-\sin(\theta)i\right)$
- 1 mark for applying compound angle formulas to show that $P(k+1)$ is true

Question 6a.

Worked solution

For two planes to be perpendicular, the scalar (dot) product of the two normal vectors must be zero.

$$\left(\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right)\cdot\left(\underset{\sim}{i}+\underset{\sim}{j}+p\underset{\sim}{k}\right)=0$$
$$1+2-p=0$$
$$p=3$$

Mark allocation: 1 mark

- 1 mark for equating the dot product of the two normal vectors equal to zero and solving for p

Question 6b.

Worked solution

We can find the cross product of the normal vectors of $\prod_1$ and $\prod_2$ to obtain a vector that is perpendicular to both normal vectors $\prod_1$ and $\prod_2$.

$$\underset{\sim}{n}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 1 & 2 & -1\\ 1 & 1 & 3\end{vmatrix}=7\underset{\sim}{i}-4\underset{\sim}{j}-\underset{\sim}{k}$$

Therefore the Cartesian equation of the plane is given by $7x-4y-z=q$, where q is a constant.

Substitute the point $(1,2,1)$ into the equation to determine the value of q.

$q=-2$

Therefore the Cartesian equation of the plane is given by $7x-4y-z=-2$.

Mark allocation: 2 marks

- 1 mark for determining the cross product of the normal vectors of $\prod_1$ and $\prod_2$:

  $\underset{\sim}{n}=7\underset{\sim}{i}-4\underset{\sim}{j}-\underset{\sim}{k}$

- 1 mark for the correct answer: $7x-4y-z=-2$

Question 6c.

Worked solution

To find the distance from A to B we must first find the coordinates of A and B.

The equation of line L can be expressed in parametric form to determine the coordinates of any point along the line.

$$(x, y, z) = (1+t, 1+3t, 1-6t)$$

To find point A, the point $(x, y, z) = (1+t, 1+3t, 1-6t)$ can be substituted into the Cartesian equation of $\prod_1$ and then solve for t.

$$\begin{aligned}(1+t)+2(1+3t)-(1-6t) &= 15\\ 2+13t &= 15\\ t &= 1\end{aligned}$$

Therefore, the coordinates of A are $(2, 4, -5)$.

To find point B, the point $(x, y, z) = (1+t, 1+3t, 1-6t)$ can be substituted into the Cartesian equation of $\prod_2$ and the equation solved for t.

$$\begin{aligned}(1+t)+(1+3t)+3(1-6t) &= 5\\ 5-14t &= 5\\ t &= 0\end{aligned}$$

Therefore the coordinates of B are $(1,1,1)$.

The vector $\overrightarrow{AB}$ is given by $-\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$.

Therefore $\left|\overrightarrow{AB}\right| = \sqrt{46}$.

Mark allocation: 3 marks

- 1 mark for determining the coordinates of A
- 1 mark for determining the coordinates of B
- 1 mark for the correct answer: $\sqrt{46}$

Question 7a.

Worked solution

The volume of one block of ice is given by $V = 3 \times 3 \times H = 9H$.

Since the height of each block of ice is independent, the total volume of ice is given by $T = 9H_1 + 9H_2 + \ldots + 9H_6$.

$$\begin{aligned}\mathrm{E}(T) &= 9\mathrm{E}(H_1) + 9\mathrm{E}(H_2) + \ldots + 9\mathrm{E}(H_6)\\ &= 108 \text{ cm}^3\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for the correct answer: 108 cm^3

Question 7b.

Worked solution

To determine the standard deviation we first must calculate the variance.

$$\text{Var}(T) = 9^2\,\text{Var}(H_1) + 9^2\,\text{Var}(H_2) + \ldots + 9^2\,\text{Var}(H_6)$$

$$= \frac{81}{6}$$

$$\sigma = \frac{3\sqrt{6}}{2}\ \text{cm}^3$$

Mark allocation:2 marks

- 1 mark for calculating the variance: $\frac{81}{6}$
- 1 mark for the correct answer: $\frac{3\sqrt{6}}{2}\ \text{cm}^3$

Tip

- *Ensure that you understand that the difference between a sum of independent random variables and a scalar multiple of a random variable. While it may be seen that* $E\left(X_1 + X_2\right) = E\left(2X\right)$*, where* X_1 *and* X *share the same distribution, the variance does not behave in the same way. That is,* $V\left(X_1 + X_2\right) \neq V\left(2X\right)$*.*

Question 7c.

Worked solution

The surface area of one cube of ice is made up of a $3\ \text{cm} \times 3\ \text{cm}$ square top and base, and four sides of $3\ \text{cm} \times h\ \text{cm}$.

Therefore,

$$A = 18 + 12H$$

$$\text{E}(A) = 18 + 12\text{E}(H) = 42\ \text{cm}^2$$

Mark allocation: 1 mark

- 1 mark for the correct answer: $42\ \text{cm}^2$

Tip

- *Drawing a diagram to represent the information provided in a question may help you understand what is required.*

Question 8a.

Worked solution

To find the speed we first must determine the velocity vector.

$\underset{\sim}{v}(t) = -2\sin(t)\underset{\sim}{i} + 4\cos(t)\underset{\sim}{j} + 2\underset{\sim}{k}$

The speed of the particle is given by the magnitude of the velocity vector.

$\left|\underset{\sim}{v}(t)\right| = \sqrt{4\sin^2(t) + 16\cos^2(t) + 4}$

The expression for the speed can be simplified using Pythagoras's theorem, as $4\sin^2(t) + 4\cos^2(t) = 4$.

Therefore, $\left|\underset{\sim}{v}(t)\right| = \sqrt{8 + 12\cos^2(t)}$.

The maximum and minimum speeds will occur when $\cos(t) = 1$ and $\cos(t) = 0$ respectively.

Therefore the maximum speed is $2\sqrt{5}$ m/s and the minimum speed is $2\sqrt{2}$ m/s.

Mark allocation: 2 marks

- 1 mark for obtaining the expression for the speed: $\left|\underset{\sim}{v}(t)\right| = \sqrt{8 + 12\cos^2(t)}$
- 1 mark for the correct answer: $2\sqrt{5}$ is the maximum speed and $2\sqrt{2}$ is the minimum speed

Tip

- *Questions relating to position vectors and speed often incorporate trigonometric functions, which can be solved using double angle and/or compound angle formulas.*

Question 8b.

Worked solution

$\underset{\sim}{a}(t) = -2\cos(t)\underset{\sim}{i} - 4\sin(t)\underset{\sim}{j}$

$v \cdot a = -12\sin(t)\cos(t) = 0$

$\sin(2t) = 0$

Solving for t over the interval $t \in [0, 2\pi]$ gives 5 solutions: $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

Therefore there are 5 times when the velocity is perpendicular to the acceleration.

Mark allocation: 2 marks

- 1 mark for obtaining $v \cdot a = -12\sin(t)\cos(t) = 0$
- 1 mark for the correct answer: 5

Question 9a.

Worked solution

The volume of the tank at any given time, t, can be expressed as $200 + 2t - 4t = 200 - 2t$.

From the description given in the equation:

Inflow of salt $= 0 \times 2 = 0$

Outflow of salt $= 4 \times \dfrac{x}{200 - 2t} = \dfrac{2x}{100 - t}$

Therefore $\dfrac{dx}{dt}$ = inflow – outflow = $0 - \dfrac{2x}{100 - t}$

$$\frac{dx}{dt} = -\frac{2x}{100 - t}$$

Mark allocation: 1 mark

- 1 mark for setting up the inflow and outflow rates and showing that $\dfrac{dx}{dt} = -\dfrac{2x}{100 - t}$

Tip

- *Representing the information provided by drawing a diagram can help in understanding what is needed.*

Question 9b.

Worked solution

$$\int \frac{1}{2x}\, dx = \int -\frac{1}{100 - t}\, dt$$

$$\frac{1}{2}\log_e |x| + c = \log_e |100 - t|$$

Substituting the point $t = 0$ and $x = 16$ into the equation gives $c = \log_e(25)$.

Therefore

$$\frac{1}{2}\log_e |x| = \log_e \left|\frac{100 - t}{25}\right|$$

$$x = \left(\frac{100 - t}{25}\right)^2$$

Mark allocation: 2 marks

- 1 mark for separating the variables and anti-differentiating to obtain $\dfrac{1}{2}\log_e |x| + c = \log_e |100 - t|$, or equivalent
- 1 mark for the correct answer: $x = \left(\dfrac{100 - t}{25}\right)^2$

Question 9c.

Worked solution

The amount of salt in the tank after 25 minutes is $x(25) = 9$ kg.

Since no salt has flowed into the tank, it follows that $16 - 9 = 7$ kg of salt has flowed out.

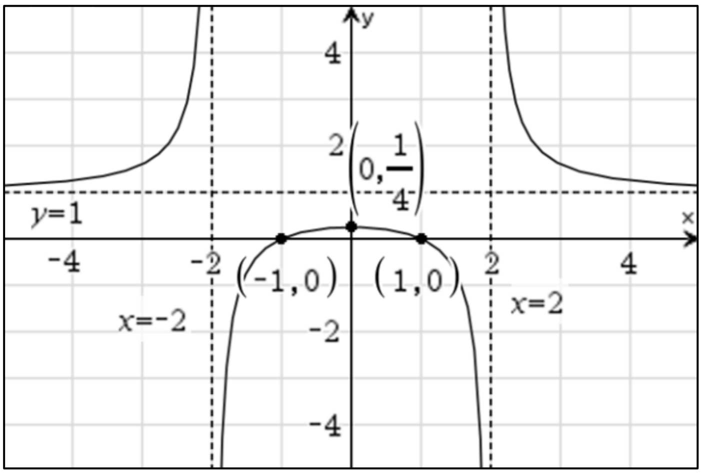
The concentration of salt remaining in the tank after 25 minutes is $\begin{aligned}\frac{9}{200-2\times 25} &= \frac{9}{150}\text{ kg/L}\\ &= \frac{3}{50}\text{ kg/L}\end{aligned}$.

Mark allocation: 2 marks

- 1 mark for deriving the amount of salt: 7 kg
- 1 mark for the correct answer for the concentration of salt: $\frac{3}{50}$ kg/L

Question 10

Worked solution



Since the degree of the denominator is equal to the degree of the numerator, long division of polynomials can be used.

Therefore $y = \frac{x^2-1}{x^2-4} = 1 + \frac{3}{x^2-4}$

It can be seen that there are vertical asymptotes at $x = 2$ and $x = -2$.

Considering the long-term behaviour as $x \to \pm\infty$, it can be seen that there is a horizontal asymptote at $y = 1$.

To find the x-intercepts consider $\frac{x^2-1}{x^2-4} = 0$, which results in $x = \pm 1$.

To find the y-intercept evaluate the function at $x = 0$, which results in $y = \frac{1}{4}$.

The graph of $y = \frac{x^2-1}{x^2-4} = 1 + \frac{3}{x^2-4}$ can also be viewed as the reciprocal graph $y = \frac{3}{x^2-4}$ that has been translated in the positive y-direction by 1 unit.

Mark allocation: 4 marks

- 1 mark for the two vertical asymptotes correctly drawn and labelled
- 1 mark for the horizontal asymptote correctly drawn and labelled
- 1 mark for the three intercepts correctly labelled
- 1 mark for drawing the correct shape of the graph (i.e. converges on the asymptotes).

Note: The graph shouldn't cross or veer away from asymptotes.

Tips

- *Vertical asymptotes, if they exist, can be obtained by setting the denominator of a rational function equal to zero.*
- *To obtain any horizontal or oblique asymptotes, consider resolving the rational function into partial fractions. If partial fractions cannot be used, consider using long division to resolve the rational function and to identify the horizontal or oblique asymptote.*
- *Consider the long-term behaviour of the graph, by considering $x \to \pm\infty$, to identify any horizontal or oblique asymptotes.*

End of Worked Solutions