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SPECIALIST MATHEMATICS TRIAL EXAMINATION 1 SOLUTIONS 2024

Question 1 (3 marks)

Evaluate $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\log_e(2x)}{x^2}\,dx$ using integration by parts.

The integration by parts formula is given on the formula sheet: $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

Let $u = \log_e(2x)$ and let $\dfrac{dv}{dx} = \dfrac{1}{x^2}$.

$\therefore \dfrac{du}{dx} = \dfrac{1}{x}$ and $v = -\dfrac{1}{x}$.

$$\therefore \int_{\frac{1}{2}}^{1} \frac{\log_e(2x)}{x^2}\,dx = \left[-\frac{1}{x}\log_e(2x)\right]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} -\frac{1}{x}\times\frac{1}{x}\,dx$$ **(1 mark)**

$$= \left[-\frac{1}{x}\log_e(2x)\right]_{\frac{1}{2}}^{1} - \left[\frac{1}{x}\right]_{\frac{1}{2}}^{1}$$ **(1 mark)**

$$= (-\log_e(2) - 0) - (1 - 2)$$

$$= 1 - \log_e(2)$$ **(1 mark)**

Question 2 (3 marks)

$$8\sin^4(x)\cot^4(x)+\cos(2x)+12\sin^2(x)=8$$

$$8\sin^4(x)\frac{\cos^4(x)}{\sin^4(x)}+\cos(2x)+12\sin^2(x)=8$$

$$8\cos^4(x)+(2\cos^2(x)-1)+12(1-\cos^2(x))=8,\qquad \sin(x)\neq 0$$

(1 mark – simplify $\cot^4(x)$ or re-write $\cos(2x)$)

$$8\cos^4(x)-10\cos^2(x)+3=0$$

Let $a=\cos^2(x)$

$8a^2-10a+3=0$ **(1 mark – recognise quadratic = 0)**

$(4a-3)(2a-1)=0$

$$a=\frac{3}{4},\ \frac{1}{2}$$

$$\therefore \cos^2(x)=\frac{3}{4},\ \frac{1}{2}$$

$$\cos(x)=\pm\frac{\sqrt{3}}{2},\ \pm\frac{1}{\sqrt{2}}$$

Given $x\in\left(\frac{\pi}{2},\pi\right)$ (quadrant 2)

$x=\frac{3\pi}{4},\ \frac{5\pi}{6}$ **(1 mark)**

Question 3 (4 marks)

Let I be the normal random variable for the volume (mL) of ice-cream dispensed.
Let T be the normal random variable for the volume (mL) of chocolate topping dispensed.

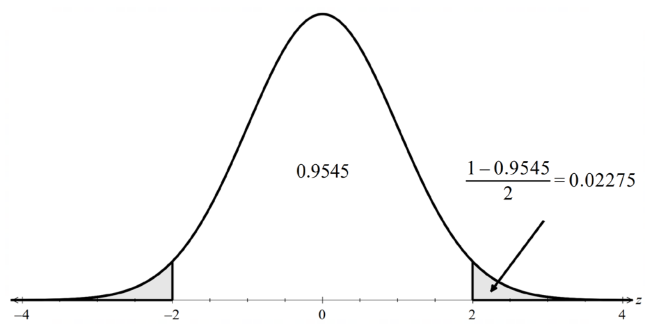
$\text{E}(I) = 125 \qquad \text{E}(T) = 20$
$\text{Var}(I) = 16 \qquad \text{Var}(T) = 9$

a. Let $V = I + T$, where V is the random variable for the volume (mL) of ice-cream and chocolate topping dispensed.

$\text{E}(V) = 125 + 20 = 145$
$\text{Var}(V) = 16 + 9 = 25$ since I and T are independent random variables
$\therefore \text{sd}(I+T) = \sqrt{25} = 5$

(1 mark – either E(*I* + *T*) or Var(*I* + *T*) correct)

$\therefore \Pr(V > 155)$
$= \Pr(Z > 2)$ since $Z = \dfrac{V - \mu_V}{\sigma_V} = \dfrac{155 - 145}{5} = 2$
$= 0.02275$
$= 0.023$ (correct to 3 decimal places) **(1 mark)**



b. $\text{Cost} = \dfrac{1}{1000}(11I + 15T)$
(note: multiply by $\dfrac{1}{1000}$ to account for conversion from mL to L)

$\text{E(Cost)} = \dfrac{1}{1000}(11\text{E}(I) + 15\text{E}(T))$ **(1 mark)**
$= \dfrac{1}{1000}(11 \times 125 + 15 \times 20)$
$= \dfrac{1}{1000}(1675)$
$= \$1.68$ (nearest cent) **(1 mark)**

Question 4 (3 marks)

Let $P(n)$ be the proposition that $7+9+11+....+(2n+1)=n^2+2n-8$ for all integers $n \geq 3$.

Step 1 Show $P(3)$ is true.

$7=3^2+2\times3-8$
$7=7$ which is true. **(1 mark)**

Step 2 Assume that $P(k)$ is true for $k \in Z$, $k \geq 3$.

$7+9+11+....+(2k+1)=k^2+2k-8$

Step 3 Prove that $P(k)$ true implies that $P(k+1)$ is true.

$7+9+11+....+(2k+1)+(2(k+1)+1)$
$=7+9+11+....+(2k+1)+(2k+3)$ **(1 mark)**
$=P(k)+(2k+3)$
$=k^2+2k-8+(2k+3)$
$=k^2+4k-5$
$=k^2+(2k+2k)+1+2-8$
$=k^2+2k+1+2k+2-8$
$=(k+1)^2+2(k+1)-8$
$=P(k+1)$

It follows that $P(k)$ true implies that $P(k+1)$ is true.
Using the principle of mathematical induction, it therefore follows that $P(n)$ is true for all integers $n \geq 3$. **(1 mark)**

Question 5 (3 marks)

a. The scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ is $\dfrac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

$$\therefore\ \frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{|\underset{\sim}{a}|}=\frac{7}{\sqrt{14}}$$

$|\underset{\sim}{a}|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$ and $\underset{\sim}{a}\cdot\underset{\sim}{b}=3\times1-m+2\times1=5-m$

$$\therefore\ \frac{7}{\sqrt{14}}=\frac{5-m}{\sqrt{14}}$$

$$7=5-m$$
$$m=-2$$

(1 mark)

b. Let $\underset{\sim}{u}$ be the vector resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$.

$$\underset{\sim}{u}=\frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\underset{\sim}{a}\cdot\underset{\sim}{a}}\underset{\sim}{a}=\frac{7}{\sqrt{14}\sqrt{14}}\underset{\sim}{a}=\frac{1}{2}\underset{\sim}{a}=\frac{1}{2}(3\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k})$$

(1 mark)

Let $\underset{\sim}{v}$ be the vector resolute of $\underset{\sim}{b}$ that is perpendicular to $\underset{\sim}{a}$.

Then $\underset{\sim}{b}=\underset{\sim}{u}+\underset{\sim}{v}\ \Rightarrow\ \underset{\sim}{v}=\underset{\sim}{b}-\underset{\sim}{u}$

$\therefore$ vector resolute of $\underset{\sim}{b}$ that is perpendicular to $\underset{\sim}{a}$ is $\underset{\sim}{v}=\underset{\sim}{b}-\frac{1}{2}\underset{\sim}{a}$

$$\begin{aligned}\underset{\sim}{b}-\frac{1}{2}\underset{\sim}{a}&=(\underset{\sim}{i}+m\underset{\sim}{j}+k)-\frac{1}{2}(3\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k})\\&=(\underset{\sim}{i}-2\underset{\sim}{j}+k)-\frac{1}{2}(3\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k})\\&=-\frac{1}{2}\underset{\sim}{i}-\frac{3}{2}\underset{\sim}{j}\end{aligned}$$

(1 mark)

Alternatively, you can use $|\underset{\sim}{a}|=\sqrt{14}$ and scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}=\dfrac{7}{\sqrt{14}}$ to determine that $\underset{\sim}{u}=\dfrac{1}{2}\underset{\sim}{a}$.

Question 6 (6 marks)

a. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$

Using implicit differentiation,

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \times \frac{dy}{dx} = 0$$ **(1 mark)**

Method 1 – Rearrange for $\frac{dy}{dx}$ first

$$\frac{dy}{dx} = \frac{-\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}}$$

$$= -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Sub in the point $(1,1)$,

$$\frac{dy}{dx} = -1$$ **(1 mark)**

Method 2 – Substitute in the point $(1,1)$ into $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \times \frac{dy}{dx} = 0$.

$$\frac{2}{3} \times 1 + \frac{2}{3} \times 1 \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1$$

b. Using formula sheet, Surface Area $= \int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$

$x = 2^{\frac{3}{2}}\cos^3(t)$ $\quad$ $y = 2^{\frac{3}{2}}\sin^3(t)$

$\frac{dx}{dt} = -3\times 2^{\frac{3}{2}}\cos^2(t)\sin(t)$ $\quad$ $\frac{dy}{dt} = 3\times 2^{\frac{3}{2}}\sin^2(t)\cos(t)$

(1 mark – either derivative correct)

$$SA = 2\pi\int_0^{\pi} 2^{\frac{3}{2}}\sin^3(t)\sqrt{(-3\times 2^{\frac{3}{2}}\cos^2(t)\sin(t))^2 + (3\times 2^{\frac{3}{2}}\sin^2(t)\cos(t))^2}\ dt$$

$$= 2\pi\times 2^{\frac{3}{2}}\int_0^{\pi}\sin^3(t)\sqrt{9\times 2^3\cos^4(t)\sin^2(t) + 9\times 2^3\sin^4(t)\cos^2(t)}\ dt$$

$$= 2\pi\times 2^{\frac{3}{2}}\int_0^{\pi}\sin^3(t)\sqrt{(9\times 2^3\cos^2(t)\sin^2(t))(\sin^2(t)+\cos^2(t))}\ dt$$

(1 mark – get to point of simplifying $\cos^2(t) + \sin^2(t)$)

$$= 2\pi\times 2^{\frac{3}{2}}\int_0^{\pi}\sin^3(t)\sqrt{(9\times 2^3\cos^2(t)\sin^2(t)}\ dt$$

$$= 2\pi\times 2^{\frac{3}{2}}\int_0^{\pi}\sin^3(t)\times 3\times 2^{\frac{3}{2}}\,|\cos(t)\sin(t)|\ dt$$

since $\sqrt{\cos^2(t)\sin^2(t)} = |\cos(t)\sin(t)|$

$$= 6\pi\times 2^3\int_0^{\pi}\sin^3(t)\,|\cos(t)\sin(t)|\ dt$$

$$= 48\pi\int_0^{\pi}\sin^3(t)\,|\cos(t)\sin(t)|\ dt$$

Given symmetry of graph, it is easier to use domain $0 \le t \le \frac{\pi}{2}$ as this gives $\cos(t) \ge 0$ and $\sin(t) \ge 0$ (therefore $|\cos(t)\sin(t)| = \cos(t)\sin(t)$) and multiply by 2.

$\therefore SA = 2\times 48\pi\int_0^{\frac{\pi}{2}}\sin^4(t)\cos(t)\ dt$ **(1 mark)**

$$= 96\pi\int_0^{\frac{\pi}{2}}\sin^4(t)\cos(t)\ dt$$

Let $u = \sin(t)$

$\frac{du}{dt} = \cos(t)$

$t = 0\ ,\ u = 0$

$t = \frac{\pi}{2}\ ,\ u = 1$

$$= 96\pi\int_0^{\frac{\pi}{2}} u^4\frac{du}{dt}dt$$

$$= 96\pi \int_0^1 u^4 \, du$$

$$= 96\pi \left[\frac{u^5}{5} \right]_0^1$$

$$= \frac{96\pi}{5} \text{ units}^2$$ **(1 mark)**

Question 7 (3 marks)

$$2\overline{z}^2 + 2z + \text{Re}(z) = -5$$

Let $z = x + yi$, $x, y \in R$.

$$2(x - yi)^2 + 2(x + yi) + x + 5 = 0$$
$$2x^2 - 4xyi - 2y^2 + 2x + 2yi + x + 5 = 0$$ **(1 mark)**

Equating real parts: $2x^2 - 2y^2 + 3x + 5 = 0$ …… (1)
Equating imaginary parts: $-4xy + 2y = 0$ ……. (2)

From (2), $y(-4x + 2) = 0$

$y = 0$ or $x = \frac{1}{2}$

- Substitute $y = 0$ into (1)

  $2x^2 + 3x + 5 = 0$
  Check discriminant, $\Delta = 3^2 - 4 \times 2 \times 5 = -31 < 0$
  $\therefore$ no real solution for x therefore $y = 0$ is rejected.

(1 mark – determine only one solution for *x*)

- Substitute $x = \frac{1}{2}$ into (1)

$$2\left(\frac{1}{2}\right)^2 - 2y^2 + 3\left(\frac{1}{2}\right) + 5 = 0$$
$$-2y^2 = -7$$
$$y^2 = \frac{7}{2}$$
$$y = \pm\sqrt{\frac{7}{2}} = \pm\frac{\sqrt{7}}{\sqrt{2}} = \pm\frac{\sqrt{14}}{2}$$

$$\therefore z = \frac{1}{2} - \frac{\sqrt{14}}{2}i, \ z = \frac{1}{2} + \frac{\sqrt{14}}{2}i$$ **(1 mark)**

Question 8 (4 marks)

$$\frac{dv}{dt}=\frac{2v}{1+t^2}$$

$\therefore \int \frac{1}{v}\,dv=\int \frac{2}{1+t^2}\,dt$ **(1 mark – attempt to separate variables)**

$\log_e|v|=2\tan^{-1}(t)+c$ **(1 mark)**

Method 1 – rearrange the constant

$$|v|=e^{2\tan^{-1}(t)+c}$$
$$=e^c e^{2\tan^{-1}(t)}$$
$$\therefore v=\pm e^c e^{2\tan^{-1}(t)}$$
$$=Ae^{2\tan^{-1}(t)} \quad \text{where } A=\pm e^c \in R\setminus\{0\}$$

Substitute in $v(0)=e$:

$$e=Ae^{2\tan^{-1}(0)}=A$$

(1 mark – attempt to find *A*)

Therefore $v=e\times e^{2\tan^{-1}(t)}$

$v=e^{2\tan^{-1}(t)+1}$ **(1 mark)**

Method 2 – find the +*c* first

$$\log_e|v|=2\tan^{-1}(t)+c$$

Given $v(0)=e$,

$$1=2\tan^{-1}(0)+c$$

$c=1$ **(1 mark – attempt to find +*c*)**

$$\log_e|v|=2\tan^{-1}(t)+1$$
$$v=\pm e^{2\tan^{-1}(t)+1}$$

Using the given condition of $v(0)=e$, leads us to reject the negative answer.

$v=e^{2\tan^{-1}(t)+1}$ **(1 mark)**

Question 9 (4 marks)

a. By inspection, $\underset{\sim}{d_1} = \sqrt{2}\underset{\sim}{i} + a\underset{\sim}{j} + 3\underset{\sim}{k}$ is a vector in the direction of the line $\underset{\sim}{r_1}$ and $\underset{\sim}{d_2} = \sqrt{2}\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ is a vector in the direction of the line $\underset{\sim}{r_2}$.

By definition, the angle between two lines is equal to the angle between these two vectors and is found using the dot product.

Using the dot product, $\underset{\sim}{d_1} \cdot \underset{\sim}{d_2} = |\underset{\sim}{d_1}||\underset{\sim}{d_2}|\cos(\theta)$,

$$\therefore\ 5 + a = \sqrt{11 + a^2} \times \sqrt{4} \times \cos(60°)$$ **(1 mark)**

$$5 + a = \sqrt{11 + a^2}$$

$$25 + 10a + a^2 = 11 + a^2$$

$$14 = -10a$$

$$a = \frac{-7}{5}$$ **(1 mark)**

Since the 'squaring' process can introduce extraneous solutions, it should be checked that $a = \frac{-7}{5}$ satisfies the original equation (which it does).

b. A normal to the plane is a vector normal to the direction of each line, that is, normal to the vectors $\underset{\sim}{d_1} = \sqrt{2}\underset{\sim}{i} - \frac{7}{5}\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{d_2} = \sqrt{2}\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

The cross product is used to determine this normal vector and hence, the equation of the plane.

Using the formula sheet for vectors $\underset{\sim}{r_1} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r_2} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$,
$\underset{\sim}{r_1} \times \underset{\sim}{r_2} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$.

$$\therefore \underset{\sim}{d_1} \times \underset{\sim}{d_2} = \left(\frac{-7}{5} \times 1 - 1 \times 3\right)\underset{\sim}{i} + (\sqrt{2} \times 3 - \sqrt{2} \times 1)\underset{\sim}{j} + \left(\sqrt{2} \times 1 - \sqrt{2} \times -\frac{7}{5}\right)\underset{\sim}{k}$$

$$= -\frac{22}{5}\underset{\sim}{i} + 2\sqrt{2}\underset{\sim}{j} + \frac{12\sqrt{2}}{5}\underset{\sim}{k}$$

$\therefore \frac{-22}{5}x + 2\sqrt{2}y + \frac{12\sqrt{2}}{5}z = k, \quad k \in R$ is the equation of the plane. **(1 mark)**

Substitute in a point on the plane e.g. $(5, \sqrt{2}, -5\sqrt{2})$ (which was found when $\mu = 0$ for line $\underset{\sim}{r_2}(\mu) = 5\underset{\sim}{i} + \sqrt{2}\underset{\sim}{j} - 5\sqrt{2}\underset{\sim}{k} + \mu(\sqrt{2}\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k})$).

(Another point that could be used as an example is $\left(10, -\sqrt{2}, \frac{5\sqrt{2}}{4}\right)$ found when $\lambda = 0$ for $\underset{\sim}{r_1}(\lambda) = 10\underset{\sim}{i} - \sqrt{2}\underset{\sim}{j} + \frac{5\sqrt{2}}{4}\underset{\sim}{k} + \lambda(\sqrt{2}\underset{\sim}{i} + a\underset{\sim}{j} + 3\underset{\sim}{k})$).

$$\frac{-22}{5} \times 5 + 2\sqrt{2} \times \sqrt{2} + \frac{12\sqrt{2}}{5} \times -5\sqrt{2} = k$$

$$-22 + 4 - 24 = k$$

$$k = -42$$

$\therefore \frac{-22}{5}x + 2\sqrt{2}y + \frac{12\sqrt{2}}{5}z = -42$ is the equation of the plane. **(1 mark)**

(alternatives include $\frac{22}{5}x - 2\sqrt{2}y - \frac{12\sqrt{2}}{5}z = 42$ or $22x - 10\sqrt{2}y - 12\sqrt{2}z = 210$)

Question 10 (7 marks)

a. $$f(x)=\frac{2x^2-4}{(x-2)^2}$$

$$\therefore f'(x)=\frac{(x-2)^2(4x)-(2x^2-4)(2(x-2))}{(x-2)^4}$$
$$=\frac{(x-2)(4x(x-2)-2(2x^2-4))}{(x-2)^4}$$

(1 mark – attempt quotient rule and simplification)

$$=\frac{4x^2-8x-4x^2+8}{(x-2)^3}$$
$$=\frac{-8x+8}{(x-2)^3}$$

Stationary point occurs when $f'(x)=0$

$$\therefore\ -8x+8=0$$
$$x=1$$

$$f(1)=\frac{2-4}{(-1)^2}=-2$$

$\therefore$ stationary point is at $(1,-2)$. **(1 mark – show that)**

b. $f''(x)=\dfrac{(x-2)^3(-8)-(-8x+8)(3(x-2)^2)}{(x-2)^6}$

$=\dfrac{(x-2)^2(-8(x-2)-3(-8x+8))}{(x-2)^6}$

$=\dfrac{16x-8}{(x-2)^4}$

A condition for a point of inflection to occur is that $f''(x)=0$

$$\frac{16x-8}{(x-2)^4}=0$$

$$\therefore\ 16x-8=0$$

$$x=\frac{1}{2}$$

(1 mark – show that)

A second condition for a point of inflection to occur is that there must be a change in concavity (that is, a change in sign of $f''(x)$).

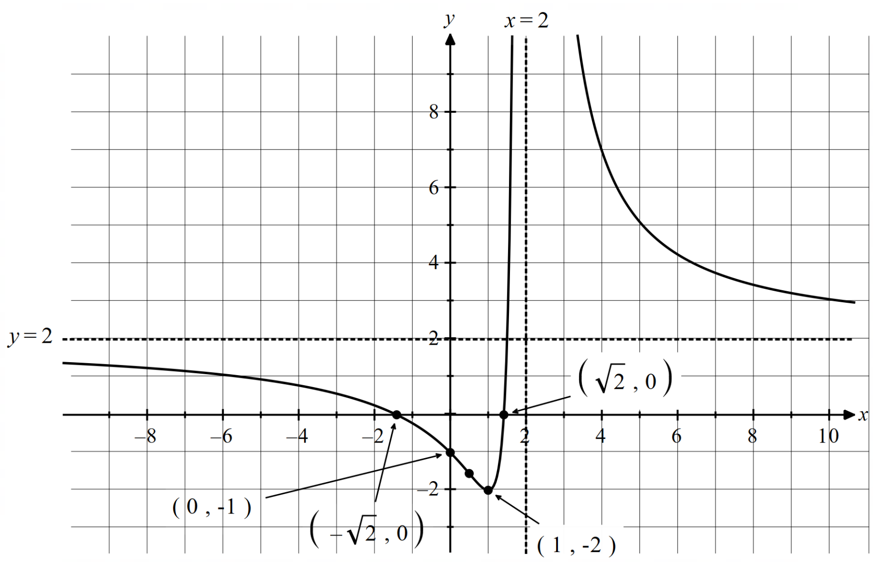
x	0	$\frac{1}{2}$	1
$f''(x)$	$=\frac{-8}{16}<0$	0	$=\frac{8}{1}>0$

(1 mark – show sign change)

c.

- Stationary point given at $(1,-2)$
- y-intercept: $f(0)=-1$
- x-intercepts: $f(x)=0$
  $2x^2-4=0$
  $x=\pm\sqrt{2}$
- asymptotes: $(x-2)^2=0 \quad \therefore\ x=2$ vertical asymptote

$$\frac{2x^2-4}{(x-2)^2}=2+\frac{8x-12}{(x-2)^2} \qquad \therefore\ y=2 \text{ horizontal asymptote}$$



(1 mark – two asymptotes with their equations)
(1 mark – correct shape including point of inflection shape indicated)
(1 mark – correct *y*-intercept, *x*-intercepts and stationary point)