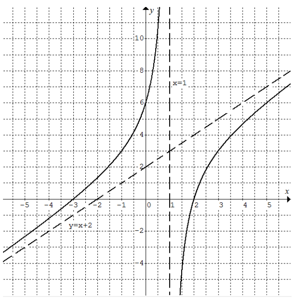
itute.com

2023 VCAA Specialist Mathematics Exam 1 Solutions

Q1a $f(x)=\frac{(x^2+x-2)-4}{x-1}=\frac{(x+2)(x-1)-4}{x-1}=x+2-\frac{4}{x-1}$

Q1b



Q2 $z^{\frac{1}{3}}=b-i$ is in the fourth quadrant, $\text{Arg}\left(z^{\frac{1}{3}}\right)=\tan^{-1}\left(\frac{-1}{b}\right)$

$\text{Arg}(z)=3\text{Arg}\left(z^{\frac{1}{3}}\right)=3\tan^{-1}\left(\frac{-1}{b}\right)=\frac{-\pi}{2}$.: $\tan^{-1}\left(\frac{-1}{b}\right)=\frac{-\pi}{6}$,

$\frac{-1}{b}=\frac{-1}{\sqrt{3}}$.: $b=\sqrt{3}$

Q3a $a=v\frac{dv}{dx}=\left(\frac{3x+2}{2x-1}\right)\left(\frac{3(2x-1)-2(3x+2)}{(2x-1)^2}\right)=-\frac{7(3x+2)}{(2x-1)^3}$

When $x=2$, $a=-\frac{56}{27}$

Q3b $v=\frac{3x+2}{2x-1}=\frac{3+\frac{2}{x}}{2-\frac{1}{x}}$, as $x\to\infty$, $v\to\frac{3}{2}$

Q4 $\frac{d}{dx}\left(x\sin^{-1}y^2\right)=0$, $\sin^{-1}y^2+\frac{2xy}{\sqrt{1-y^4}}\frac{dy}{dx}=0$

.: $\frac{dy}{dx}=-\frac{\sin^{-1}y^2\times\sqrt{1-y^4}}{2xy}$. At $\left(6,\frac{1}{\sqrt{2}}\right)$, $\frac{dy}{dx}=-\frac{\pi\sqrt{6}}{144}$

Q5 Integration by parts:

$3\int_1^2 x^2\log_e x\,dx=\left[x^3\log_e x\right]_1^2-\int_1^2 x^2\,dx=\left[x^3\log_e x\right]_1^2-\left[\frac{x^3}{3}\right]_1^2$

$\int_1^2 x^2\log_e x\,dx=\frac{1}{3}\left(8\log_e 2-\frac{7}{3}\right)$

Q6a Mean $=20+8+12=40$, s.d. $=\sqrt{6^2+\left(\sqrt{3}\right)^2+5^2}=8$

Q6b $\text{E}(\overline{X})=\mu_w=8$, $\text{sd}(\overline{X})=\frac{\sigma_w}{\sqrt{n}}=\frac{\sqrt{3}}{\sqrt{12}}=\frac{1}{2}$

$a=\frac{7.75-8}{\frac{1}{2}}=-\frac{1}{2}$ and $b=\frac{8.5-8}{\frac{1}{2}}=1$

Q7 $\frac{dx}{dt}=\frac{t}{2}$, $\frac{dy}{dt}=\sqrt{3}$

Surface area $A=2\pi\int_0^2\sqrt{3}\,t\sqrt{\left(\frac{t}{2}\right)^2+\left(\sqrt{3}\right)^2}\,dt$

Let $u=\left(\frac{t}{2}\right)^2+3$, $A=4\sqrt{3}\pi\int_3^4 u^{\frac{1}{2}}\,du=\pi\left(\frac{64\sqrt{3}}{3}-24\right)$

Q8 Given $f(x)=x\,e^{2x}$, prove that $f^{(n)}(x)=\left(2^n x+n2^{n-1}\right)e^{2x}$, $n\in Z^+$

Proof by mathematical induction:

$f^{(1)}(x)=(2x+1)e^{2x}$ is true for $n=1$

Assume $f^{(k)}(x)=\left(2^k x+k2^{k-1}\right)e^{2x}$ is true for $n=k$, then

$f^{(k+1)}(x)=2^k e^{2x}+2\left(2^k x+k2^{k-1}\right)e^{2x}$

$=\left(2^k+2\times2^k x+2\times k2^{k-1}\right)e^{2x}$

$=\left(2^k+2^{k+1}x+k2^k\right)e^{2x}$

$=\left(2^{k+1}x+(k+1)2^k\right)e^{2x}$

.: the statement $f^{(n)}(x)=\left(2^n x+n2^{n-1}\right)e^{2x}$ is true for $n=k+1$

.: the statement is true for all $n\in Z^+$

Q9a $D(0,2,0)$

Q9b

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(-1-1)\tilde{i}+(-2-3)\tilde{j}+(4-{}^{-}2)\tilde{k}=-2\tilde{i}-5\tilde{j}+6\tilde{k}$

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=(0-1)\tilde{i}+(2-3)\tilde{j}+(0-{}^{-}2)\tilde{k}=-\tilde{i}-\tilde{j}+2\tilde{k}$

Q9c $\tilde{n}=\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\tilde{i}&\tilde{j}&\tilde{k}\\-2&-5&6\\-1&-1&2\end{vmatrix}=-4\tilde{i}-2\tilde{j}-3\tilde{k}$ Let $\tilde{a}=\overrightarrow{OD}=2\tilde{j}$

Plane: $-4x-2y-3z=\tilde{a}.\tilde{n}=-4$, simplify to $4x+2y+3z=4$

Q9d $C(a,-1,5)$ is on the plane .: $4a+2(-1)+3(5)=4$.: $a=-\frac{9}{4}$

Q9e Area of parallelogram

$=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|=\left|-4\tilde{i}-2\tilde{j}-3\tilde{k}\right|=\sqrt{(-4)^2+(-2)^2+(-3)^2}=\sqrt{29}$

Q10a $5-6\sin^2 t=5-3(1-\cos 2t)=2+3\cos 2t$

Q10b $x=2+3\cos 2t$ and $y=1+3\sin 2t$

.: $(x-2)=3\cos 2t$ and $(y-1)=3\sin 2t$.: $(x-2)^2+(y-1)^2=3^2$

The path of the particle is a circle of radius 3 centred at $(2,1)$

Q10c Period of circular motion $=\frac{2\pi}{2}=\pi$

Distance travelled is a fraction $\frac{\frac{3\pi}{4}}{2\pi(3)}=\frac{1}{8}$ of the circumference and

the time taken is .: $\frac{\pi}{8}$.: $a=\frac{\pi}{8}$

Q10d A position vector passing through the centre $(2,1)$ of the circle is always perpendicular to the velocity vector.

.: $\frac{1+3\sin 2t}{2+3\cos 2t}=\frac{1}{2}$.: $6\sin 2t=3\cos 2t$, $\tan 2t=\frac{1}{2}$

$2t=\tan^{-1}\left(\frac{1}{2}\right)+n\pi$.: $t=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)+\frac{n\pi}{2}$ where $n=0,1,2,\ldots$

Please inform mathline@itute.com re conceptual and/or mathematical errors.