# VCE Specialist Maths Exam Questions - 2023

## Exam 1

### Question 1 (4 marks)

Consider the function $f$ with the rule $f(x) = \dfrac{x^2 + x - 6}{x - 1}$

**a.** Show that the rule for the function $f$ can be written as $f(x) = x + 2 - \dfrac{4}{x - 1}$

Easiest to just work backwards:

$$x + 2 - \frac{4}{x-1} = \frac{(x+2)(x-1) - 4}{x-1} = \frac{x^2 + 2x - x - 2 - 4}{x-1} = \frac{x^2 + x - 7}{x-1} \text{ as required}$$

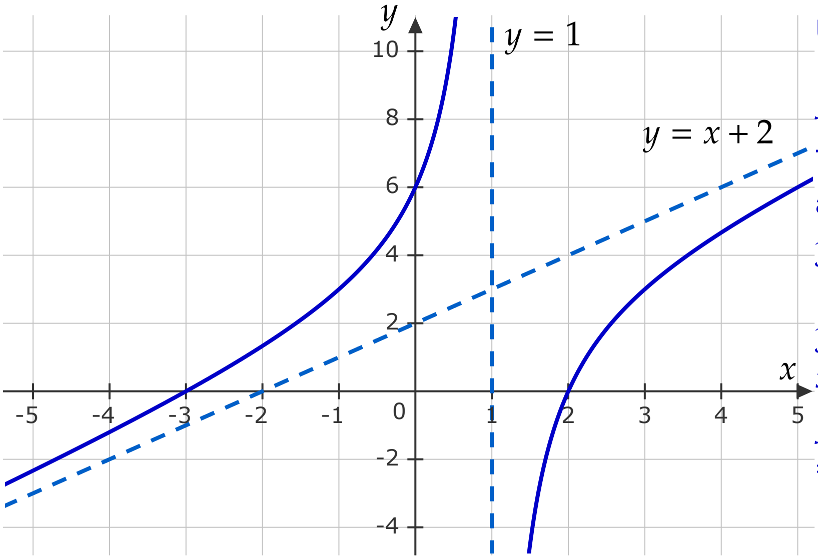
Or could also use long division, eg

$$\begin{aligned} x^2 + x - 6 &= (x-1)(x+2) - 4 \\ &= x^2 - x + 2x - 2 - 4 \end{aligned} \implies \frac{x^2 + x - 6}{x-1} = \frac{(x-1)(x+2) - 4}{x-1} = x + 2 - \frac{4}{x-1}$$

or

$$\begin{array}{r} x\ +2 \\ x-1 \overline{\big)\, x^2 + x - 6} \\ \underline{x^2 - x\phantom{-6}} \\ 2x - 6 \\ \underline{2x - 2} \\ -4 \end{array} \qquad \implies \frac{x^2 + x - 6}{x-1} = x + 2 - \frac{4}{x-1}$$

**b.** Sketch the graph of $f$ on the axes below, labelling any asymptotes with their equations.




Using the form from part a

$f(x) = x + 2 - \dfrac{4}{x-1}$

This has a VA at $x = 1$

and a diagonal asymptotate along

$y = x + 2$

$y$-intercept: $f(0) = 6$

$x$-intercept:

$f(x) = 0 \implies x^2 + x - 6 = 0$

$\implies x = -3,\ 2$

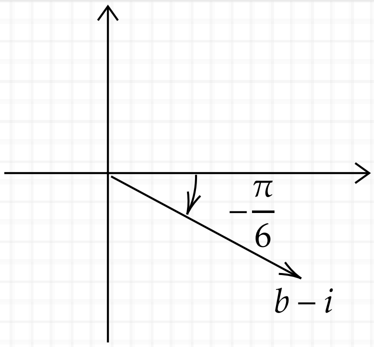
### Question 2 (3 marks)

Consider the complex number $z = (b - i)^3$, where $b \in \mathbb{R}^+$

Find $b$ given that $\arg(z) = -\dfrac{\pi}{2}$.

$$-\frac{\pi}{2} = \arg(z) = 3\arg(b-i) \implies \arg(b-i) = -\frac{\pi}{6}$$

$$\implies \arctan\left(\frac{1}{b}\right) = \frac{\pi}{6} \implies b = \sqrt{3}$$




**Question 3 (3 marks)**

A particle moves along a straight line. When the particle is $x$ m from a fixed point $O$, its velocity, $v$ m s$^{-1}$, is given by $v = \frac{3x+2}{2x-1}$, where $x \geqslant 1$

**a.** Find the acceleration of the particle, in m s$^{-2}$, when $x = 2$.

$$a = v\frac{dv}{dx} = \frac{3x+2}{2x-1} \times \frac{3(2x-1) - 2(3x+2)}{(2x-1)^2}$$

$$\xrightarrow{x=2} \frac{8}{3} \times \frac{3\times 3 - 2\times 8}{3^2} = -\frac{8\times 7}{3^3} = -\frac{56}{27}$$

**b.** Find the value that the velocity of the particle approaches as $x$ becomes very large.

It's just a rectangular hyperbola, that students are used to from Methods...

Take the limit $v = \frac{3x+2}{2x-1} \to \frac{3}{2}$ as $x \to \infty$

If uncomfortable with that, then split the fraction using long division

$v = \frac{3}{2} + \frac{7}{2}\frac{1}{2x-1} \to \frac{3}{2}$ as $x \to \infty$, but that is more work!

**Question 4 (3 marks)**

Consider the relation $x \arcsin\left(y^2\right) = \pi$.

Use implicit differentiation to find $\frac{dy}{dx}$ at the point $\left(6, \frac{1}{\sqrt{2}}\right)$.

Give your answer in the form $-\pi\frac{\sqrt{a}}{b}$, where $a, b \in \mathbb{Z}^+$

Hit the equation with $\frac{d}{dx}$: $\arcsin\left(y^2\right) + x \times 2y\frac{dy}{dx}\frac{1}{\sqrt{1-\left(y^2\right)^2}} = 0$

$$\implies \frac{dy}{dx} = -\arcsin\left(y^2\right)\frac{\sqrt{1-y^4}}{2xy}$$

$$@\left(6, \frac{1}{\sqrt{2}}\right): \ \frac{dy}{dx} = -\arcsin\left(\frac{1}{2}\right)\frac{\sqrt{1-\frac{1}{4}}}{2\times 6\times\frac{1}{\sqrt{2}}} = -\frac{\pi}{6}\times\frac{\frac{1}{2}\sqrt{3}}{12}\times\frac{\sqrt{2}}{1} = -\pi\frac{\sqrt{6}}{144}$$

**Question 5 (3 marks)**

Evaluate $\int_1^2 x^2 \ln(x)dx$

Want to hit the $\ln(x)$ with a derivative to turn it into $\frac{1}{x}$, so rewrite $x^2$ as $\frac{1}{3}\frac{d}{dx}x^3$

$$\begin{aligned}\int_1^2 x^2\ln(x)dx &= \frac{1}{3}\int_1^2\left(\frac{d}{dx}x^3\right)\ln(x)dx\\ &= \frac{1}{3}\left[x^3\ln(x)\right]_1^2 - \frac{1}{3}\int_1^2 x^3\frac{d}{dx}\ln(x)dx\\ &= \frac{1}{3}\left[(2)^3\ln(2)-0\right]-\frac{1}{3}\int_1^2 x^2dx\\ &= \frac{8}{3}\ln(2)-\frac{1}{9}\left[x^3\right]_1^2 = \frac{8}{3}\ln(2)-\frac{1}{9}[8-1]\\ &= \frac{8}{3}\ln(2)-\frac{7}{9}\end{aligned}$$

**Question 6 (4 marks)**

Josie travels from home to work in the city. She drives a car to a train station, waits, and then rides on a train to the city. The time, $X_c$ minutes, taken to drive to the station is normally distributed with a mean of 20 minutes ($\mu_c = 20$) and standard deviation of 6 minutes ($\sigma_c = 6$). The waiting time, $X_w$ minutes, for a train is normally distributed with a mean of 8 minutes ($\mu_w = 8$) and standard deviation of $\sqrt{3}$ minutes ($\sigma_w = \sqrt{3}$). The time, $X_t$ minutes, taken to ride on a train to the city is also normally distributed with a mean of 12 minutes ($\mu_t = 12$) and standard deviation of 5 minutes ($\sigma_t = 5$). The three times are independent of each other.

**a.** Find the mean and standard deviation of the total time, in minutes, it takes for Josie to travel from home to the city.

Total time is the drive plus the wait, plus the train. They are all normally distributed, so the total time $T \sim \mathrm{N}(\mu_T, \sigma_T)$ has

mean: $\mu_T = \mu_c + \mu_w + \mu_t = 20 + 8 + 12 \implies \boxed{\mu_T = 40 \text{ minutes}}$

variance: $\sigma_T^2 = \sigma_c^2 + \sigma_w^2 + \sigma_t^2 = 6^2 + 3 + 5^2 = 64$

$\Rightarrow$ standard deviation: $\boxed{\sigma_T = 8 \text{ minutes}}$

**b.** Josie's waiting time for a train on each work day is independent of her waiting time for a train on any other work day. The probability that, for 12 randomly chosen work days, Josie's average waiting time is between 7 minutes 45 seconds and 8 minutes 30 seconds is equivalent to $\Pr(a < Z < b)$, where $Z \sim \mathrm{N}(0, 1)$ and $a$ and $b$ are real numbers.
Find the values of $a$ and $b$.

Mean waiting time ($n = 12$): $\widehat{X_w} \sim \mathrm{N}\left(\mu_w, \dfrac{\sigma_w}{\sqrt{12}}\right) = \mathrm{N}\left(8, \dfrac{1}{2}\right)$

Want to find $p = \Pr(7.75 < \widehat{T} < 8.5)$

Let $Z = \dfrac{\widehat{T} - 8}{1/2} \sim \mathrm{N}(0, 1) \implies p = \Pr\left(2(7.75 - 8) < Z < 2(8.5 - 8)\right) = \Pr\left(-\dfrac{1}{2} < Z < 1\right)$

So, $a = -\dfrac{1}{2}$, $b = 1$

**Question 7 (4 marks)**

The curve defined by the parametric equations

$x = \dfrac{t^2}{4} - 1,\ y = \sqrt{3}\,t$, where $0 \leqslant t \leqslant 2$

is rotated about the $x$-axis to form an open hollow surface of revolution.
Find the surface area of the surface of revolution.

Give your answer in the form $\pi\left(\dfrac{a\sqrt{b}}{c} - d\right)$ where $a,\ b,\ c$ and $d \in \mathbb{Z}^+$.

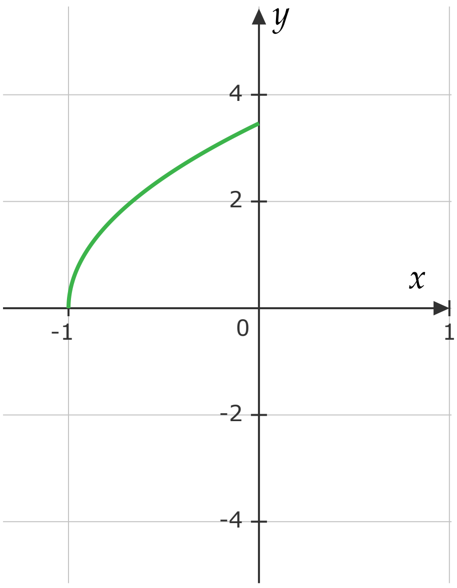
We'll just calculate the surface area of the outside of the created paraboloid
(other interpretations might be possible, will have to wait to see what VCAA accepts):

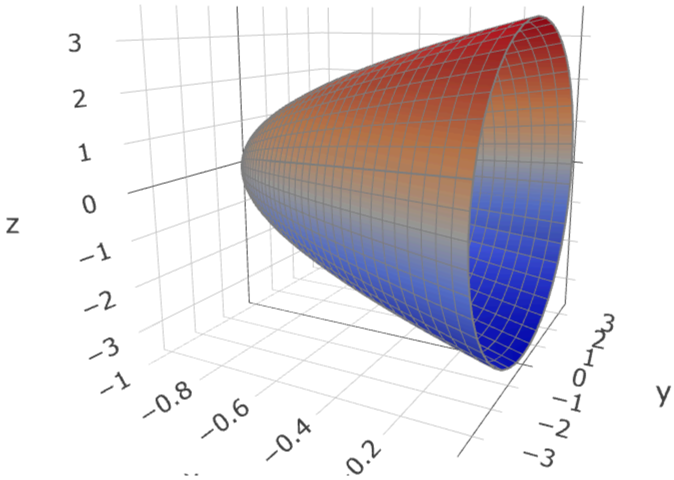
$$A = \int_0^2 2\pi y\sqrt{\dot{x}^2 + \dot{y}^2}dt = 2\pi\sqrt{3}\int_0^2 t\sqrt{\left(\tfrac{t}{2}\right)^2 + 3}\,dt$$

$$= 4\pi\sqrt{3}\int_3^4 \sqrt{u}\,du, \quad \text{where } u = \left(\frac{t}{2}\right)^2 + 3,\ du = \tfrac{1}{2}t\,dt$$

$$= 4\pi\sqrt{3} \times \left[\frac{2}{3}u^{\frac{3}{2}}\right]_3^4 = \frac{8\pi}{\sqrt{3}}\left(8 - 3^{\frac{3}{2}}\right)$$

$$= \pi\left(\frac{64\sqrt{3}}{3} - 24\right) \quad \text{in the required form}$$




Alternatively, could convert to Cartesian form first

$$x = \frac{\left(\frac{y}{\sqrt{3}}\right)^2}{4} - 1 = \frac{y^2}{12} - 1,\ \ 0 \leqslant y \leqslant 2\sqrt{3}$$

$$\implies y = 2\sqrt{3}\sqrt{x+1},\ \ -1 \leqslant x \leqslant 0 \qquad \implies \frac{dy}{dx} = \sqrt{\frac{3}{x+1}}$$

$$A = \int_{-1}^{0} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = 4\sqrt{3}\pi\int_{-1}^{0}\sqrt{x+1}\sqrt{1+\frac{3}{x+1}}dx$$

$$= 4\sqrt{3}\pi\int_{-1}^{0}\sqrt{x+4}dx = 4\sqrt{3}\pi\left[\frac{2}{3}(x+4)^{\frac{3}{2}}\right]_{-1}^{0} = \frac{8\pi}{\sqrt{3}}\left[(4)^{\frac{3}{2}} - 3^{\frac{3}{2}}\right] = \frac{8\pi}{\sqrt{3}}\left(8 - 3^{\frac{3}{2}}\right)$$

Of course, the required form $\pi\left(\frac{a\sqrt{b}}{c} - d\right)$ where $a,\ b,\ c$ and $d \in \mathbb{Z}^+$ is not unique

$$A = \pi\left(\frac{64\sqrt{3}}{3} - 24\right) = \pi\left(\frac{128\sqrt{27}}{18} - 24\right) = \ldots$$

but with the implicit requirement to simplify factors of 2, 3, 5 etc it is ok.

**Question 8 (4 marks)**

A function $f$ has the rule $f(x) = x\,e^{2x}$.
Use mathematical induction to prove that $f^{(n)}(x) = \left(2^n x + n\,2^{n-1}\right)e^{2x}$ for $n \in \mathbb{Z}^+$,
where $f^{(n)}(x)$ represents the n[th] derivative of $f(x)$.

Base case: $n = 1$
Calculate the 1st derivative: $f'(x) = e^{2x} + x \times 2e^{2x} = (1+2x)e^{2x} = (2x+1)e^{2x}$
Compare to the provided formula: $f^{(1)}(x) = \left(2^1 x + 1 \times 2^0\right)e^{2x} = (2x+1)e^{2x}$

These match, so the claim is true for $n = 1$

Assume that there is some $n = k \in \mathbb{Z}^+$ for which the statement holds.
Then examine the $n = k + 1$ case:

$RHS = \left(2^{k+1}x + (k+1)2^k\right)e^{2x}$

$$\begin{aligned} LHS = f^{(k+1)}(x) &= \frac{d}{dx}f^{(k)}(x) = \frac{d}{dx}\left(\left(2^k x + k\,2^{k-1}\right)e^{2x}\right) \\ &= \left(2^k + 2 \times 2^k x + 2 \times k\,2^{k-1}\right)e^{2x} \\ &= \left(2^k + 2^{k+1}x + k\,2^k\right)e^{2x} \\ &= \left(2^{k+1}x + (k+1)\,2^k\right)e^{2x} \end{aligned}$$

ie, if true for some $k$, then true for some $k + 1$

So, by the principle of mathematical induction, $f^{(n)}(x) = \left(2^n x + n\,2^{n-1}\right)e^{2x}$ for all $n \in \mathbb{Z}^+$

**Question 9 (6 marks)**

A plane contains the points $A(1, 3, -2)$, $B(-1, -2, 4)$ and $C(a, -1, 5)$, where $a$ is a real constant. The plane has a $y$-axis intercept of 2 at the point $D$.

**a.** Write down the coordinates of point $D$. 1 mark

The point $D$ is at $(0,\ 2,\ 0)$

**b.** Show that $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are $-2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ and $-\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ respectively 1 mark

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix} = -2\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$ [a good check on part **a**]

**c.** Hence find the equation of the plane in Cartesian form. 2 marks

Find a normal vector to the plane:

$$\mathbf{n} \propto \overrightarrow{AB} \times \overrightarrow{AD} = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix} \times \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -10+6 \\ -6+4 \\ 2-5 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -3 \end{bmatrix}$$

Choose $\mathbf{n} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

The equation of the plane is $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$,

choose $\mathbf{r}_0 = \overrightarrow{OB}$, so $\mathbf{n} \cdot \mathbf{r}_0 = 4(-1) + 2(-2) + 3(4) = 4$

[could also choose $\mathbf{r}_0 = \overrightarrow{OD}$, so $\mathbf{n} \cdot \mathbf{r}_0 = 4(0) + 2(2) + 3(0) = 4$, etc]

$$\boxed{4x+2y+3z=4}$$

**d.** Find $a$ 1 mark

All points on the plane satisfy $\mathbf{n}\cdot\mathbf{r}=4$. so sub in $\overrightarrow{OB}$ to get

$$4(a)+2(-1)+3(5)=4a+13=4 \implies a=-\frac{9}{4}$$

**e.** $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are adjacent sides of a parallelogram.
Find the area of this parallelogram 1 mark

area $=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|=\left|\mathbf{n}\right|=\sqrt{4^2+2^2+3^2}=\sqrt{29}$ square units

**Question 10 (6 marks)**

The position vector of a particle at time t seconds is given by
$\mathbf{r}(t)=\left(5-6\sin^2 t\right)\mathbf{i}+(1+6\sin t\cos t)\mathbf{j}$, where $t\geqslant 0$

**a.** Write $5-6\sin^2 t$ in the form $\alpha+\beta\cos(2t)$, where $\alpha,\beta\in\mathbb{Z}^+$

Use the identity $\cos(2t)=1-2\sin^2 t \implies 2\sin^2 t=1-\cos(2t)$
$\implies 5-6\sin^2 t=5-3(1-\cos(2t))=2+3\cos(2t)$

**b.** Show that the Cartesian equation of the path of the particle is $(x-2)^2+(y-1)^2=9$.

$$\mathbf{r}(t)=(2+3\cos(2t))\mathbf{i}+(1+3\sin(2t))\mathbf{j}=x\,\mathbf{i}+y\,\mathbf{j}$$
$$\implies \cos(2t)=\frac{x-2}{3},\ \sin(2t)=\frac{y-1}{3}$$
$$\implies 1=\cos^2(2t)+\sin^2(2t)=\left(\frac{x-2}{3}\right)^2+\left(\frac{y-1}{3}\right)^2$$
$$\implies (x-2)^2+(y-1)^2=9 \text{ as required}$$

**c.** The particle is at point $A$ when $t=0$ and at point $B$ when $t=a$, where $a$ is a positive real constant.

If the distance travelled along the curve from $A$ to $B$ is $\frac{3\pi}{4}$, find $a$.

**Option 1:**
The trajectory is a circle of radius 3 and circumference $6\pi$, orbiting with a period of $\pi$

The distance travelled is $\frac{3\pi}{4}=\frac{1}{8}\times 6\pi$, so it is $\frac{1}{8}^{\text{th}}$ of the way around the circle.

Thus $a = \frac{\pi}{8}$.

**Option 2:**

dist trav $= \frac{3\pi}{4} = \int_0^a \sqrt{\dot{x}^2 + \dot{y}^2} dt = \int_0^a \sqrt{(-6\sin(2t))^2 + (6\cos(2t))^2} dt$

$\implies \frac{3\pi}{4} = \int_0^a 6dt = 6a$

$\implies a = \frac{1}{6}\frac{3\pi}{4} = \frac{\pi}{8}$

**d.** Find all values of $t$ for which the position vector of the particle, $\mathbf{r}(t)$, is perpendicular to its velocity vector, $\dot{\mathbf{r}}(t)$.

$\mathbf{r}(t) = (2 + 3\cos(2t))\mathbf{i} + (1 + 3\sin(2t))\mathbf{j}$
$\dot{\mathbf{r}}(t) = -6\sin(2t)\mathbf{i} + 6\cos(2t)\mathbf{j}$

Require $0 = \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = (2 + 3\cos(2t))(-6\sin(2t)) + (1 + 3\sin(2t))(6\cos(2t))$

$\implies 0 = -2\sin(2t) + \cos(2t)$

$\implies \tan(2t) = \frac{1}{2}$

$\implies t = \frac{1}{2}\arctan\left(\frac{1}{2}\right) + \frac{n\pi}{2},\ n \in \{0, 1, 2, 3, \dots\}$ [Note:$t \geqslant 0$]

## Exam 2: Section A - MCQs

### Question 1 (C)

Consider the following statement:
'If my football team plays badly, then they are not training enough.'
The contrapositive of the statement is:

Original Statement is of the form $A \implies B$,
where $A =$ my football team plays badly, $B =$ they are not training enough
The contrapositive is $\neg B \implies \neg A$, which can be read as
If they are training enough, then my football team does not play badly.

A very basic question from the proof topic - only surprising thing is that there were no more proofs in the exam paper! Only this and the proof by induction in Exam 1.

### Question 2 (B)

The graph of $y = \dfrac{x^3}{ax^2 + bx + c}$ has asymptotes given by $y = 2x + 1$ and $x = 1$.
The values of $a,\ b,$ and $c$ are, respectively

The VA at $x = 1$ implies the fraction can be written as $y = \dfrac{x^3}{a(x-1)(x-c/a)}$ and $b = -(a+c)$
Need to use long division to match up the oblique asymptote. This is easily done with the CAS using `propFrac`, but note we don't actually need the remainder:

$$y = \frac{x + 1 + c/a}{a} + \frac{\left(a^2 + ac + c^2\right)x - (a+c)c}{a^2(x-1)(x-c/a)},$$

Matching the diagonal asymptote as $x$ gets large

$$\frac{x + 1 + c/a}{a} = 2x + 1 \implies a = \frac{1}{2} \text{ and } c = -\frac{1}{4} \implies b = -\frac{1}{4}$$

Note, that there is another vertical asymptote at $x = \dfrac{c}{a} = -\dfrac{1}{2}$.
Potentially confusing wording in the question as it only mentioned two asymptotes - which could imply a repeated root in the denominator.

Without CAS you can fairly quickly find the quotient in the algebraic division and ignore the remainder. E.g., use the "long division" method of rewriting the numerator to match the denominator:

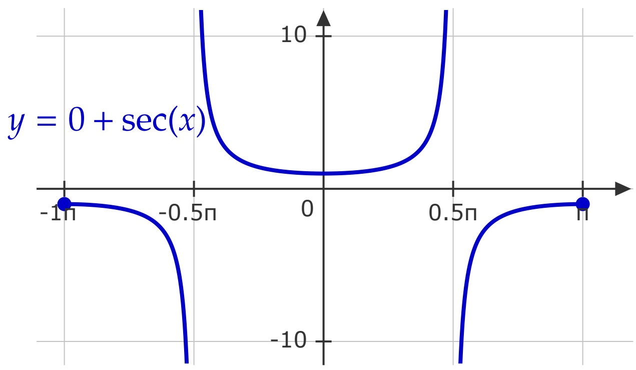
$$\frac{x^3}{a(x-1)(x-c/a)} = \frac{(x-1)(x-c/a)(x+D) + \text{remainder}}{a(x-1)(x-c/a)}$$

The $D$ just needs to cancel off the quadratic terms, and the remainder is a linear expression that cancels out the linear terms from expanding the brackets. Easy enough to find $D$

$$\frac{x^3}{a(x-1)(x-c/a)} = \frac{(x-1)(x-c/a)(x+1+c/a) + \text{remainder}}{a(x-1)(x-c/a)} = \frac{x+1+c/a}{a} + \frac{\text{remainder}}{a(x-1)(x-c/a)}$$

**Question 3 (E)**

In the interval $-\pi \leqslant x \leqslant \pi$, the graph of $y = a + \sec(x)$, where $a \in \mathbb{R}$, has two $x$-intercepts when




A quick sketch shows that $a$ should be negative enough so the middle crosses twice or $a$ should be positive enough so that the outer bits cross

$a < -1$ or $a \geqslant 1$

**Question 4 (B)**

If $z = -(2a+1) + 2ai$, where $a$ is a non-zero real constant, then $\dfrac{4a}{1+\overline{z}}$ is equal to

$$\begin{aligned}\frac{4a}{1+\overline{z}} &= \frac{4a}{1+(-(2a+1)-2ai)} \\ &= \frac{4a}{-2a-2ai} = \frac{2}{-1-i} \\ &= \frac{2}{\sqrt{2}\,\mathrm{cis}(-3\pi/4)} = \sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)\end{aligned}$$

**Question 5 (E)**

Let $z$ be a complex number where $\mathrm{Re}(z) > 0$ and $\mathrm{Im}(z) > 0$.
Given $|\overline{z}| = 4$ and $\arg\left(z^3\right) = -\pi$, then $z^2$ is equivalent to

$$\arg\left(z^3\right) = -\pi + 2n\pi$$

$$\implies \arg(z) = -\frac{\pi}{3} + \frac{2n\pi}{3} = \frac{\pi}{3} + 2m\pi \;\left(\text{restrict } n = 1 + 3m \because \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\right)$$

$$\implies z = 4\,\mathrm{cis}\left(\frac{\pi}{3}\right)$$

$$z^2 = 16\,\mathrm{cis}\left(\frac{2\pi}{3}\right) = 4 \times 4\,\mathrm{cis}\left(\pi - \frac{\pi}{3}\right) = 4\,\mathrm{cis}(\pi) \times 4\,\mathrm{cis}\left(-\frac{\pi}{3}\right) = -4\overline{z}$$

In general, $\arg$ is a multi-valued function such that $\arg(z) = \left\{\mathrm{Arg}(z) + 2n\pi, n \in \mathbb{Z}\right\}$.

So the statement that "$\arg(z^3) = -\pi$" should probably read $-\pi \in \arg(z^3)$ and the constraints on $z$ mean that $\arg(z^3)$ is a strict subset of $\{-\pi + 2n\pi, n \in \mathbb{Z}\}$, which is what the $m$ in the working above makes more explicit.

**Question 6 (C)**

Consider the following pseudocode.

```
define f(x,y) = e^(xy)
x ← 0
y ← 0
h ← 0.5
n ← 0
while n ⩾ 0  // infinite loop
  y ← y + h×f(x, y)
  x ← x + h
  n ← n + 1
  print y
end while
```

After how many iterations will the pseudocode print 2.709?

Can just trace the algorithm in a trace-table / desk-check...
but really, just using an euler() program on the CAS is easiest.

euler(exp(x*y),x,y,{0,2},0,0.5) $\rightarrow$

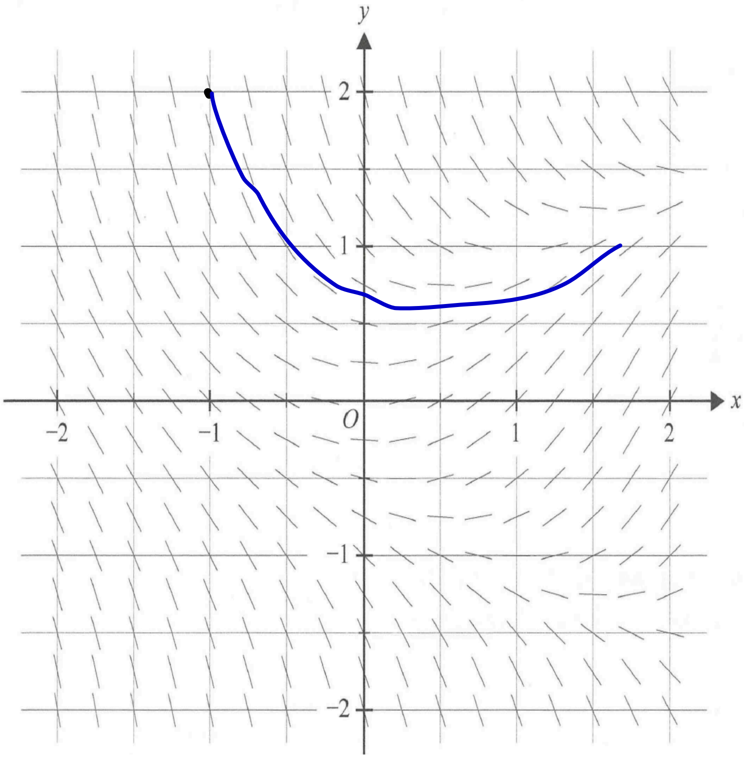
| $x_k$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $y_k$ | 0 | 0.5 | 1.142 | 2.7085 | 31.777 |

So, 3 iterations.

Note: A really basic Euler method algorithm question, and a safe 1st pseudocode question for VCAA. Although, they really mucked up their indentation - it's fixed above!

**Question 7 (D)**

The direction field for a differential equation is shown below.
On a certain solution curve of this differential equation, $y = 2$ when $x = -1$.
The value of $y$ on the same solution curve when $x = 1.5$ is closest to




A. -0.5
B. 0
C. 0.5
D. 1.0
E. 1.5

Following the slope field from $(-1, 2)$ ends up somewhere around $y = 1$ when $x = 1.5$

[near x=1.25 my curve should have had a sligtly greater gradient]

Note that this is really a slope field, not a direction field....

**Question 8 (A)**

Initially a spa pool is filled with 8000 litres of water that contains a quantity of dissolved chemical.
It is discovered that too much chemical is contained in the spa pool water. To correct this situation, 20 litres of well-mixed spa pool water is pumped out every minute while 15 litres of fresh water is pumped in each minute.
Let $Q$ be the number of kilograms of chemical that remains dissolved in the spa pool after $t$ minutes.
The differential equation relating $Q$ to $t$ is

15 L/min

$V_0 = 8000$ L

$C(t) = Q/V$

20 L/min

First, note that the volume is decreasing over time

$V(t) = 8000 - 5t, \quad 0 \leqslant t \leqslant 1600$

Then the DE for $Q$ is

$$\frac{dQ}{dt} = Q_{\text{in}} - Q_{\text{out}} = 0 - C(t) \times 20$$

$$= \frac{-20Q}{8000 - 5t} = \frac{4Q}{t - 1600}$$

**Question 9 (D)**

The position of a particle moving in the Cartesian plane, at time $t$, is given by the parametric equations

$x(t) = \dfrac{6t}{t+1}, \quad y(t) = \dfrac{-8}{t^2+4}$, where $t \geqslant 0$

What is the slope of the tangent to the path of the particle when $t = 2$?

Want $\left.\frac{dy}{dx}\right|_{t=2} = \frac{\dot{y}(2)}{\dot{x}(2)} = \left.\frac{16t}{\left(t^2+4\right)^2} \div \frac{6}{(t+1)^2}\right|_{t=2} = \frac{1/2}{2/3} = \frac{3}{4}$

**Question 10 (A)**

If $I_n = \int_0^1 (1-x)^n e^x dx$, where $n \in \mathbb{N}$, then for $n \geqslant 1$, $I_n$ equals

$$I_n = \int_0^1 (1-x)^n \left(\frac{d}{dx} e^x\right) dx = \left[(1-x)^n e^x\right]_0^1 - \int_0^1 \left(\frac{d}{dx}(1-x)^n\right) e^x dx$$
$$= -1 - \int_0^1 \left(-n(1-x)^{n-1}\right) e^x dx$$
$$= -1 + nI_{n-1}$$

Just an integration by parts question - but not sure if generating recursion relations from this is in the study design (it is in the Cambridge textbook and is common in HSC Ext 2 exams... and is a good tool to have!)

**Question 11 (E)**

The area of the curved surface generated by revolving part of the curve with equation $y = \cos^{-1}(x)$ from $\left(0, \frac{\pi}{2}\right)$ to $(1, 0)$ about the $y$-axis can be found by evaluating

$$A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy, \qquad \text{note: } x = \cos(y),\ \frac{dx}{dy} = -\sin(y)$$
$$= 2\pi \int_0^{\pi/2} \cos(y) \sqrt{1 + (-\sin(y))^2} dy$$
$$= 2\pi \int_0^1 \sqrt{1+u^2}\, du, \text{ where } u = \sin(y),\ du = \cos(y)dy,$$

**Question 12 (A)**

The acceleration, $a$ ms$^{-2}$, of a particle that starts from rest and moves in a straight line is described by $a = 1 + v$, where $v$ ms$^{-1}$ is its velocity after $t$ seconds.
The velocity of the particle after $\log_e(e+1)$ seconds is

$$a = \frac{dv}{dt} = 1 + v \Longrightarrow \int_0^v \frac{1}{1+v} dv = \int_0^{\ln(e+1)} dt$$
$$\Longrightarrow \ln(1+v) - 0 = \ln(e+1) - 0$$
$$\Longrightarrow v = e$$

**Question 13 (E)**

A tourist in a hot air balloon, which is rising vertically at 2.5 ms$^{-1}$, accidentally drops a phone over the side when the phone is 80 metres above the ground.

Assuming air resistance is negligible, how long in seconds, correct to two decimal places, does it take for the phone to hit the ground?

Ignoring air resistance means it is a constant acceleration problem, so SUVAT it...
Choosing down to be negative, we have the values
$a = -9.8$, $u = 2.5$, $v =$ 🤷, $s = -80$, $t = ?$
$s = ut + \frac{1}{2}at^2 \implies -80 = 2.5\,t - 4.9\,t^2 \implies t = 4.303757...$
[Distractors are if you ignore the initial velocity, or get the signs switched, etc]

**Question 14 (B)**

Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - \mathbf{j}$ and $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
If $\mathbf{n}$ is a unit vector such that $\mathbf{a} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} = 0$, then $|\mathbf{c} \cdot \mathbf{n}|$ is equal to

$\mathbf{i} = \frac{\mathbf{a}+\mathbf{b}}{2}$, $\mathbf{j} = \frac{\mathbf{a}-\mathbf{b}}{2}$, don't really need these, just use the fact that $\mathbf{a}$ and $\mathbf{b}$ span the $x$-$y$ plane
Then we see that $\mathbf{a} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} = 0 \Leftrightarrow \mathbf{i} \cdot \mathbf{n} = \mathbf{j} \cdot \mathbf{n} = 0$,
so the unit vector $\mathbf{n}$ must be either $\pm\mathbf{k}$

Then $|\mathbf{c} \cdot \mathbf{n}| = |\mathbf{c} \cdot \mathbf{k}| = 3$

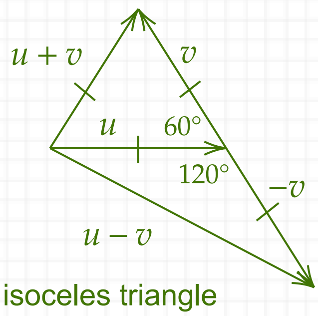
[I think the exam writer forgot where the \cdot button is...]

**Question 15 (D)**

If the sum of two unit vectors is a unit vector,
then the magnitude of the difference of the two vectors is

$|u + v| = 1$
$\implies |u+v|^2 = 1 + 2u \cdot v + 1 = 2 + 2u \cdot v = 1$
$\implies 2u \cdot v = -1$

$\implies |u - v|^2 = 1 - 2u \cdot v + 1 = 2 - 2u \cdot v = 3$
$\implies |u - v| = \sqrt{3}$




Note: this question can also be about an equilateral and a $120°$ isoceles triangle

**Question 16 (D)**

A student throws a ball for his dog to retrieve. The position vector of the ball, relative to an origin $O$ at ground level $t$ seconds after release, is given by
$\mathbf{r}_B(t) = 5t\mathbf{i} + 7t\mathbf{j} + \left(15t - 4.9t^2 + 1.5\right)\mathbf{k}$
Displacement components are measured in metres, where $\mathbf{i}$ is a unit vector to the east, $\mathbf{j}$ is a unit vector to the north and $\mathbf{k}$ is a unit vector vertically up.

The total **vertical** distance, in metres, travelled by the ball before it hits the ground is closest to

OK, so can ignore the $\mathbf{i}$ and $\mathbf{j}$ components.
At $t = 0$ the ball is thrown from 1.5 meters high with initial velocity 15 m/s up.
It travels up to a max of 12.97959 m high after 1.53061 seconds,
then comes all the way down to 0 where it hits the ground.
Total vertical distance is $2 \times 12.97959 - 1.5 \approx 24.459$

**Question 17 (C)**

Consider the vectors **a** $= \begin{bmatrix} \alpha \\ 1 \\ -1 \end{bmatrix}$, **b** $= \begin{bmatrix} 3 \\ \beta \\ 4 \end{bmatrix}$, and **c** $= \begin{bmatrix} 2 \\ -7 \\ \gamma \end{bmatrix}$, where $\alpha, \beta, \gamma \in \mathbb{R}$. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \alpha \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 3 \\ \beta \\ 4 \end{bmatrix} = \begin{bmatrix} 4+\beta \\ -3-4\alpha \\ \alpha\beta - 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ \gamma \end{bmatrix} \implies \beta = -2,\ \alpha = 1,\ \gamma = -5$$

Probably just as quick to solve this by hand as to type it into the CAS...

**Question 18 (C)**

What value of $k$, where k E R, will make the following planes perpendicular?
$\Pi_1 :\ 2x - ky + 3z = 1$
$\Pi_2 :\ 2kx + 3y - 2z = 4$

Two planes are perpendicular iff their normal vectors are perpendicular. So, require

$$0 = \begin{bmatrix} 2 \\ -k \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2k \\ 3 \\ -2 \end{bmatrix} = 4k - 3k - 6 \implies k = 6$$

**Question 19 (B)**

A company accountant knows that the amount owed on any individual unpaid invoice is normally distributed with a mean of \$800 and a standard deviation of \$200.
What is the probability, correct to three decimal places, that in a random sample of 16 unpaid invoices the total amount owed is more than \$13 500?

$U \sim \mathrm{N}(\mu = 800,\ \sigma = 200)$ be a random variable describing amount owed on unpaid invoices.
Total amount: $T = U_1 + U_2 + \ldots + U_{16} \sim \mathrm{N}(\mu = 16 \times 800,\ \sigma = 4 \times 200)$
$\Pr(T > 13500) \approx 0.1907869$

**Question 20 (A)**

The lifespan of a certain electronic component is normally distributed with a mean of $\mu$ hours and a standard deviation of $\sigma$ hours.

Given that a 99% confidence interval, based on a random sample of 100 such components, is $(10500, 15500)$, the value of $\sigma$ is closest to

$$(10500, 15500) = \left(\bar{x} - \frac{z_*\sigma}{\sqrt{n}}, \bar{x} + \frac{z_*\sigma}{\sqrt{n}}\right),\ z_* = 2.5758293,\ n = 100$$

$$15500 - 10500 = 5000 = 2\frac{z_*\sigma}{\sqrt{n}} = \frac{2 \times 2.5758}{10}\sigma \implies \sigma = \frac{5000 \times 5}{2.5758} \approx 9705.7$$

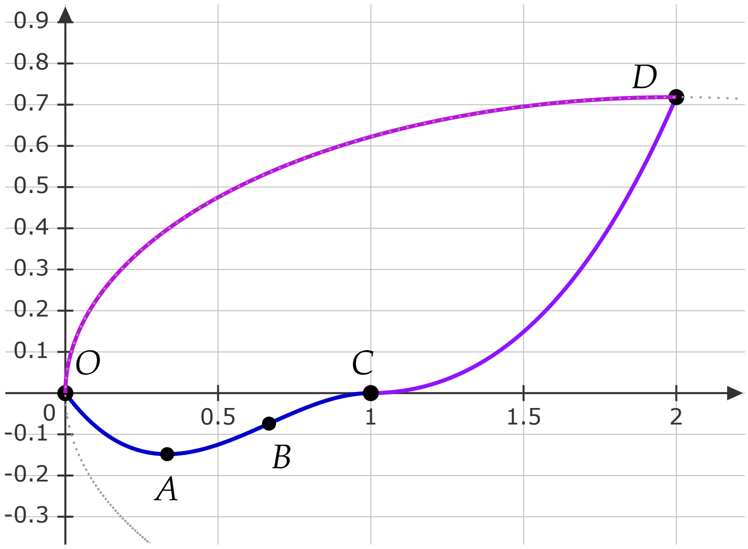
## Exam 2: Section B - ERQs

### Question 1 (10 marks)

Viewed from above, a scenic walking track from point $O$ to point $D$ is shown below.
Its shape is given by

$$f(x) = \begin{cases} -x(x+a)^2 & 0 \leqslant x \leqslant 1 \\ e^{x-1} - x + b & 1 < x \leqslant 2 \end{cases}$$

The minimum turning point of section $OABC$ occurs at point $A$. Point $B$ is a point of inflection and the curves meet at point $C(1, 0)$. Distances are measured in kilometres.




**a.** Show that $a = -1$ and $b = 0$.

At $C(1, 0)$:
$f(1) = -1(1+a)^2 = 0 \implies a = -1$
$e^{1-1} - 1 + b = 1 - 1 + b = b = 0 \implies b = 0$

**b.** Verify that the two curves meet smoothly at point $C$.

Let's assume "smoothly" means differentiable (and therefore continuous)[1].
Already constructed to be continuous ("the curves meet at point $C(1,0)$"),
so let's look at the derivative

$$\frac{d}{dx}\left(-x(x-1)^2\right) = -(x-1)^2 - 2x(x-1) = -(x-1)(3x-1) \overset{x=1}{=} 0$$

$$\frac{d}{dx}\left(e^{x-1} - x\right) = e^{x-1} - 1 \overset{x=1}{=} e^0 - 1 = 0$$

So, $f'(x)$ is well defined at $x = 1$ with $f'(1) = 0$

**c.i.** Find the coordinates of point $A$.

Find the turning point that occurs somewhere in $0 < x < 1$:

$$\frac{d}{dx}\left(-x(x-1)^2\right) = -(x-1)(3x-1) = 0 \implies x = 1,\ \frac{1}{3}$$

The minimum is at $\left(\frac{1}{3}, -\frac{4}{27}\right)$

**c.ii.** Find the coordinates of point $B$.

$$\frac{d^2}{dx^2}\left(-x(x-1)^2\right) = 4 - 6x \implies x = \frac{2}{3}$$

POI at $B\left(\frac{2}{3}, -\frac{2}{27}\right)$

The return track from point $D$ to point $O$ follows an elliptical path given by
$x = 2\cos t + 2,\ y = (e-2)\sin t$, where $t \in \left[\frac{\pi}{2}, \pi\right]$

**d.** Find the Cartesian equation of the elliptical path.

[1] Differentiable implies Continuous, which is fairly easy to prove. But first note that the contrapositive is (Not Continuous) implies (Not Differentiable), which means it is often useful to check/enforce continuity first, as if a function is not continuous then it can't be differentiable.

Proof: Assume differentiablility of the function $f$ over its whole domain:

I.e., $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$ is finite and well defined everywhere

Note the identity $f(x) - f(a) = (x-a)\frac{f(x) - f(a)}{x - a}$ for all $x \neq a$

Taking the limit as $x \to a$ of the identity, we have

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}\left((x-a)\frac{f(x) - f(a)}{x - a}\right) = \lim_{x \to a}(x-a)\lim_{x \to a}\frac{f(x) - f(a)}{x - a} = 0 \times f'(a)$$

Because $f'(a)$ is well defined everywhere, we have $\lim_{x \to a}(f(x) - f(a)) = 0$

$\Rightarrow f$ is continuous over its domain.

See https://teachingcalculus.com/2019/09/17/differentiability-implies-continuity/ for the standard proof and some good examples of continuous $\not\Rightarrow$ differentiable etc. .

$$\cos t = \frac{x-2}{2},\ \sin t = \frac{y}{e-2} \implies \left(\frac{x-2}{2}\right)^2 + \left(\frac{y}{e-2}\right)^2 = 1,\ x \in [0,\ 2]$$

**e.** Sketch the elliptical path from $D$ to $O$ on the diagram above. Done!

**f.i.** Write down a definite integral in terms oft that gives the length of the elliptical path from $D$ to $O$.

$$L = \int_{\pi/2}^{\pi} \sqrt{\dot{x}^2 + \dot{y}^2}dt = \int_{\pi/2}^{\pi} \sqrt{4\sin^2 t + (e-2)^2\cos^2 t}\, dt$$

An elliptic integral of the second kind...

**f.ii.** Find the length of the elliptical path from $D$ to $O$.
Give your answer in kilometres correct to three decimal places.

$2.2553474790366 \approx 2.255$

Seems reasonable, as a direct path would be approx $\sqrt{2^2 + 0.7^2} \approx 2.119$

**Question 2 (10 marks)**

Let $w = \text{cis}\left(\frac{2\pi}{7}\right)$

**a.** Verify that $w$ is a root of $z^7 - 1 = 0$

Arghh... you have roots of polynomials and solutions to equations! Anyway,

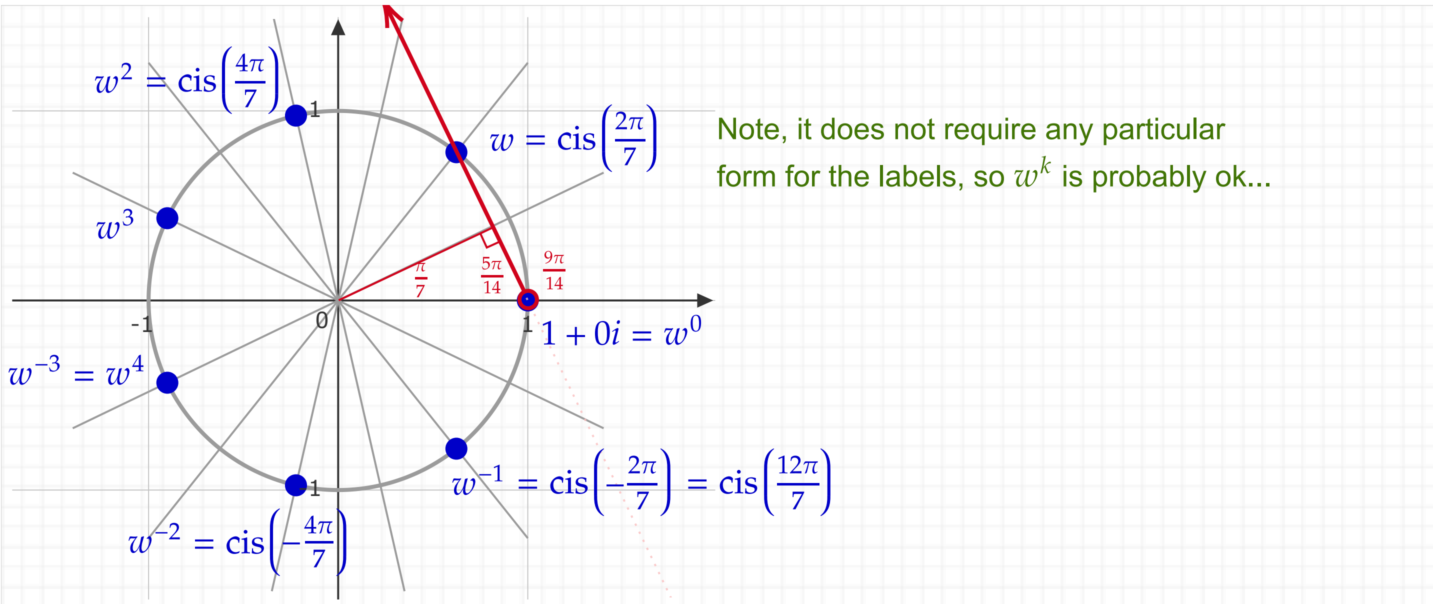
$$\text{cis}\left(\frac{2\pi}{7}\right)^7 = \text{cis}\left(7 \times \frac{2\pi}{7}\right) = \text{cis}(2\pi) = 1$$

**b.** List the other roots of $z^7 - 1 = 0$ in polar form

$$z^7 = 1 \implies z = \text{cis}\left(\frac{2\pi}{7}\right),\ \text{cis}\left(\frac{4\pi}{7}\right),\ \text{cis}\left(\frac{6\pi}{7}\right),\ \text{cis}\left(\frac{8\pi}{7}\right),\ \text{cis}\left(\frac{10\pi}{7}\right),\ \text{cis}\left(\frac{12\pi}{7}\right),\ 1$$

(Didn't require principle argument etc...)

**c.** On the Argand diagram below, plot and label the points that represent all the roots of $z^7 - 1$




**d.i.** On the Argand diagram above, sketch the ray that originates at the real root of $z^7 - 1 = 0$ and passes through the point represented by $w = \text{cis}\left(\frac{2\pi}{7}\right)$.

**d.ii.** Find the equation of this ray in the form $\text{Arg}(z - z_0) = \theta$, where $z_0 \in \mathbb{C}$, and $\theta$ is measured in radians in terms of $\pi$. [awkward and unnecessary phrasing]

$$z_0 = 1,\ \theta = \frac{\pi}{2} + \frac{\pi}{7} = \frac{9\pi}{14} \implies \text{Arg}(z-1) = \frac{9\pi}{14}$$

$$[\text{Arg}(x + iy - 1) = \frac{9\pi}{14} \implies \arctan\left(\frac{y}{x-1}\right) = \frac{9\pi}{14} \implies y = \tan\left(\frac{9\pi}{14}\right)(x-1)]$$

**e.** Verify that the equation $z^7 - 1 = 0$ can be expressed in the form $(z-1)\left(z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right) = 0$

This is just a finite geometric series - a standard result.
Could use long division to take out the factor of $z - 1$ from $z^7 - 1$...
but instead, let's just expand the factorised form:

$$\begin{aligned}
&(z-1)\left(z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right) \\
&= z\left(z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right) - \left(z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right) \\
&= \left(z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z^1\right) - \left(z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right) \\
&= z^7 - 1 = 0
\end{aligned}$$

**f. i.** Express $\operatorname{cis}\left(\frac{2\pi}{7}\right) + \operatorname{cis}\left(\frac{12\pi}{7}\right)$ in the form $A\cos(B\pi)$, where $A, B \in \mathbb{R}^+$.

$$\operatorname{cis}\left(\frac{2\pi}{7}\right) + \operatorname{cis}\left(\frac{12\pi}{7}\right) = \operatorname{cis}\left(\frac{2\pi}{7}\right) + \operatorname{cis}\left(-\frac{2\pi}{7}\right) = 2\cos\left(\frac{2}{7}\pi\right)$$

**f.ii** Given that $\operatorname{cis}\left(\frac{2\pi}{7}\right)$ satisfies $(z-1)\left(z^6 + z^5 + z^4 + z^3 + z^2 + z^1 + 1\right) = 0$,

use De Moivre's theorem to show that $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$

Note that $w = \operatorname{cis}\left(\frac{2\pi}{7}\right) \neq 1$,

so it must be (by the null factor theorem) that $w^6 + w^5 + w^4 + w^3 + w^2 + w^1 + 1 = 0$
Let's use $w^7 = 1$ to rewrite the terms as $w^{-1} + w^{-2} + w^{-3} + w^3 + w^2 + w^1 + 1 = 0$
and then pair them up as $\left(w^1 + w^{-1}\right) + \left(w^2 + w^{-2}\right) + \left(w^3 + w^{-3}\right) + 1 = 0$
De Moivre's theorem then gives

$$\begin{aligned}
&\left(\operatorname{cis}\left(\frac{2\pi}{7}\right) + \operatorname{cis}\left(-\frac{2\pi}{7}\right)\right) + \left(\operatorname{cis}\left(\frac{4\pi}{7}\right) + \operatorname{cis}\left(-\frac{4\pi}{7}\right)\right) + \left(\operatorname{cis}\left(\frac{6\pi}{7}\right) + \operatorname{cis}\left(-\frac{6\pi}{7}\right)\right) + 1 = 0 \\
&\Longrightarrow 2\cos\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{4\pi}{7}\right) + 2\cos\left(\frac{6\pi}{7}\right) + 1 = 0 \\
&\Longrightarrow \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}
\end{aligned}$$

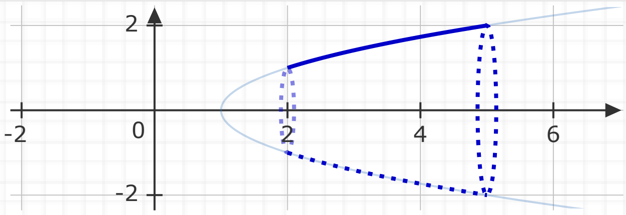
[Has this type of question been in a VCE exam before? I've seen similar in HSC Ext 2 papers, eg 2022 Q13c]

**Question 3 (10 marks)**

The curve given by $y^2 = x - 1$, where $2 \leqslant x \leqslant 5$, is rotated about the $x$-axis to form a solid of revolution. [Read as "the area under the curve.... is rotated... to form a solid]

**a.i.** Write down the definite integral, in terms of $x$, for the volume of this solid of revolution.

$$V = \int_2^5 \pi y^2 dx = \pi \int_2^5 (x-1)dx$$




**a.ii.** Find the volume of the solid of revolution.

$$V = \pi\left[\frac{1}{2}x^2 - x\right]_2^5 = \pi\left(\frac{25}{2} - 5 - 2 + 2\right) = \frac{15\pi}{2}$$

**b.i.** Express the curved surface area of the solid in the form $\pi \int_a^b \sqrt{Ax - B}\, dx$

where $a, b, A, B$ are all positive integers.

$$A = \int_2^5 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_2^5 y \sqrt{1 + \left(\frac{1}{2y}\right)^2} dx, \text{ nb: } 2y\frac{dy}{dx} = 1$$

$$= 2\pi \int_2^5 \sqrt{y^2 + \frac{1}{4}}\, dx = \pi \int_2^5 \sqrt{4y^2 + 1}\, dx$$

$$= \pi \int_2^5 \sqrt{4(x-1) + 1}\, dx$$

$$= \pi \int_2^5 \sqrt{4x - 3}\, dx$$

**b.ii.** Hence or otherwise, find the curved surface area of the solid correct to three decimal places.

$$A = \pi \int_2^5 \sqrt{4x - 3}\, dx = \pi\left[\frac{(4x-3)^{3/2}}{6}\right]_2^5 \approx 30.846489697 \approx 30.846$$

The total surface area of the solid consists of the curved surface area plus the areas of the two circular discs at each end.

The 'efficiency ratio' of a body is defined as its total surface area divided by the enclosed volume.

**c.** Find the efficiency ratio of the solid of revolution correct to two decimal places.

The end-caps at $x = 2$ and $x = 5$ have squared radii $y^2 = 1$ and $y^2 = 4$ respectively. So,

$$\frac{\text{TSA}}{V} = \frac{30.846 + 1\pi + 4\pi}{15\pi/2} \approx 1.9758323 \approx 1.98$$

**d.** Another solid of revolution is formed by rotating the curve given by $y^2 = x - 1$ about the $x$-axis for $2 \leqslant x \leqslant k$, where $k \in \mathbb{R}$. This solid has a volume of $24\pi$.
Find the efficiency ratio for this solid, giving your answer correct to two decimal places.

$V = 24\pi = \pi \int_2^k (x-1)dx \implies k = 8$ [reject option of $k = -6$ as sqrt graph is not defined for $x < 1$]

$$A = \pi \int_2^8 \sqrt{4x-3}\, dx \approx 75.916293006713$$

$$\frac{\text{TSA}}{V} = \frac{75.9163 + 1\pi + 7\pi}{24\pi} \approx 1.34020444 \approx 1.34$$

[Seems ok, note for growing similar (which these aren't) figures,
the efficiency ratio scales as $\frac{s^2}{s^3} = \frac{1}{s}$]

**Question 4 (10 marks)**

A fish farmer releases 200 fish into a pond that originally contained no fish.

The fish population, $P$, grows according to the logistic model, $\dfrac{dP}{dt} = P\left(1 - \dfrac{P}{1000}\right)$,

where $t$ is the time in years after the release of the 200 fish.

**a.** The above logistic differential equation can be expressed as

$\displaystyle\int \frac{A}{P} + \frac{B}{1 - P/1000}dP = \int dt$, where $A,\ B \in \mathbb{R}$

Find the values of $A$ and $B$

This is just a separation of variables then partial fractions

Separation of variables: $\displaystyle\int \frac{dP}{P\left(1 - \frac{P}{1000}\right)} = \int dt$

$$\int \frac{A}{P} + \frac{B}{1 - P/1000} = \int \frac{A(1 - P/1000) + BP}{P\left(1 - \frac{P}{1000}\right)}dP$$

$$\boxed{A = 1,\ B = \frac{1}{1000}}$$

One form of the solution for $P$ is $P = \dfrac{1000}{1 + D\,e^{-t}}$, where $D$ is a real constant.

**b.** Find the value of $D$.

This is fixed by the initial condition

$$P(0) = 200 = \frac{1000}{1 + D\,e^{0}}$$

$$\implies 1 + D = 5$$

$$\implies \boxed{D = 4}$$

The farmer releases a batch of $n$ fish into a second pond, pond 2, which originally contained no fish. The population, $Q$, of fish in pond 2 can be modelled by $Q = \dfrac{1000}{1 + 9e^{-1.1t}}$,

where $t$ is the time in years after the $n$ fish are released.

**c.** Find the value of $n$.

$$n = Q(0) = \frac{1000}{1 + 9} = 100$$

**d.** Find the value of $Q$ when $t = 6$. Give your answer correct to the nearest integer.

$$Q(6) = \frac{1000}{1 + 9e^{-6.6}} \approx 987.90477 \approx 988$$

**e.i.** Given that $\frac{dQ}{dt} = \frac{11}{10}Q\left(1 - \frac{Q}{1000}\right)$, express $\frac{d^2Q}{dt^2}$ in terms of Q

$$\frac{d^2Q}{dt^2} = \frac{d}{dt}\frac{dQ}{dt} = \frac{11}{10}\frac{d}{dt}\left(Q\left(1 - \frac{Q}{1000}\right)\right) = \frac{11}{10}\left[\frac{dQ}{dt}\left(1 - \frac{Q}{1000}\right) + Q\left(-\frac{dQ}{dt}\frac{1}{1000}\right)\right]$$
$$= \frac{11}{10}\frac{dQ}{dt}\left[1 - \frac{Q}{500}\right]$$
$$= \left(\frac{11}{10}\right)^2 Q\left(1 - \frac{Q}{1000}\right)\left(1 - \frac{Q}{500}\right)$$

**e.ii.** Hence or otherwise, find the size of the fish population in pond 2
and the value of $t$ when the rate of growth of the population is a maximum.
Give your answer for $t$ correct to the nearest year.

**Hence:** $\frac{d^2Q}{dt^2} = \left(\frac{11}{10}\right)^2 Q\left(1 - \frac{Q}{1000}\right)\left(1 - \frac{Q}{500}\right) = 0 \implies Q = 0, 1000, 500$

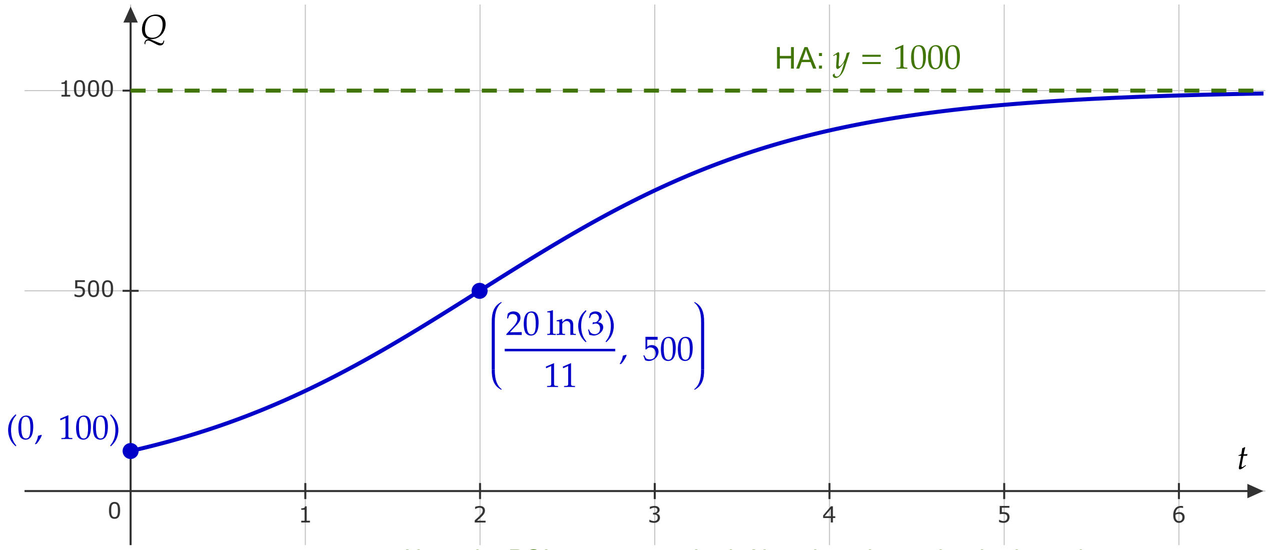
The $Q = 0$ and $Q = 1000$ cases occur as $t \to -\infty$ and $t \to \infty$ respectively.
This leaves the $Q = 500$ case as the max rate of growth of the population.

**Otherwise:** $\frac{dQ}{dt} = \frac{11}{10}Q\left(1 - \frac{Q}{1000}\right)$

This growth rate is a quadratic wrt $Q$, and the max rate occurs between the two zeros,
i.e., $Q = 500$. Sub into the DE's solution

$$Q = 500 = \frac{1000}{1 + 9e^{-1.1t}} \implies 1 + 9e^{-1.1t} = 2$$
$$\implies t = -\frac{1}{1.1}\ln\left(\frac{1}{9}\right) = \frac{10\ln(9)}{11} \approx 1.9974... \approx 2$$

**f.** Sketch the graph of $Q$ versus $t$ on the set of axes below. Label any axis intercepts
and any asymptotes with their equations




Note: the POI was not required. Also, there is another horizontal asymptote at $y = 0$, but that's not required given the domain of $Q$ is $t \geqslant 0$

The farmer wishes to take 5 .5% of the fish from pond 2 each year. The modified logistic differential equation that would model the fish population, $Q$, in pond 2 after $t$ years in this situation is

$$\frac{dQ}{dt} = \frac{11}{10}Q\left(1 - \frac{Q}{1000}\right) - 0.055Q$$

**g.** Find the maximum number of fish that could be supported in pond 2 in this situation.

$\frac{dQ}{dt} = 0 \implies Q = 0$ or $Q = 950$ -- the maximum number of fish would be 950

**Question 5 (11 marks)**

The points with coordinates $A(1,1,2)$, $B(1,2,3)$ and $C(3,2,4)$ all lie in a plane $\Pi$.

**a.** Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and hence show that the area of triangle $ABC$ is 1.5 square units.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{bmatrix}3\\2\\4\end{bmatrix} - \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix}$$

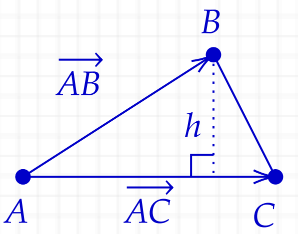
Area of $\triangle ABC$

$$\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left\|\begin{bmatrix}0\\1\\1\end{bmatrix} \times \begin{bmatrix}2\\1\\2\end{bmatrix}\right\| = \frac{1}{2}\left\|\begin{bmatrix}2-1\\2-0\\0-2\end{bmatrix}\right\| = \frac{1}{2}\left\|\begin{bmatrix}1\\2\\-2\end{bmatrix}\right\|$$

$$= \frac{1}{2}\sqrt{1^2+2^2+2^2} = \frac{1}{2}\sqrt{9}$$

$$= \frac{3}{2} = 1.5$$

**b.** Find the shortest distance from point B to the line segment AC.




**Option 1**: We already have the area of the triangle, and the required distance is just the altitude of $B$ in the triangle.

$$A = \frac{3}{2} = \frac{1}{2}|AC|\,h \implies h = \frac{3}{|AC|} = \frac{3}{\sqrt{2^2+1^2+2^2}} = 1$$

**Option 2**: Just use the formula that the (min) distance between a point and a line is the perpendicular projection of the vector between the point and any point on the line. E.g.

$$h = \left|\overrightarrow{AB} \times \widehat{AC}\right| = \frac{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}{\left|\overrightarrow{AC}\right|} = \frac{3}{3} = 1$$

A second plane, $\psi$, has the Cartesian equation $2x - 2y - z = -18$.

**c.** At what acute angle does the line given by $\mathbf{r}(t) = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, $t \in \mathbb{R}$, intersect the plane $\psi$? Give your answer in degrees correct to the nearest degree.

Normal to the plane: $\mathbf{n} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, direction vector of line: $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

Angle $\theta$ between normal and line is $\cos\theta = \hat{\mathbf{n}} \cdot \hat{\mathbf{v}} = \dfrac{2+4-2}{3 \times 3} = \dfrac{4}{9}$

Angle between plane and line is the complementary angle

$$\frac{\pi}{2} - \arccos\left(\frac{4}{9}\right) = \arcsin\left(\frac{4}{9}\right) \approx 26.3878° \approx 26°$$

A line $L$ passes through the origin and is normal to the plane $\psi$. $L$ intersects $\psi$ at a point $D$.

**d.** Write down an equation of the line L in parametric form.

$$L : \mathbf{r}(t) = \mathbf{0} + t\,\mathbf{n} = t(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \implies \begin{cases} x = 2t \\ y = -2t \\ z = -t \end{cases}$$

(other parameterisations are possible - this is just the simplest given the provided info)

**e.** Find the shortest distance from the origin to the plane $\psi$.

Can just use the formula: dist $= \dfrac{|d|}{|\mathbf{n}|} = \dfrac{|-18|}{|3|} = 6$ (where $\mathbf{n} \cdot \mathbf{r} = ax + by + cz = d$)
This follows from a simple vector resolution argument.

**f.** Find the coordinates of point $D$.

$D$ is the closes point on the plane to the origin (because $L$ is normal to $\psi$)
Can find its coordinates by substituting its equation into the planes:
$2(2t) - 2(-2t) - (-t) = -18 \implies t = -2 \implies D = (-4, 4, 2)$
Check: $|\overrightarrow{OD}| = \sqrt{4^2 + 4^2 + 2^2} = \sqrt{36} = 6$ ✓

Could also get this by moving 6 units along $\hat{\mathbf{n}}$ starting at $O$:
$6\,\widehat{-\mathbf{n}} = -2\,\mathbf{n} = -2(2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = -4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$
Note, need to be careful you move in the right direction and end up on the plane
$-2\mathbf{n} \cdot \mathbf{n} = -18$

**Question 6 (9 marks)**

A forest ranger wishes to investigate the mass of adult male koalas in a Victorian forest. A random sample of 20 such koalas has a sample mean of 11.39 kg.
It is known that the mass of adult male koalas in the forest is normally distributed with a standard deviation of 1 kg.

**a.** Find a 95% confidence interval for the population mean (the mean mass of all adult male koalas in the forest). Give your values correct to two decimal places.

$$\overline{x} = 11.39,\ n = 20,\ \sigma_X = 1 \Rightarrow\ ME = z_* \sigma_{\widehat{X}} a \approx 1.959964 \times \frac{1}{\sqrt{20}} \approx 0.43826$$

$CI:\ (\overline{x} - ME,\ \overline{x} + ME) \approx (10.95,\ 11.83)$

Can use the Confidence Interval function on the CAS

`zInterval 1,11.39,20,0.95: stat.results` → $(10.95,\ 11.83)$

**b.** Sixty such random samples are taken and their confidence intervals are calculated.
In how many of these confidence intervals would the actual mean mass of all adult male koalas in the forest be expected to lie?

A 95% confidence interval says that if 100 such were calculated, the population mean would lie with in it 95% of the time.
So, 95% of 60 = 57

The ranger wants to decrease the width of the 95% confidence interval by 60% to get a better estimate of the population mean.

**c.** How many adult male koalas should be sampled to achieve this?

The CI scales as $\dfrac{1}{\sqrt{n}}$.

So need $\dfrac{1}{\sqrt{n_1}} \div \dfrac{1}{\sqrt{20}} = 1 - 0.60 = 0.4 \Longrightarrow n_1 = 125$

I.e. 125 adult male koalas should be sampled to reduce CI width by 60% cf with original sample of only 20 koalas.

It is thought that the mean mass of adult male koalas in the forest is 12 kg. The ranger thinks that the true mean mass is less than this and decides to apply a one-tailed statistical test. A random sample of 40 adult male koalas is taken and the sample mean is found to be 11.6 kg.

**d.** Write down the null hypothesis, $H_0$, and the alternative hypothesis, $H_1$, for the test.

$H_0:\ \mu = 12$
$H_1:\ \mu < 12$
(both measured in kg and assuming normal distribution with $\sigma = 1$)

The ranger decides to apply the one-tailed test at the 1% level of significance and assumes the mass of adult male koalas in the forest is normally distributed with a mean of 12 kg and a standard deviation of 1 kg.

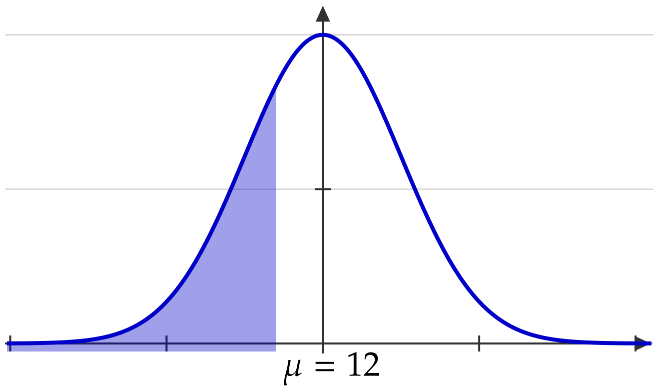
**e.i.** Find the $p$ value for the test correct to four decimal places.

$p = \Pr(\overline{X} \leqslant 11.6 \mid H_0) \approx 0.0057$ (`normCdf(-∞,11.6,12,1/sqrt(40))`)

**e.ii.** Draw a conclusion about the null hypothesis in part **d.** from the $p$ value found above, giving a reason for your conclusion.

The value of $p = 0.0057 < 1\%$, so we can reject the null hypothesis

**f.** What is the critical sample mean (the smallest sample mean for $H_0$ **not** to be rejected) in this test? Give your answer in kilograms correct to three decimal places.




Need to find $x_*$ s.t. $0.01 = \Pr(\overline{X} \leqslant x_* \mid H_0)$

$\implies x_* \approx 11.63217 \approx 11.633$

[`invNorm(0.01,12, 1/sqrt(40))`]

Note: Have to round up to have $x_*$ **not** to not reject $H_0$

Suppose that the true mean mass of adult male koalas in the forest is 11.4 kg, and the standard deviation is 1 kg. The level of significance of the test is still 1%.

**g.** What is the probability, correct to three decimal places, of the ranger making a type II error in the statistical test?

$\Pr(\overline{X} \geqslant x_* \mid H_1) \approx 0.0702918 \approx 0.070$

if you use the rounded answer from part **f**

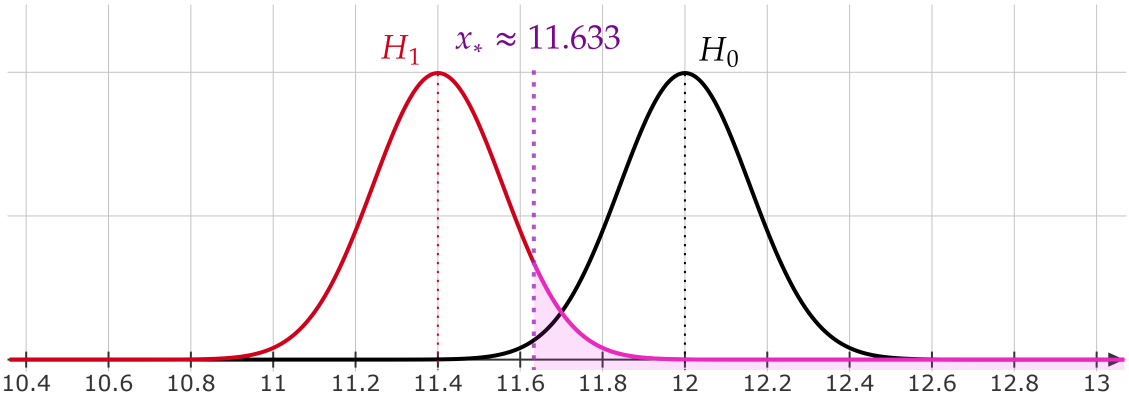
[`normCdf(11.633,∞,11.4,1/sqrt(40) →0.070291814185603`]

If you use the full precision from part **f**, you get a slightly different answer

$\Pr(\overline{X} \geqslant x_* \mid H_1) \approx 0.0709998 \approx 0.071$

[`normCdf(11.63217210392,∞,11.4,1/sqrt(40) → 0.070999830180144`]

**h.** The frequency curves for the sampling distributions associated with $H_0$ and $H_1$ are shown below. [Note: On the exam, $H_0$ and $H_1$ were labelled the wrong way!]




Label the critical sample mean on the diagram and shade the region that represents the type II error.... and done :)