SPECIALIST MATHEMATICS 2023

Unit 3

Key Topic Test 13 – Differentiation Applications
Technology Free

Recommended writing time*: 45 minutes
Total number of marks available: 30 marks

SOLUTIONS

Question 1

a. $f(x) = \ln(x^2 + 4)$
$f'(x) = \frac{2x}{x^2+4}$

1 mark

b. $f'(x) = 0 \rightarrow x = 0$
$(0, \ln(4))$
$f''(x) = \frac{2(x^2+4)-2x(2x)}{(x^2+4)^2} = \frac{-2x^2+8}{(x^2+4)^2}$
$f''(0) = \frac{8}{16} > 0$
Hence (0, ln(4)) is a point of local minimum.

3 marks

c. $f''(x) = 0 \rightarrow 2x^2 = 8$
$x = \pm 2$
$(2, \ln(8)) \quad and \quad (-2, \ln(8))$

2 marks

Question 2

$\sin^2(x) + \cos^2(y) = \frac{x}{y}$

$2\sin(x)\cos(x) + 2\cos(y)\ (-\sin(y))\frac{dy}{dx} = \frac{y - x\frac{dy}{dx}}{y^2}$

Sub $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

$2 \times \frac{1}{2} - 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}\frac{dy}{dx} = \frac{\frac{\pi}{3} - \frac{\pi}{4}\frac{dy}{dx}}{\frac{\pi^2}{9}}$

$1 - \frac{\sqrt{3}}{2}\frac{dy}{dx} = \frac{3}{\pi} - \frac{9}{4\pi}\frac{dy}{dx}$

$\frac{dy}{dx}\left(\frac{9}{4\pi} - \frac{\sqrt{3}}{2}\right) = \frac{3}{\pi} - 1$

$\frac{dy}{dx}\left(\frac{9}{4\pi} - \frac{2\sqrt{3}\pi}{4\pi}\right) = \frac{3}{\pi} - \frac{\pi}{\pi}$

$\frac{dy}{dx}\left(\frac{9 - 2\sqrt{3}\pi}{4\pi}\right) = \frac{3 - \pi}{\pi}$

$\frac{dy}{dx} = \frac{4\pi(3 - \pi)}{\pi(9 - 2\sqrt{3}\pi)}$

$m_T = \frac{12 - 4\pi}{9 - 2\sqrt{3}\pi}$

$m_N = \frac{9 - 2\sqrt{3}\pi}{12 - 4\pi}$

4 marks

Question 3

$2x^2y + 3x = 2y$

$y = 1 \rightarrow 2x^2 + 3x - 2 = 0 \rightarrow (2x-1)(x+2) = 0 \rightarrow x = \frac{1}{2}, -2$

$\left(\frac{1}{2},\ 1\right)$ or $(-2, 1)$

$4xy + 2x^2\frac{dy}{dx} + 3 = 2\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-3-4xy}{2x^2-2}$

$m_T = \frac{10}{3}$ or $\frac{5}{6}$

Equation of tangent at $\left(\frac{1}{2}, 1\right)$ *is*

$y - 1 = \frac{10}{3}\left(x - \frac{1}{2}\right) \rightarrow y = \frac{10}{3}x - \frac{2}{3}$

Equation of tangent at $(-2,1)$ *is*

$y - 1 = \frac{5}{6}(x+2) \rightarrow y = \frac{5}{6}x + \frac{8}{3}$

Solve $y = \frac{10}{3}x - \frac{2}{3}$ and $y = \frac{5}{6}x + \frac{8}{3}$ simultaneously

$\frac{5}{6}x + \frac{8}{3} = \frac{10}{3}x - \frac{2}{3} \rightarrow x = \frac{4}{3}$

Point of intersection $\left(\frac{4}{3}, \frac{34}{9}\right)$

5 marks

Question 4

a. $V = Area \times depth$

$\frac{a}{2} \times \pi r^2 = \pi r^2 \times 2$

$\frac{a}{2} = 2 \rightarrow a = 4$

1 mark

b. $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$\frac{dr}{dt} = \frac{1}{4\pi r} \times 9.5$

$\frac{dr}{dt} = \frac{1}{4\pi \times 25} \times 9.5 = \frac{19}{2000\pi}$ mm/minute

2 marks

c. $\frac{dV}{dr} = 4\pi r$

1 mark

Question 5

a. $f(x) = x^2e^x$

$f'(x) = 2xe^x + x^2e^x = e^x(2x + x^2)$

$f'(x) = 0 \rightarrow 2x + x^2 = 0 \rightarrow x = 0,\ x = -2$

Stationary points are $(0, 0)$ *and* $(-2,\ 4e^{-2})$

2 marks

b. $f''(x) = e^x(2x + x^2 + 2 + 2x) = e^x(x^2 + 4x + 2)$
At $(0, 0)$, $f''(x) = 2 > 0$
Hence (0, 0) is a point of local minimum.
At $(-2, 4e^{-2})$, $f''(x) = e^{-2}(4 - 8 + 2) = -\frac{2}{e^2} < 0$
Hence $(-2,\ 4e^{-2})$ is a point of local maximum.
$f(x)$ is strictly increasing on $(-\infty, -2) \cup (0, \infty)$

3 marks

c. $f''(x) = e^x(x^2 + 4x + 2)$
$f''(x) = 0 \rightarrow x^2 + 4x + 2 = 0$
$x = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2}$
Inflection points are
$\left(-2 + \sqrt{2}, (6 - 4\sqrt{2})e^{-2+\sqrt{2}}\right)$ and $\left(-2 - \sqrt{2}, (6 + 4\sqrt{2})e^{-2-\sqrt{2}}\right)$

3 marks

Question 6

$f(x) = \sin^{-1}\left(\frac{4}{x}\right)$
$f'(x) = \frac{1}{\sqrt{1 - \frac{16}{x^2}}} \times -\frac{4}{x^2}$
$f'(x) = \frac{-4}{x\sqrt{x^2 - 16}}$, $x^2 - 16 > 0 \rightarrow\ x < -4$ or $x > 4$
$f'(x) < 0$ when $x > 0$
Combined with the domain restriction this means that
$f'(x) < 0$ when $x > 4$

3 marks