The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2023

Trial Written Examination 2 - SOLUTIONS

SECTION A – Multiple-choice questions

ANSWERS

1	2	3	4	5	6	7	8	9	10
A	D	B	B	A	D	C	C	D	E

11	12	13	14	15	16	17	18	19	20
B	D	E	A	B	A	B	C	C	A

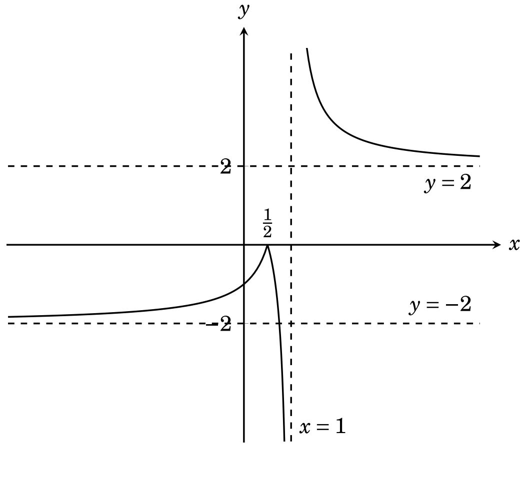
SOLUTIONS

Question 1 **Answer is A**

If $x \geq \frac{1}{2}$ then $\frac{|2x-1|}{x-1} = \frac{2x-1}{x-1} = 2 + \frac{1}{x-1}$

If $x < \frac{1}{2}$ then $\frac{|2x-1|}{x-1} = \frac{1-2x}{x-1} = -2 - \frac{1}{x-1}$

The graph of $y = \frac{|2x-1|}{x-1}$ has straight line asymptotes $x = 1$, $y = 2$ and $y = -2$.



Question 2 **Answer is D**

Note that

$$-1 \le x^2 \le 1$$
$$\Rightarrow 0 \le x^2 \le 2$$
$$\Rightarrow -\sqrt{2} \le x \le \sqrt{2}$$

When $x = 0$, $x^2 - 1 = -1$ and $\arctan\left(x^2 - 1\right) + \frac{\pi}{2} = 0$.

Therefore, the implied domain of $f(x) = \dfrac{1}{\arctan\left(x^2 - 1\right) + \frac{\pi}{2}}$ is $\left[-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right]$

Question 3 **Answer is B**

Using a double angle formula,

$$\tan(2x) = -1$$
$$\Rightarrow \frac{2\tan(x)}{1 - \tan^2(x)} = -1$$

Solving for $\tan(x)$ gives $\tan(x) = 1 \pm \sqrt{2}$.

But $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and so $\tan(x) > 0$.

Therefore, $\tan(x) = 1 + \sqrt{2}$.

Question 4 **Answer is B**

The converse of a statement $P \Rightarrow Q$ is $Q \Rightarrow P$. Therefore, the converse of the statement

If n is divisible by 4 then n is divisible by 2

is

If n is divisible by 2 then n is divisible by 4

Question 5 Answer is A

In the inductive step, we assume that

$$1^2+2^2+3^2+\ldots+k^2>\frac{k^3}{3}$$

and deduce that

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2>\frac{(k+1)^3}{3}$$

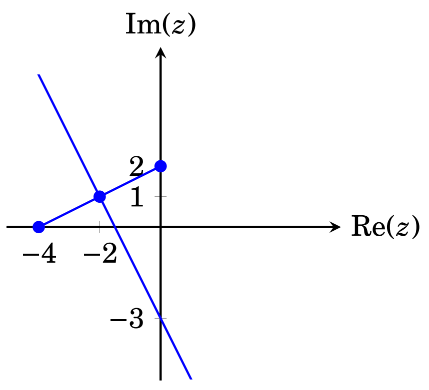
Question 6 Answer is D

Graphically: The gradient of the line segment joining the points $(-4,0)$ and $(0,2)$ is $\frac{1}{2}$. The gradient of the line perpendicular to this line segment is -2.

The midpoint of the line segment joining $(-4,0)$ and $(0,2)$ is $(-2,1)$.
Therefore, the equation of the perpendicular bisector of the line segment joining $(-4,0)$ and $(0,2)$ is

$$\begin{aligned} y &= -2(x+2)+1 \\ &= -2x-3 \end{aligned}$$

It is often useful to draw a quick diagram:



Algebraically: Let $z=x+yi$. Then

$$\begin{aligned} &|z+4|=|z-2i| \\ &\Rightarrow |(x+4)+yi|=|x+(y-2)i| \\ &\Rightarrow \sqrt{(x+4)^2+y^2}=\sqrt{x^2+(y-2)^2} \\ &\Rightarrow x^2+8x+16+y^2=x^2+y^2-4y+4 \\ &\Rightarrow 4y=-8x-12 \\ &\Rightarrow y=-2x-3 \end{aligned}$$

Question 7 **Answer is C**

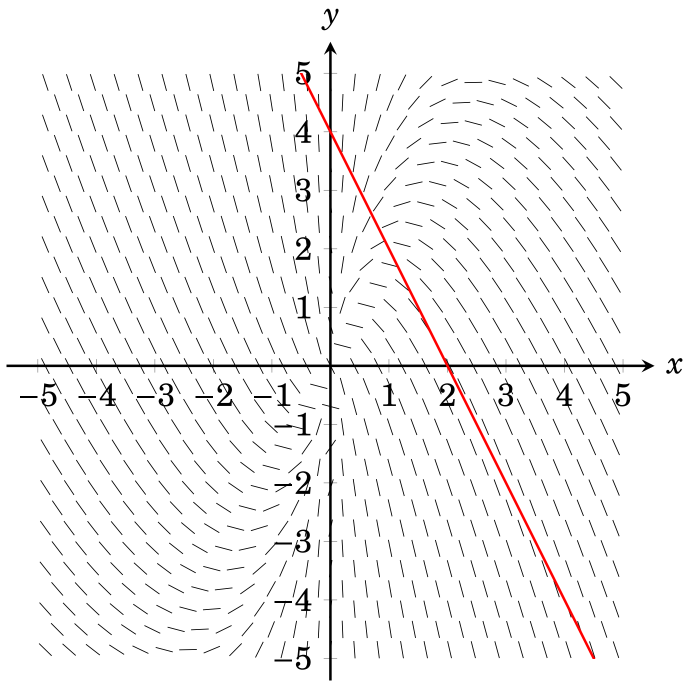
Let $z = x + iy$. Then

$$\begin{aligned}(z+4i)(\overline{z}-4i) &= 16 \\ \Rightarrow (x+(y+4)i)(x-(y+4)i) &= 16 \\ \Rightarrow x^2+(y+4)^2 &= 16\end{aligned}$$

The set of points $(z+4i)(\overline{z}-4i)=16$ is a circle of radius 4 in the complex plane, centered at $(0,-4)$.

Question 8 **Answer is C**

The tangent to the solution curve at $(2,0)$ must pass through the point $(0,4)$ (for example). Only option **C** has this property:



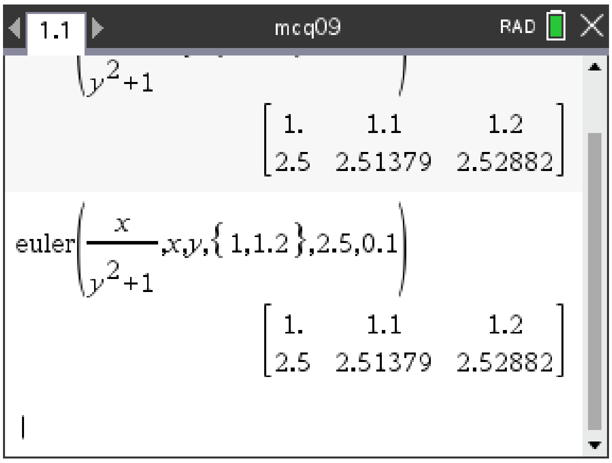
Question 9 Answer is D

Use a table to keep track of the variables:

i	x	y	f
0	1	2.5	0.1379
1	1.1	2.5138	0.1503
2	1.2	2.5288	0.1623

Note that the pseudocode is describing an application of Euler's method to solve the differential equation $\frac{dy}{dx} = \frac{x}{y^2+1}$, $y(1) = 2.5$ with a step size of 0.1.

The result could be found using CAS:



The output when `i=2` is `(1.2,2.5288)`

Question 10 Answer is E

From the vector equation of the line we have

$$x = 4 + 2t$$
$$y = -1 - 2t$$
$$z = 5 + 3t$$

Substituting these equations into the Cartesian equation of the plane gives

$$2(4+2t) - 3(-1-2t) + 4(5+3t) = 20.$$

Solving this equation for t gives $t = -\frac{1}{2}$.

Then, $\underset{\sim}{r}\left(-\frac{1}{2}\right) = 3\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{k}$ and the point of intersection of the line and the plane is $\left(3, 0, \frac{7}{2}\right)$.

Question 11 Answer is B

The parametric equations of the line are

$$x = 1 + 3t$$
$$y = -2 + 5t$$
$$z = -1 - 2t$$

and so a vector normal (perpendicular) to the plane is $\underset{\sim}{n} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$.
The equation of the plane is $3x + 5y - 2z = d$. To find d, substitute the point $P(1,1,-3)$ into the plane equation:

$$3(1) + 5(1) - 2(-3) = 14.$$

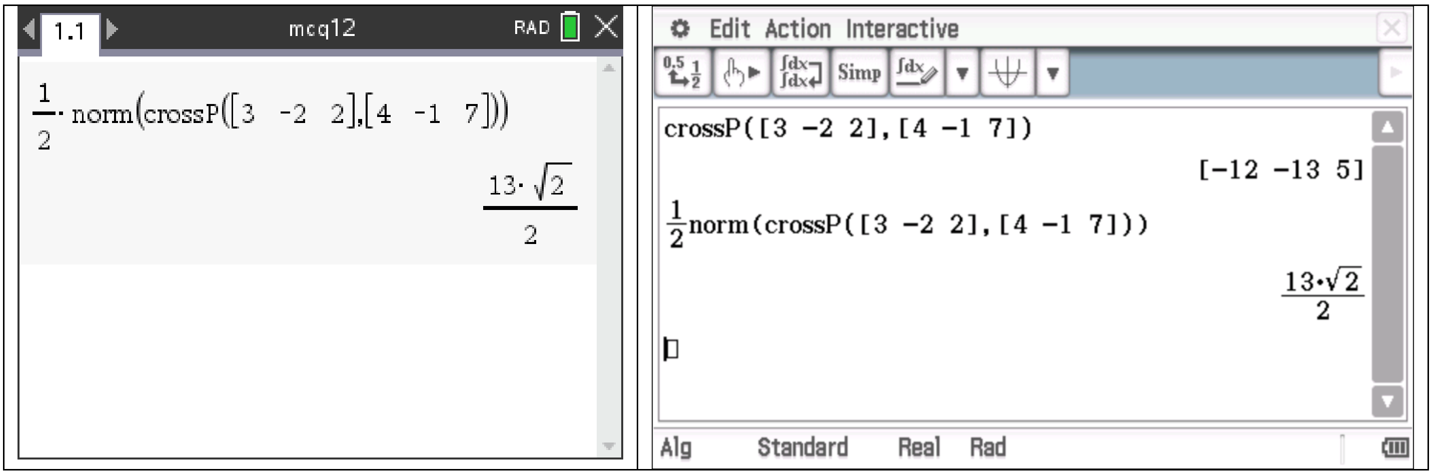
The equation of the plane is $3x + 5y - 2z = 14$.

Question 12 Answer is D

The area can be found using the cross product:

$$\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{13}{\sqrt{2}}$$

Use CAS to find this:



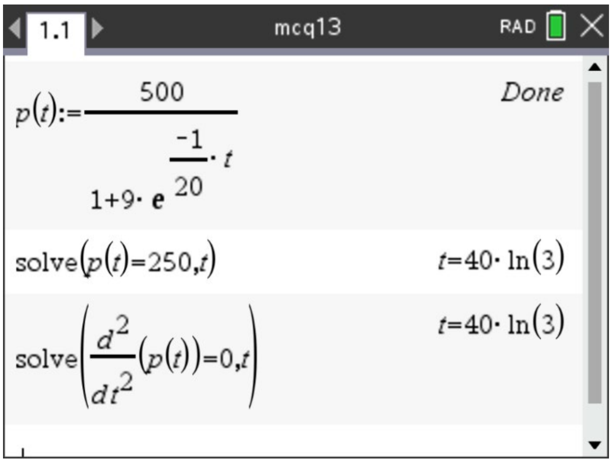
Question 13 **Answer is E**

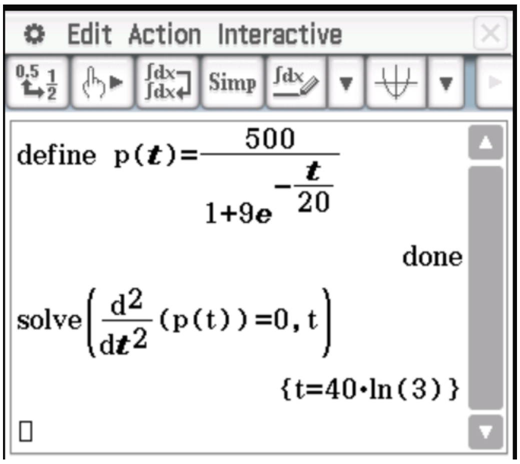
The asymptote is $P = 500$ and so $a = 500$. Furthermore, $P(0) = \dfrac{500}{1+b} = 50$ and so $b = 9$.

The rate of change of P is greatest when $P = 250$ (halfway between $P = 0$ and $P = 500$).

Solving $\dfrac{500}{1+9e^{-\frac{1}{20}t}} = 250$ gives $t = 20\log_e(9)$.

Alternatively, solve $\dfrac{d^2P}{dt^2} = 0$ to obtain the same result.





Question 14 **Answer is A**

This can be done by hand:

$$\frac{dx}{dt} = 2\sqrt{2}\sin(t) + 2\sqrt{2}t\cos(t)$$

$$\frac{dy}{dt} = 2\sqrt{2}\cos(t) - 2\sqrt{2}t\sin(t)$$

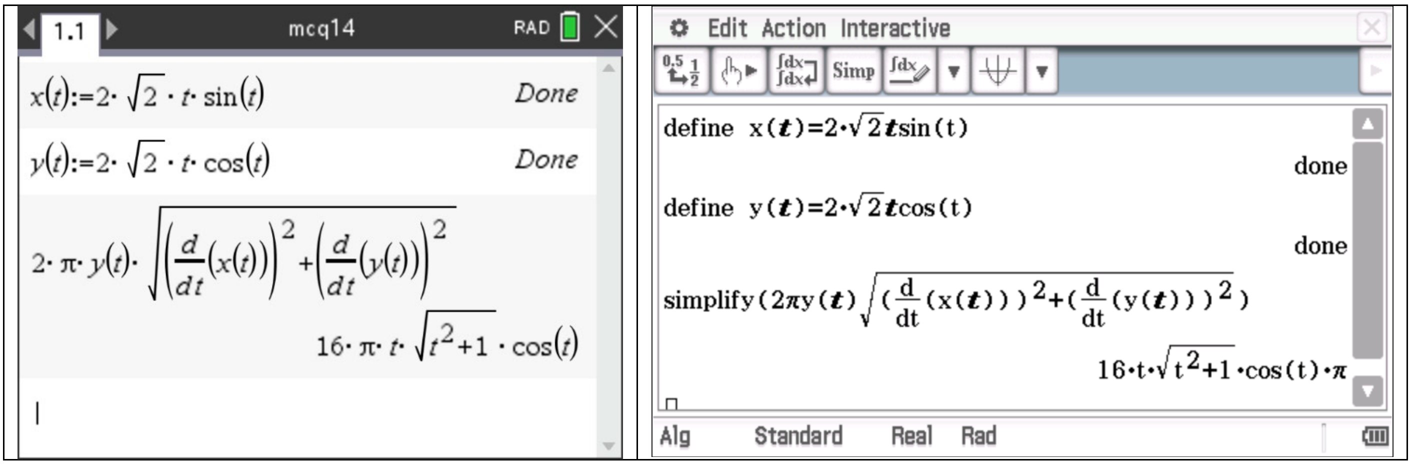
Then

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 8\sin^2(t) + 16t\sin(t)\cos(t) + 8t^2\cos^2(t) \\ &\quad + 8\cos^2(t) - 16t\sin(t)\cos(t) + 8t^2\sin^2(t) \\ &= 8\left(\sin^2(t) + \cos^2(t)\right) + 8t^2\left(\sin^2(t) + \cos^2(t)\right) \\ &= 8 + 8t^2\end{aligned}$$

From the formula sheet, the surface area generated by rotating the graph about the x-axis is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\pi \cdot 2\sqrt{2}t\cos(t)\sqrt{8+8t^2}\,dt = 16\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t\sqrt{1+t^2}\cos(t)dt$$

The integrand can also be found using CAS:



Question 15 Answer is B

Let $u = x^5$ and $\frac{dv}{dx} = \sin(3x)$

Then $\frac{du}{dx} = 5x^4$ and $v = -\frac{1}{3}\cos(3x)$.

Then by integration by parts:

$$\int x^5 \sin(3x)dx = -\frac{1}{3}x^5\cos(3x) + \frac{5}{3}\int x^4\cos(3x)dx$$

Question 16 Answer is A

Use the acceleration equivalent formula $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \sqrt{x}$$

$$\Rightarrow \frac{1}{v^2} = \int \sqrt{x}\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}} + c$$

When $x = 1$, $v = 2$ and so

$$2 = \frac{2}{3} + c \Rightarrow c = 2 - \frac{2}{3} = \frac{4}{3}$$

When $x = 16$:

$$\frac{1}{2}v^2 = \frac{2}{3}(16)^{\frac{3}{2}} + \frac{4}{3}$$

$$= \frac{2}{3}\times 4^3 + \frac{4}{3}$$

$$= \frac{132}{3} = 44$$

Therefore $v^2 = 88$ and the speed of the body is $|v| = \sqrt{88} = 2\sqrt{22}$.

Question 17 **Answer is B**

Differentiate to find

$$\dot{\underset{\sim}{r}}(t) = 6t\underset{\sim}{i} + 2e^{2t}\underset{\sim}{j} - 3\underset{\sim}{k}$$
$$\Rightarrow \dot{\underset{\sim}{r}}(0) = 2\underset{\sim}{j} - 3\underset{\sim}{k}$$

and so $\left|\dot{\underset{\sim}{r}}(0)\right| = \sqrt{4+9} = \sqrt{13}$

Question 18 **Answer is C**

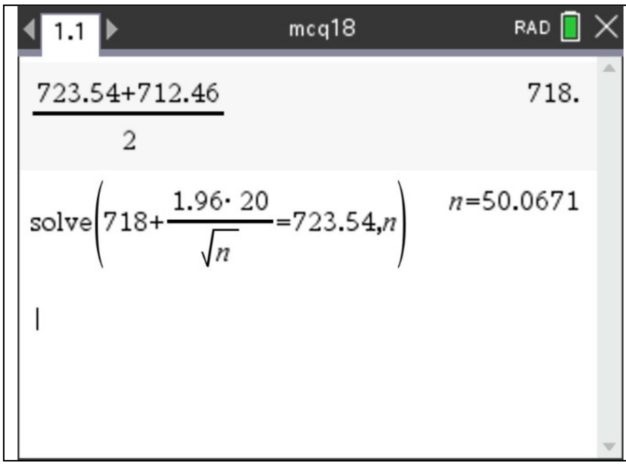
The sample mean is

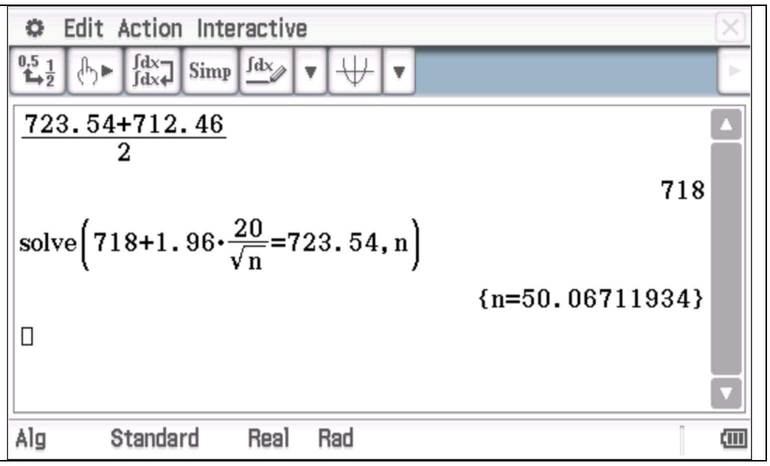
$$\bar{x} = \frac{723.54 + 712.46}{2} = 718$$

For a 95% confidence interval, $z \approx 1.96$ and so

$$718 + 1.96 \times \frac{20}{\sqrt{n}} = 723.54$$

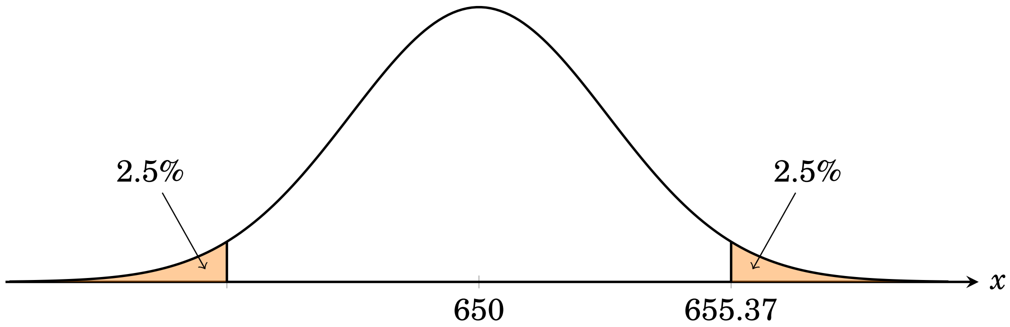
This gives $n = 50$ (correct to the nearest integer).





Question 19 **Answer is C**

For a two-sided test at the 5% significance level, the area in each of the tails is 0.025:



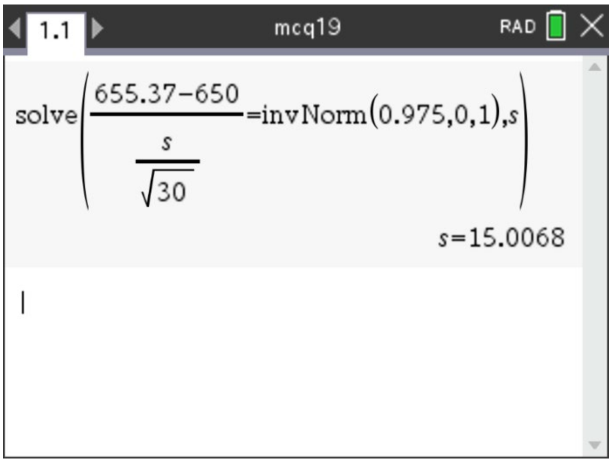
Therefore

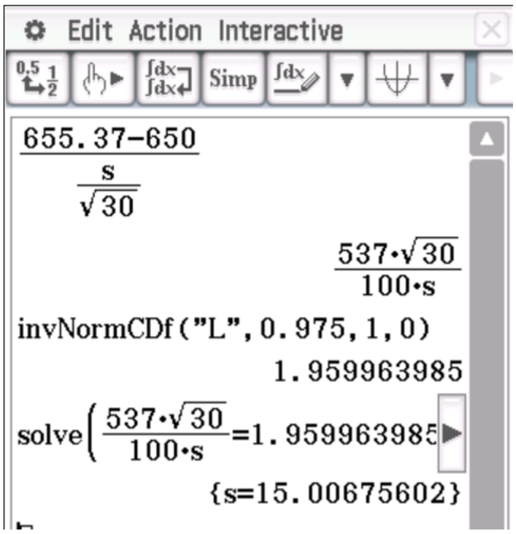
$$\Pr\left(\bar{X} < 655.37\right) = 0.975$$

$$\Rightarrow \Pr\left(Z < \frac{655.37 - 650}{\frac{\sigma}{\sqrt{30}}}\right) = 0.975$$

$$\Rightarrow \frac{655.37 - 650}{\frac{\sigma}{\sqrt{30}}} = 1.96$$

$$\Rightarrow \sigma \approx 15$$





Question 20 **Answer is A**

Let $A_i \sim \mathrm{N}\left(80, 5^2\right)$, $i = 1, 2, 3$ and $M_j \sim \mathrm{N}\left(60, 8^2\right)$, $j = 1, 2, 3, 4$ be the random variables representing the masses of apples and mandarins respectively.

Let X be the normal random variable that represents the masses of four mandarins minus the masses of three apples:

$X = M_1 + M_2 + M_3 + M_4 - \left(A_1 + A_2 + A_3\right)$.

Then

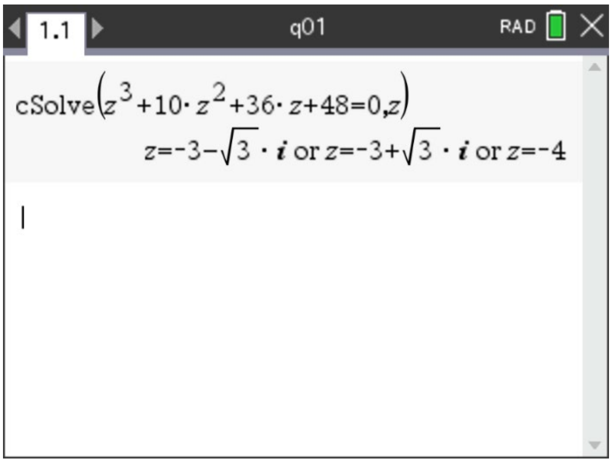
$$\begin{aligned} \mathrm{E}(X) &= 4 \times 60 - 3 \times 80 = 0 \\ \mathrm{Var}(X) &= 4 \times 64 + 3 \times 25 = 331 \\ \mathrm{sd}(X) &= \sqrt{331} \end{aligned}$$

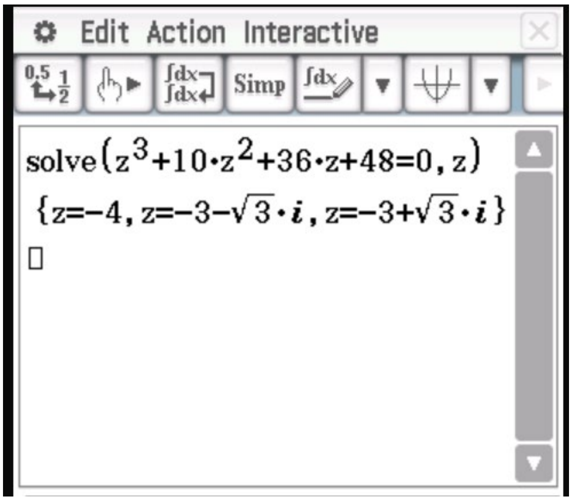
SECTION B

Question 1

a.

The other solutions are $z = -4$ and $z = -3 - \sqrt{3}i$. Use the CAS command `cSolve`:





[A1]

b.

As $z = -3 + \sqrt{3}i$ and $z = -4$ all lie on the circumference of a circle $|z - a| = r$ where $a \in R$, then

$$\left|-3 + \sqrt{3}i - a\right| = r$$

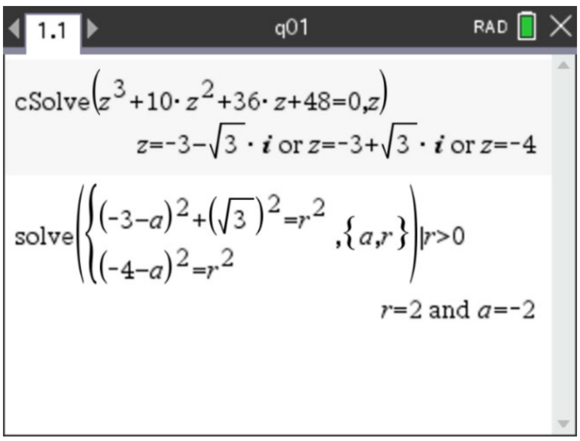
$$\left|-4 - a\right| = r$$ **[M1]**

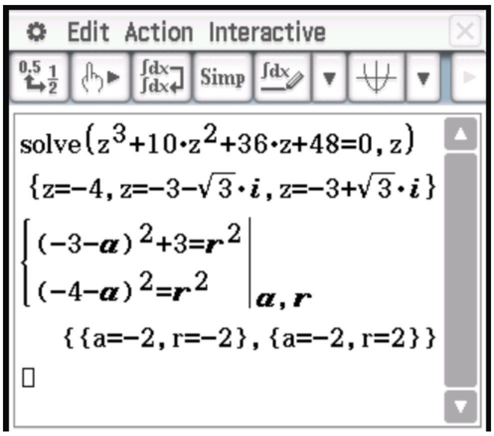
and so

$$(3 + a)^2 + 3 = r^2$$

$$(4 + a)^2 = r^2$$

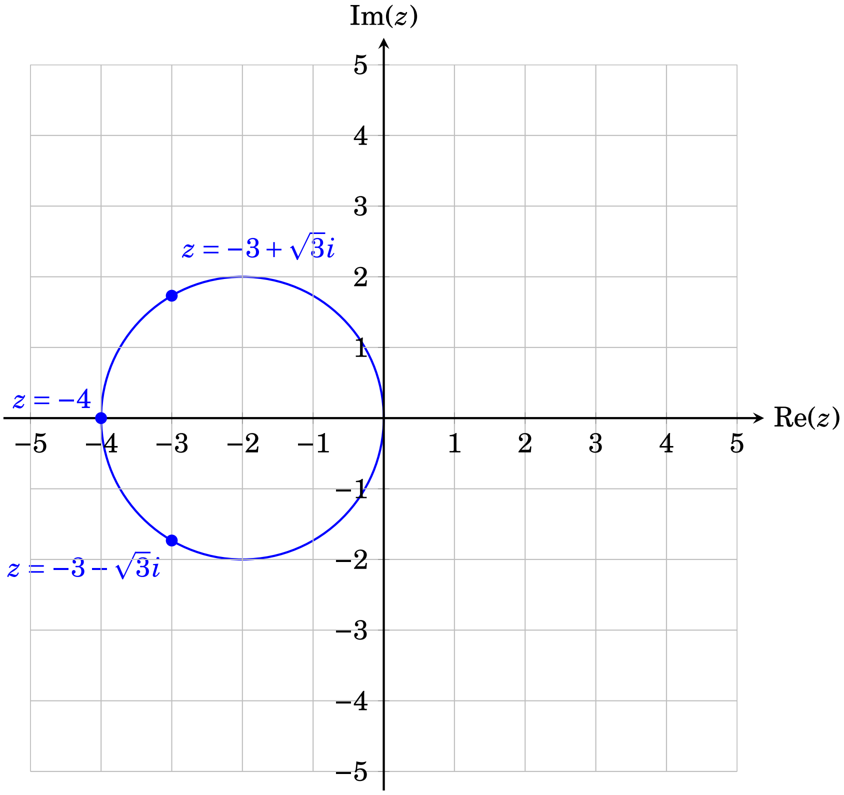
Solving gives $a = -2$ and $r = 2$. **[A1]**





c.

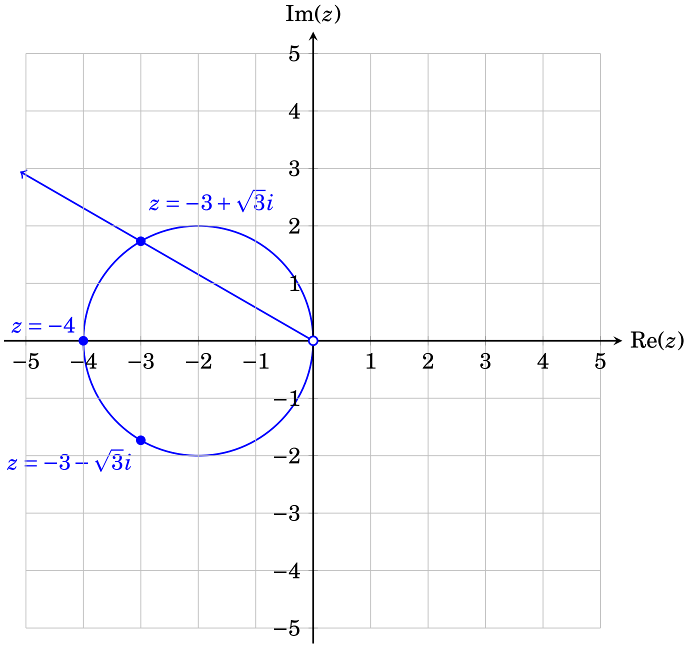
The circle is shown below:



[A1 - circle]
[A1 - points]

d.

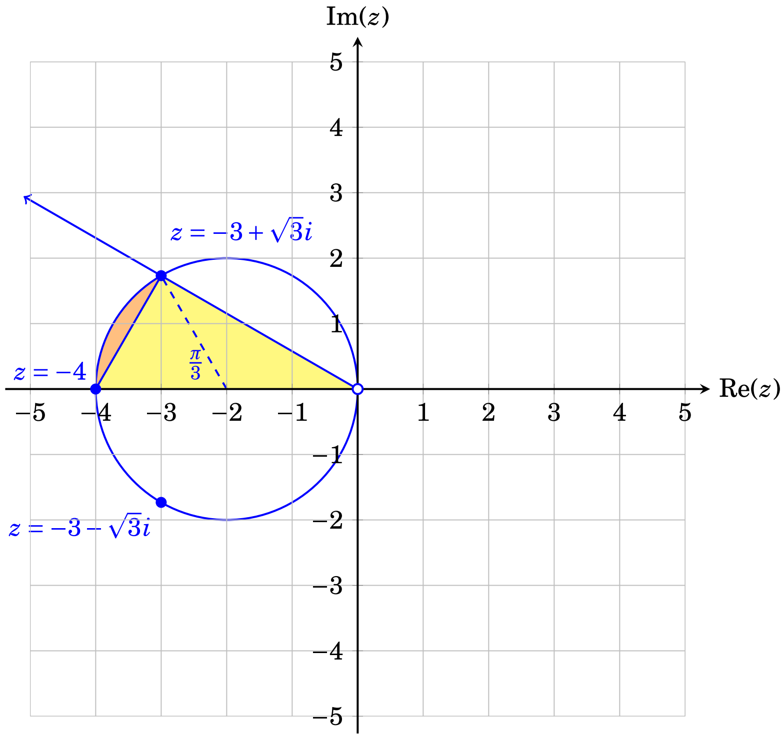
$\text{Arg}(z) = \dfrac{5\pi}{6}$. The ray is shown on the diagram from **c.** below:



[A1]

e.

The area required is shown on the diagram below:



The area can be found (for example) by considering the area of the triangle with vertices at the origin and at the points $z=-4$ and $z=-3+\sqrt{3}i$, plus the area of the minor segment as shown above.

$$A=\frac{1}{2}\cdot 4\cdot\sqrt{3}+\frac{1}{2}\cdot 2^2\left(\frac{\pi}{3}-\sin\left(\frac{\pi}{3}\right)\right)$$
$$=2\sqrt{3}+\frac{2\pi}{3}-2\cdot\frac{\sqrt{3}}{2}$$
$$=\frac{2\pi}{3}+\sqrt{3}$$

[M1 – formula]
[A1]

Question 2

a.

The line l_1 has the same direction as the normal vector of the plane Π_1 and passes through the point $A(3,2,-1)$. Therefore a vector equation of the line is

$\underset{\sim}{r}(t) = 3\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k} + \left(3\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)t$ **[A1]**

b.

We have

$\overrightarrow{DE} = -5\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$

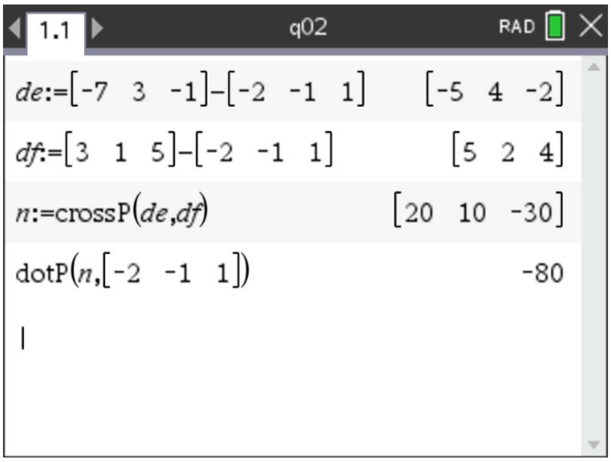
$\overrightarrow{DF} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

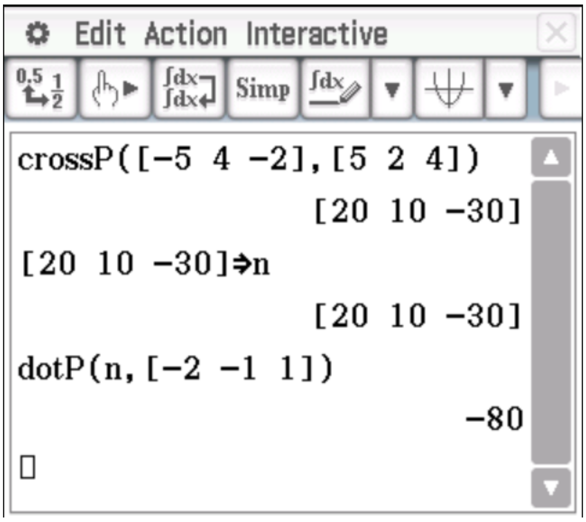
and so

$\overrightarrow{DE} \times \overrightarrow{DF} = 20\underset{\sim}{i} + 10\underset{\sim}{j} - 30\underset{\sim}{k} = 10\left(2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\right).$ **[M1]**

A vector normal to the plane Π_2 is $2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ and a Cartesian equation of the plane is $2x + y - 3z = d$.

Substituting $D(-2,-1,1)$ (for example) gives $2x + y - 3z = -8$.





Therefore, a Cartesian equation for plane Π_2 is $2x + y - 3z = -8$.

(Other equivalent answers are acceptable) **[A1]**

c. i.

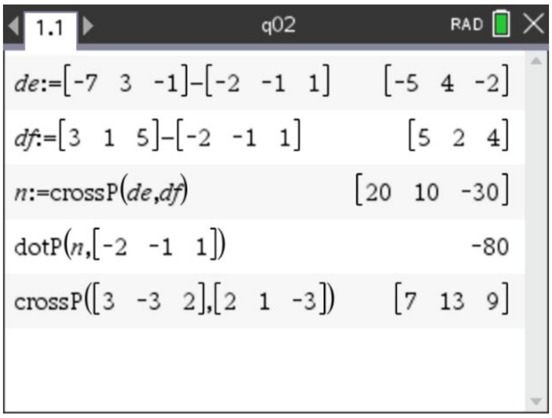
Substitute the point $P(4,-1,5)$ into the Cartesian equation for the planes Π_1 and Π_2 and confirm that the equation is satisfied:

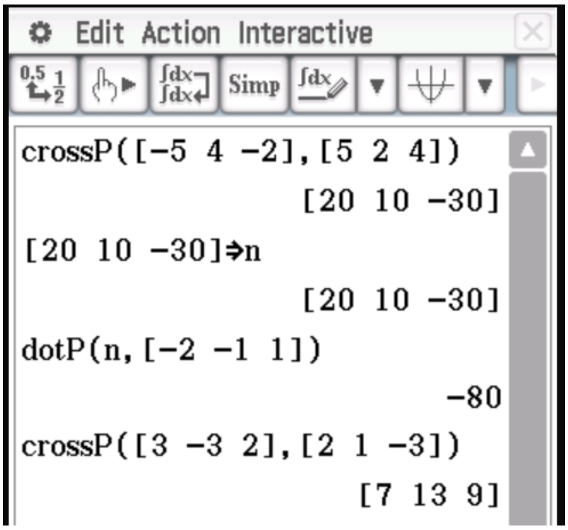
$3(4) - 3(-1) + 2(5) = 25$ and $2(4) + (-1) - 3(5) = -8$ **[A1* – shown]**

c. ii.

The line contained in both Π_1 and Π_2 is perpendicular to the normal vectors to each of the planes. That is, the line l_2 is parallel to

$$\left(3\underset{\sim}{i}-3\underset{\sim}{j}+2\underset{\sim}{k}\right)\times\left(2\underset{\sim}{i}+\underset{\sim}{j}-3\underset{\sim}{k}\right)=7\underset{\sim}{i}+13\underset{\sim}{j}+9\underset{\sim}{k}$$ **[M1]**





The line passes through the point $(4,-1,5)$ and so a Cartesian equation for the line is

$$\frac{x-4}{7}=\frac{y+1}{13}=\frac{z-5}{9}$$ **[A1]**

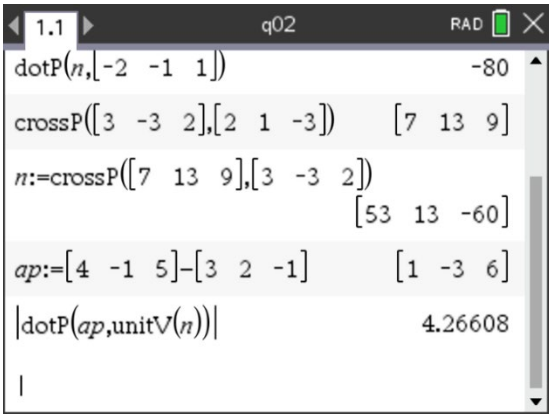
d.

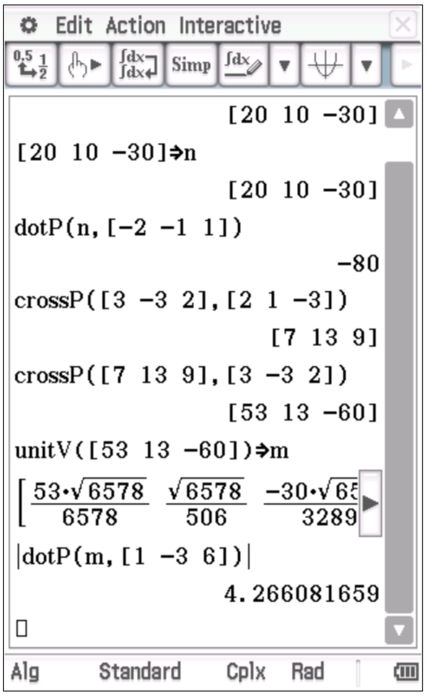
Let $\underset{\sim}{n}$ be a vector perpendicular to both lines:

$$\underset{\sim}{n}=\left(7\underset{\sim}{i}+13\underset{\sim}{j}+9\underset{\sim}{k}\right)\times\left(3\underset{\sim}{i}-3\underset{\sim}{j}+2\underset{\sim}{k}\right)$$
$$=53\underset{\sim}{i}+13\underset{\sim}{j}-60\underset{\sim}{k}$$ **[M1]**

$A(3,2,-1)$ lies on l_1 and $P(4,-1,5)$ lies on l_2. Then $\overrightarrow{AP}=\underset{\sim}{i}-3\underset{\sim}{j}+6\underset{\sim}{k}$. **[H1]**

The distance between the lines is $\left|\overrightarrow{AP}\cdot\hat{\underset{\sim}{n}}\right|=4.27$ **[A1]**





Question 3

a.

The terminal velocity occurs when the acceleration is zero. Therefore

$$g-\frac{5}{4}v=0$$

$$\Rightarrow v=\frac{4g}{5}=7.84$$ **[A1]**

b.

Since $\dfrac{dv}{dt}=g-\dfrac{5}{4}v$,

$$\int \frac{dv}{g-\frac{5}{4}v}=\int dt$$

$$\Rightarrow -\frac{4}{5}\log_e\left(g-\frac{5}{4}v\right)=t+c$$ **[M1]**

$$\Rightarrow g-\frac{5}{4}v=Ae^{-\frac{5}{4}t}$$

When $t=0$, $v=0$ and so $A=g$. Therefore

$$\frac{5}{4}v=g\left(1-e^{-\frac{5}{4}t}\right)$$

$$\Rightarrow v=\frac{4g}{5}\left(1-e^{-\frac{5}{4}t}\right)$$

[A1]

c.

Since $\frac{dv}{dt} = g - \frac{5}{4}v$, the time taken for the parachutist to reach a speed of 7.5 ms^{-1} is

$$\int_0^{7.5} \frac{dv}{g - \frac{5}{4}v} \approx 2.51$$ **[A1]**

d.

Find when $v = \frac{3g}{5}$:

$$\frac{3g}{5} = \frac{4g}{5}\left(1 - e^{-\frac{5}{4}t}\right)$$

$$\Rightarrow \frac{3}{4} = 1 - e^{-\frac{5}{4}t}$$

$$\Rightarrow -\frac{5}{4}t = \log_e\left(\frac{1}{4}\right)$$

$$\Rightarrow t = \frac{4}{5}\log_e(4)$$

We have **[M1]**

$$\frac{dx}{dt} = \frac{4g}{5}\left(1 - e^{-\frac{5}{4}t}\right)$$

and so the distance the parachutist falls is

$$x = \int_0^{\frac{4}{5}\log_e(4)} \frac{4g}{5}\left(1 - e^{-\frac{5}{4}t}\right)dt \approx 3.99$$ **[A1]**

The parachutist falls 3.99 m, correct to two decimal places.

Alternatively, note that

$$v\frac{dv}{dx} = g - \frac{5}{4}v$$

[M1]

$$\Rightarrow x = \int_0^{\frac{3g}{5}} \frac{v}{g - \frac{5}{4}v}dv \approx 3.99$$

[A1]

e.

The parachutist reaches the ground at time t_1, where

$$\int_0^{t_1} \frac{4g}{5}\left(1-e^{-\frac{5}{4}t}\right)dt = 1500.$$

By CAS, $t_1 \approx 192.127$ seconds. **[A1]**

The helicopter is initially at rest and reaches a speed of 50 ms^{-1} in 30 seconds. Using the constant acceleration formula $v = u + at$ we have

$$50 = 30a$$
$$\Rightarrow a = \frac{5}{3}\ \text{ms}^{-2}$$

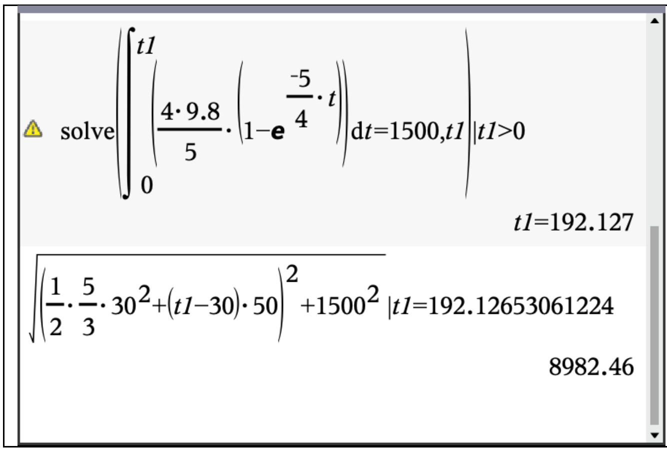
The distance travelled by the helicopter in the time between $t = 0$ and $t = t_1$ is

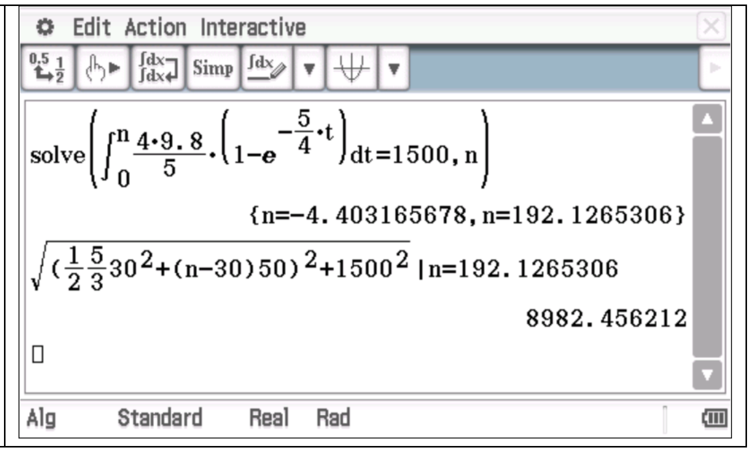
$$\frac{1}{2}\cdot\frac{5}{3}\cdot 30^2 + (t_1 - 30)\cdot 50 \approx 8856.33\ \text{m}$$ **[H1]**

The distance between the helicopter and the parachutist at the moment when she reaches the ground is

$$\sqrt{8856.33^2 + 1500^2} \approx 8982.46\ \text{m}$$

Therefore, the distance is 8982 m, to the nearest metre. **[A1]**





Question 4

a. i.

Logistic equation: $\frac{dx}{dt} = kx\left(1-\frac{x}{K}\right)$.

k is the growth parameter.

By inspection of $\frac{dP}{dt} = 0.15P\left(1-\frac{P}{30}\right)$: $k = 0.15$.

Answer: 0.15. **[A1]**

a. ii.

Logistic equation: $\frac{dx}{dt} = kx\left(1-\frac{x}{K}\right)$.

K is the carrying capacity (the sustainable number that can be sustained by the environment).

By inspection of $\frac{dP}{dt} = 0.15P\left(1-\frac{P}{30}\right)$: $K = 30$.

Answer: 30, 000. **[A1]**

Note: Must multiply by 1,000 because P is measured in units of thousands.

a. iii.

- Maximum rate of change occurs at the point of inflection of the logistic curve.

Method 1:

- Therefore $P = \frac{K}{2} = 15$.

Method 2:

$P = P$-coordinate of turning point of $\frac{dP}{dt} = 0.15P\left(1-\frac{P}{30}\right)$.

Answer: 15,000. **[A1]**

a. iv.

- The year $1872 \Rightarrow t = 22$.
- Population $28,500 \Rightarrow P = 28.5$.
- The year $1850 \Rightarrow t = 0$.
- Use a CAS to solve $\dfrac{dP}{dt} = 0.15P\left(1 - \dfrac{P}{30}\right)$ subject to $P(22) = 28.5$:

$$P = \frac{570}{19 + e^{-\frac{3}{20}(t-22)}}.$$

Note: Forms of solution (including forms using decimal approximations) are possible.

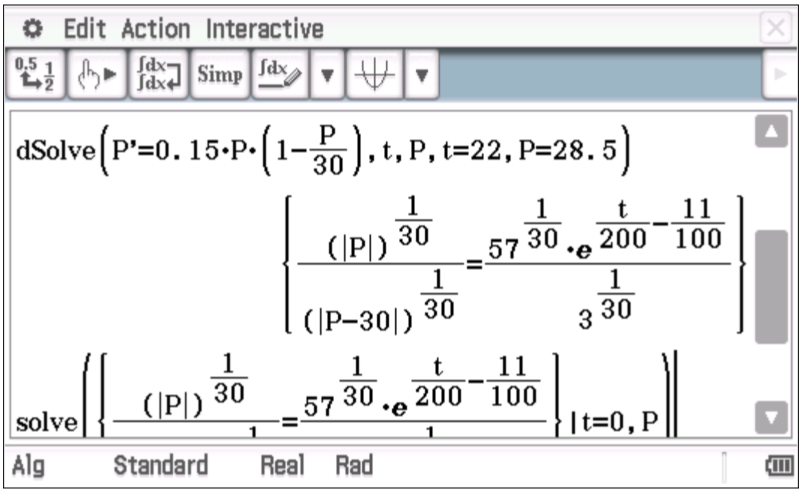
- Substitute $t = 0$ and solve for P:

$P = 12.361$ (correct to three decimal places).

Answer: 12,361. **[A1]**

Note on using the ClassPad:

The ClassPad does not directly give the solution for P in terms of t:

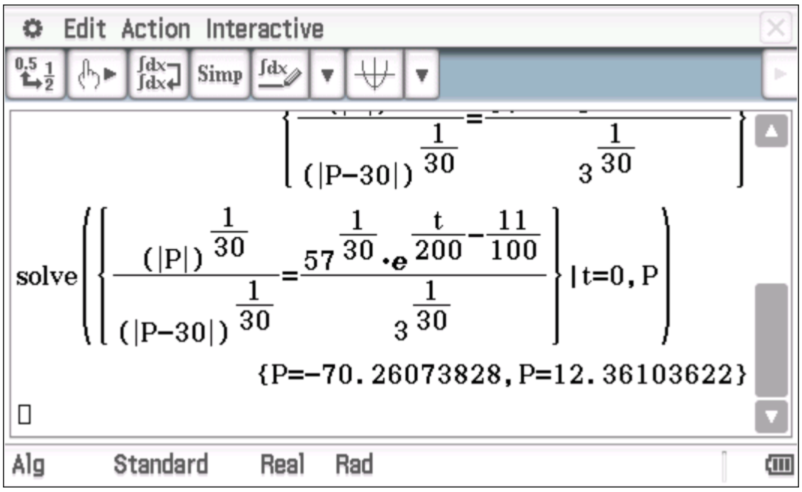


Option 1: Substitute $t = 0$ and solve for P using an appropriate restriction on P:

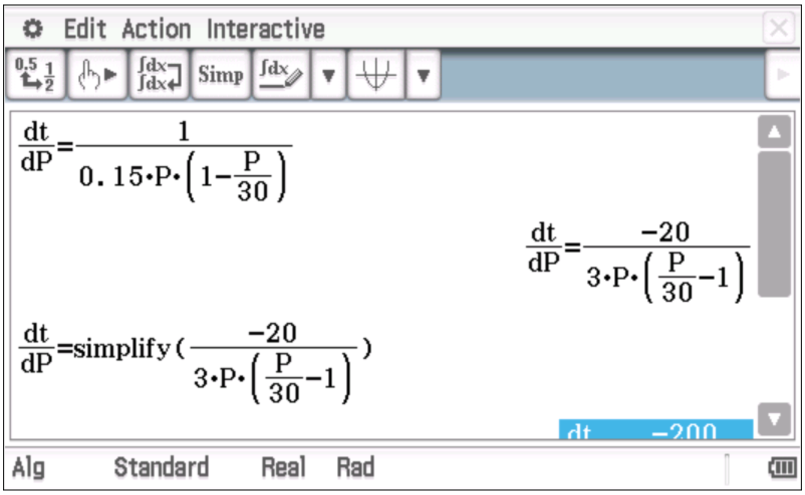
$$\text{solve}\left(\frac{(|P|)^{\frac{1}{30}}}{(|P-30|)^{\frac{1}{30}}} = \frac{57^{\frac{1}{30}} \cdot e^{\frac{0}{200} - \frac{11}{100}}}{3^{\frac{1}{30}}}, P, 0, 0, 100\right)$$

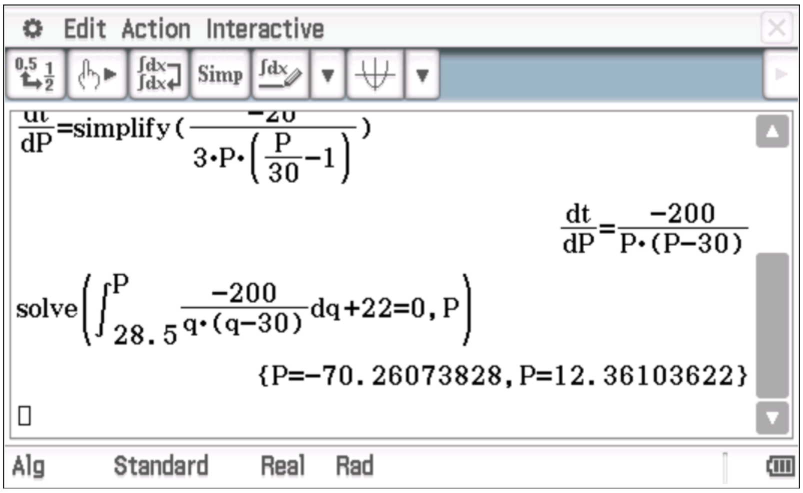
$$\{P = 12.36103622\}$$

Option 2: Substitute $t = 0$ and solve for P:



Option 3: Use the integral solution:





$P = 12.361$ is chosen since $P > 0$.

b. i.

Answer: $\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - n$. **[A1]**

b. ii.

Use a CAS to solve $\frac{dP}{dt} = 0 = 0.15P\left(1 - \frac{P}{30}\right) - n$ for P:

Case 1: $P = 5\left(3 + \sqrt{9 - 8n}\right)$. **[M1]**

$5\left(3 + \sqrt{9 - 8n}\right) = 18 \quad \Rightarrow n = 1.08$.

Case 2: $P = 5\left(3 - \sqrt{9 - 8n}\right)$.

$5\left(3 - \sqrt{9 - 8n}\right) = 18$ has no solution.

Answer: $n = 1.08$. **[A1]**

Note:

- Under the '1850 – 1900' model:

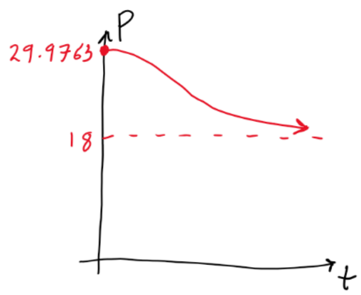
The year $1900 \Rightarrow t = 50$ and $P(50) = 29.976$ (correct to three decimal places).

Therefore:

- $5\left(3 + \sqrt{9 - 8n}\right) = 29.976 \quad \Rightarrow n = 0.004$ (correct to three decimal places)

corresponds to the population remaining constant.

- **Case 1** ($n = 1.08$) corresponds to P decreasing from 29.976 towards an equilibrium value of 18:



- The minimum equilibrium value of P is 15 and occurs when $9 - 8n = 0 \Rightarrow n = \frac{9}{8}$.

- If $n > \frac{9}{8}$ the population will decrease to zero.

- If $0 \le n < 0.004$ the population will increase to an equilibrium value.

The maximum equilibrium value of P is 30 and occurs when $n = 0$.

b. iii.

- Use a CAS to solve $\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - 1.290$ subject to $P(0) = 29.976$:

$P = 15 - \sqrt{33t}\,\text{an}\left(\frac{\sqrt{33}t}{200} - \arctan\left(\frac{624\sqrt{3}}{125\sqrt{11}}\right)\right)$.

Note: Other forms of solution (including forms using decimal approximations) are possible.

- Use the above solution to solve $P = 12$ for t:

$t = 58.7$ (correct to one decimal place) which corresponds to during the year1958.

Answer: 1958. **[A1]**

Note:

Calculation of the estimate given in the question of the polar bear population in 1900:

Under the '1850 – 1900' model:

The year $1900 \Rightarrow t = 50$ and $P(50) = 29.976$ (correct to three decimal places).

Question from the cutting room floor:

b. iv. Find the largest value of n for which polar bears will not become extinct. Give your answer in thousands per year. 2 marks

Solution:

From **part b i.**: $\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - n$, $n \geq 0$.

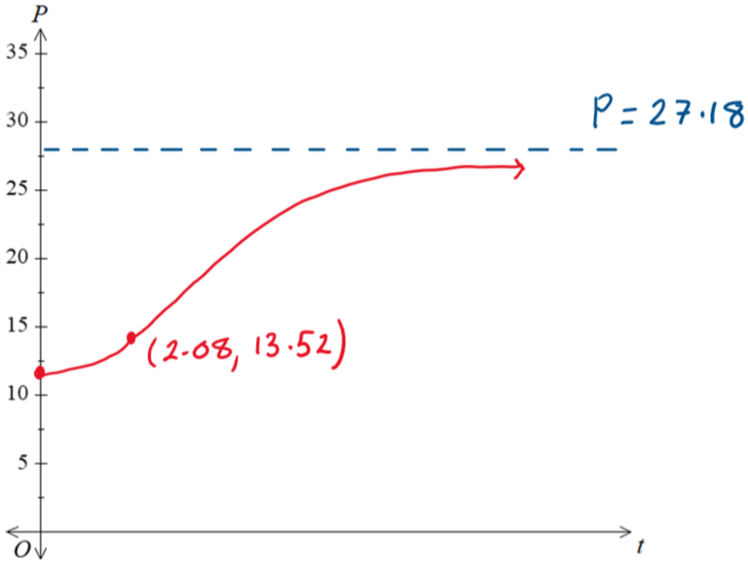
It is required that $\frac{dP}{dt} \geq 0$ for $P > 0$.

$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - n$ has a maximum turning point at $P = 15$.

When $P = 15$: $\frac{dP}{dt} = 0.15(15)\left(1 - \frac{15}{30}\right) - n \quad = 1.125 - n$.

$1.125 - n \geq 0 \quad \Rightarrow n \leq 1.125$. **Answer:** $n = 1.125$.

c. Answer



Shape: **[A1]**

P-intercept must be consistent with $P(0) = 11.720$.

Horizontal asymptote: $P = 27.18$. **[A1]**

***P*-coordinate of point of inflection:** $P = 13.52$. **[A1]**

***t*-coordinate of point of inflection:** $t = 2.08$. **[H1]**

Calculations:

- Horizontal asymptote:

Solve $\frac{dP}{dt} = 0 = 0.15P\left(1 - \frac{P}{30}\right) - 0.2\log_e\left(\frac{P}{4}\right)$ to get the equilibrium value of P.

From a CAS: $P = 27.18$ (correct to two decimal places).

- P-coordinate of point of inflection:

Solve $\frac{d^2P}{dt^2} = 0 \quad \Rightarrow \frac{d}{dP}\left(0.15P\left(1 - \frac{P}{30}\right) - 0.2\log_e\left(\frac{P}{4}\right)\right) = 0$.

Note: $\frac{d^2P}{dt^2} = \frac{d}{dt}\left(\frac{dP}{dt}\right) = \underbrace{\frac{d}{dP}\left(\frac{dP}{dt}\right) \times \frac{dP}{dt}}_{\text{Chain Rule}}$.

From a CAS: $P = 13.52$ (correct to two decimal places).

- t-coordinate of point of inflection:

Use the integral solution.

$$\frac{dt}{dP} = \frac{1}{0.15P\left(1-\frac{P}{30}\right)-0.2\log_e\left(\frac{P}{4}\right)}, \text{ where } P(0)=11.720$$

$$\Rightarrow t = \int_{11.720}^{P} \frac{1}{0.15w\left(1-\frac{w}{30}\right)-0.2\log_e\left(\frac{w}{4}\right)}\, dw.$$

Substitute $P = 13.520797$ (using more accuracy than the final answer requires in order to avoid rounding error). From a CAS:

$$t = \int_{11.720}^{13.520797} \frac{1}{0.15w\left(1-\frac{w}{30}\right)-0.2\log_e\left(\frac{w}{4}\right)}\, dw = 2.08 \text{ (correct to two decimal places).}$$

Note: The existence of a point of inflection is justified below.

- Shape:

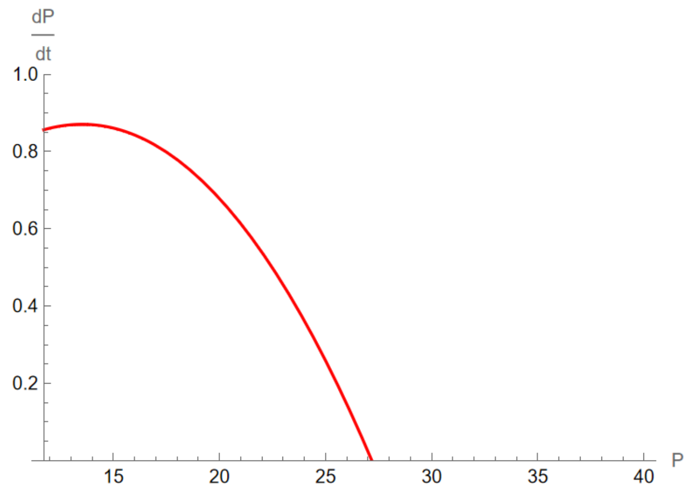
Method 1: Use a phase diagram.

- Consider a graph of $\frac{dP}{dt}$ versus P over the domain $P \in [11.720,\ 27.180]$.

Note on domain:

$P = 11.720$ is the initial population.

$P = 27.180$ is the equilibrium value of P.



- Initially ($t = 0$): $\frac{dP}{dt} > 0$ by inspection therefore P increases from 11.720.

- $\frac{dP}{dt} > 0$ as $P \to 27.180$ therefore $P = P(t)$ is an increasing function.

- $\frac{dP}{dt}$ has a turning point at $P = 27.180$ (correct to three decimal places)

therefore the graph of $P = P(t)$ has a point of inflection at $P = 27.180$.

- $P = 27.180$ is an equilibrium solution and is a horizontal asymptote of $P = P(t)$.

Note: The integral solution to the differential equation suggests that $P \neq 27.180$:

$$\frac{dt}{dP} = \frac{1}{0.15P\left(1-\frac{P}{30}\right)-0.2\log_e\left(\frac{P}{4}\right)} \quad \Rightarrow t = \int_{11.720}^{P} \frac{1}{0.15w\left(1-\frac{w}{30}\right)-0.2\log_e\left(\frac{w}{4}\right)}\,dw$$

and from a CAS: $\displaystyle \lim_{P\to 27.180} \int_{11.720}^{P} \frac{1}{0.15w\left(1-\frac{w}{30}\right)-0.2\log_e\left(\frac{w}{4}\right)}\,dw = \text{large}.$

Method 2: Some CAS (such as *Mathematica*) can solve the differential equation

$$\frac{dP}{dt} = 0.15P\left(1-\frac{P}{30}\right)-0.2\log_e\left(\frac{P}{4}\right) \text{ subject to } P(0) = 11.720$$

numerically and plot this solution.

The t-coordinate of the point of inflection can be found by some CAS by solving $P(t) = 14.3007$ from the numerical solution.

Note:

Calculation of the estimate given in the question of the polar bear population in 1960:

Under the 'large scale hunting' model:

The year $1960 \Rightarrow t = 60$.

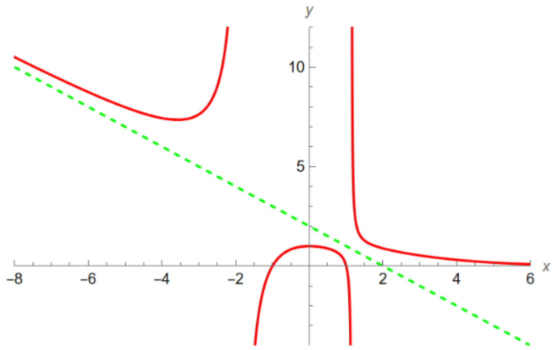
From a CAS: $P(60) = 11.720$ (correct to three decimal places).

Question 5

a. i.

By inspection of a graph of $y = f_2(x) = \dfrac{x^2 - 1}{e^x - x - 2}$ (plot using a CAS)

it can be seen that $y = f_2(x)$ has a diagonal asymptote as $x \to -\infty$:



Definition:

A line $y = mx + c$ is a diagonal asymptote of the function

$y = g(x)$ as $x \to +\infty$ if $\lim\limits_{x \to +\infty} \left(g(x) - [mx + c] \right) = 0$.

Similarly when $x \to -\infty$: $\lim\limits_{x \to -\infty} \left(g(x) - [mx + c] \right) = 0$.

Note: If $m = 0$ then the line is a horizontal asymptote.

Method 1:

- From a CAS: $f'(x) = -\dfrac{1 - e^x + 4x - 2xe^x + x^2 + x^2 e^x}{\left(e^x - x - 2\right)^2}$.

From a CAS: $\lim\limits_{x \to -\infty} f'(x) = -1 = m$. **[A1]**

> **Note 1:** $\lim\limits_{x \to \pm\infty} f'(x)$ may not exist when a diagonal asymptote exists
>
> (in which case, **Method 2** should be used). Example: $f(x) = \dfrac{\sin\left(e^x\right)}{x} + x$
>
> has the diagonal asymptote $y = x$ as $x \to \pm\infty$ but $\lim\limits_{x \to \infty} f'(x)$ does not exist.
>
> **Note 2:** If $f(x)$ has a diagonal asymptote $y = mx + c$ and $\lim\limits_{x \to \pm\infty} f'(x)$ exists,
> then m is given by the above equation. The converse is **not** true.

- Substitute $m = -1$ into the definition:

$\lim\limits_{x \to -\infty} \left(f(x) - [-x + c] \right) = 0 \quad \Rightarrow \lim\limits_{x \to -\infty} \left(f(x) + x - c \right) = 0$

$\Rightarrow \lim\limits_{x \to -\infty} \left(f(x) + x \right) = \lim\limits_{x \to -\infty} (c) \quad \Rightarrow \lim\limits_{x \to -\infty} \left(f(x) + x \right) = c$

$\Rightarrow c = 2$. **[A1]**

Check: $\lim\limits_{x\to-\infty}\left(\dfrac{x^2-1}{e^x-x-2}-[-x+2]\right)=0$ using a CAS.

Method 2:

It follows from the definition that:

- $\lim\limits_{x\to+\infty}\dfrac{g(x)}{x}=m$. .... (1)
- $\lim\limits_{x\to+\infty}\left(g(x)-mx\right)=c$. .... (2)

Similarly when $x\to-\infty$:

- $\lim\limits_{x\to-\infty}\dfrac{g(x)}{x}=m$. .... (1')
- $\lim\limits_{x\to-\infty}\left(g(x)-mx\right)=c$. .... (2')

Proof of (1):

$\lim\limits_{x\to+\infty}\left(g(x)-[mx+c]\right)=0 \quad \Rightarrow \lim\limits_{x\to+\infty}\left(\dfrac{g(x)-[mx+c]}{x}\right)=0 \quad \Rightarrow \lim\limits_{x\to+\infty}\dfrac{g(x)}{x}=\lim\limits_{x\to+\infty}\dfrac{mx+c}{x}$

$\Rightarrow \lim\limits_{x\to+\infty}\dfrac{g(x)}{x}=\lim\limits_{x\to+\infty}\left(m+\dfrac{c}{x}\right) \quad \Rightarrow \lim\limits_{x\to+\infty}\dfrac{g(x)}{x}=m+\lim\limits_{x\to+\infty}\dfrac{c}{x} \quad \Rightarrow \lim\limits_{x\to+\infty}\dfrac{g(x)}{x}=m$.

Proof of (2):

$\lim\limits_{x\to+\infty}\left(g(x)-[mx+c]\right)=0 \quad \Rightarrow \lim\limits_{x\to+\infty}\left(g(x)-mx\right)-\lim\limits_{x\to+\infty}(c)=0$

$\Rightarrow \lim\limits_{x\to+\infty}\left(g(x)-mx\right)-\lim\limits_{x\to+\infty}\left(c\right)=0 \quad \Rightarrow \lim\limits_{x\to+\infty}\left(g(x)-mx\right)-c=0 \quad \Rightarrow \lim\limits_{x\to+\infty}\left(g(x)-mx\right)=c$.

Note: If $g(x)$ has a diagonal asymptote $y=mx+c$, then *m* and *c* are given by the above equations. The converse is **not** true.

Since $y=f(x)$ has a diagonal asymptote as $x\to-\infty$, use equations (1') and (2').

From a CAS: $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}=-1=m$. **[A1]**

Substitute $m=-1$: $\lim\limits_{x\to-\infty}\left(f(x)+x\right)=c$.

From a CAS: $\lim\limits_{x\to-\infty}\left(f(x)+x\right)=2$. **[A1]**

Connection to Method 1: Theorem: If $\lim_{x \to \pm\infty} f'(x)$ exists then $\lim_{x \to \pm\infty} f'(x) = \lim_{x \to -\infty} \frac{f(x)}{x}$.

Method 3:

$$f_2(x)-[mx+c] \quad = \frac{x^2-1}{e^x-x-2}-[mx+c]$$

$$= \frac{x^2(m+1)+x(2m+c)-e^x(mx-c)}{e^x-x-2}$$

using a CAS or 'by hand'.

Then $\lim\limits_{x\to-\infty}\left(f_2(x)-[mx+c]\right) \quad = \lim\limits_{x\to-\infty}\frac{x^2(m+1)+x(2m+c)-e^x(mx-c)}{e^x-x-2}=0$

is required. This can only happen if

$\lim\limits_{x\to-\infty}\left(x^2(m+1)+x(2m+c)\right)=0$ (since $\lim\limits_{x\to-\infty} e^x=0$)

and this can only happen if

$m+1=0$

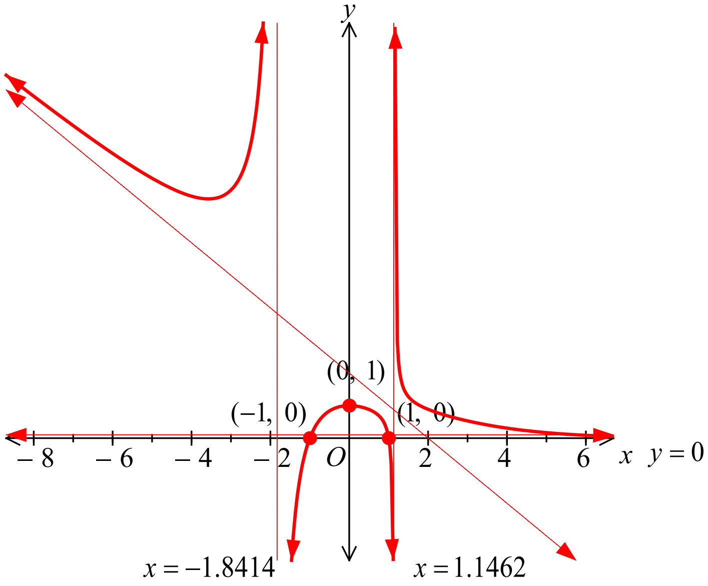
$2m+c=0$.

Therefore $m=-1$ and $c=2$.

Note: Some CAS (such as *Mathematica*) can directly solve $\lim\limits_{x\to-\infty}\left(f_2(x)-[mx+c]\right)=0$ for *m* and *c*.

a. ii.

Answer:



Marking scheme:

- Correct shape and diagonal asymptote is shown: **[A1]**

  The diagonal asymptote is not required to pass through $x = 2$.

- Vertical and horizontal asymptotes: $x = -1.8414$, $x = 1.1462$, $y = 0$. **[A1]**

- Axial intercepts: $(1,\ 0)$, $(-1,\ 0)$, $(0,\ 1)$. **[A1]**

Calculations:

- Shape: Plot a graph of $y = f_2(x) = \dfrac{x^2 - 1}{e^x - x - 2}$ using a CAS.

- Horizontal asymptote: $\lim\limits_{x \to +\infty} \dfrac{x^2 - 1}{e^x - x - 2} = 0$.

- Vertical asymptotes: Use a CAS to solve $e^x - x - 2 = 0$.

$x = -1.8414$, $x = 1.1462$.

- Diagonal asymptote:

Existence is known from **part c.** and the behaviour of the graph suggests the shape.

b. i.

Apply the definition: $\lim\limits_{x\to+\infty} f_k(x) = \lim\limits_{x\to+\infty} \dfrac{x^2-1}{e^x - x - k} = 0$.

Answer: $y = 0$. **[A1]**

b. ii.

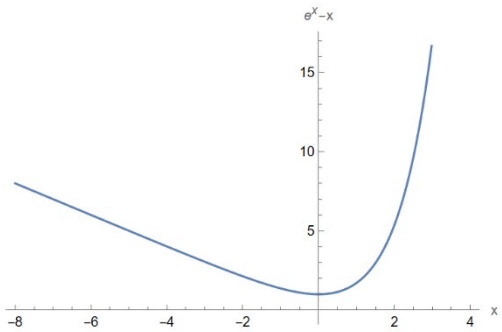
- Values of x for which $f_k(x) = \dfrac{x^2-1}{e^x - x - k}$ is undefined are required.
- It is therefore required that $e^x - x - k = 0$ has two real solutions

provided $x^2 - 1 \neq 0$.

Note: If $e^x - x - k = 0$ and $x^2 - 1 = 0$ then $f_k(x)$ has the indeterminate form $f_k(x) = \dfrac{0}{0}$ (which indicates a 'hole') rather than being undefined.

- $e^x - x - k = 0 \quad \Rightarrow e^x - x = k$.

From a graph of $y = e^x - x$ (plot using a CAS) it is seen that $e^x - x = k$ has two real solutions when the line $y = k$ lies above the tangent to the graph of $y = e^x - x$ at its turning point:



Use a CAS to get the coordinates of the turning point: $(0,\ 1)$.

Therefore $e^x - x - k = 0$ has two solutions when $k > 1$. **[A1]**

- But values of k such that

$e^x - x - k = 0$ and $x^2 - 1 = 0 \Rightarrow x = \pm 1$ **[M1]**

must be rejected:

Case 1: $x = 1 \quad \Rightarrow e - 1 - k = 0 \quad \Rightarrow k = e - 1 > 1$.

Case 2: $x = -1 \quad \Rightarrow e^{-1} + 1 - k = 0 \quad \Rightarrow k = \frac{1}{e} + 1 > 1$.

Therefore $k = e - 1$ and $k = \frac{1}{e} + 1$ are rejected.

Answer: $k > 1 \setminus \left\{ e - 1, \ \frac{1}{e} + 1 \right\}$. **[A1]**

Note:

The graph of $f_k(x) = \frac{x^2 - 1}{e^x - x - k}$ has a 'hole' at $x = 1$ when $k = e - 1$ and a 'hole' at $x = -1$ when $k = \frac{1}{e} + 1$.

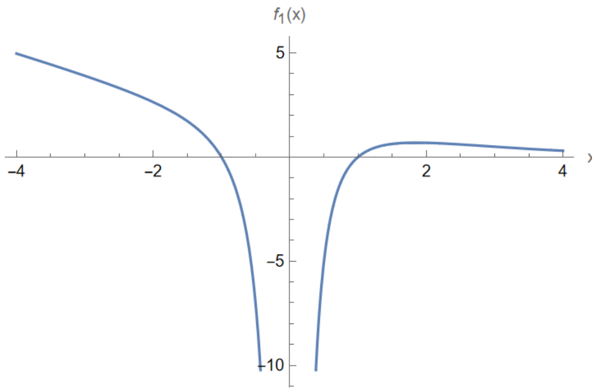
c. i.

- From a CAS: $f_k'(x) = -\dfrac{1 - e^x - 2e^x x + 2kx + x^2 + e^x x^2}{\left(e^x - k - x\right)^2}$.

Substitute $x = 0$: $f_k'(0) = \dfrac{0}{(1-k)^2} = 0$ **provided** $(1-k)^2 \neq 0$. **[A1]**

- Therefore the case $k = 1$ must be investigated.

The graph of $y = f_1(x) = \dfrac{x^2 - 1}{e^x - x - 1}$ (plot using a CAS) has a vertical asymptote $x = 0$:



$\lim\limits_{x \to 0} \dfrac{x^2 - 1}{e^x - x - 1} = -\infty$.

Answer: $k \in R \setminus \{1\}$. **[A1]**

c. ii.

- Evaluate $f_k''(x)=0$ at $x=0$ using a CAS:

$$\frac{1}{(1-k)^2}+\frac{2}{1-k}=0.$$

- Solve $\frac{1}{(1-k)^2}+\frac{2}{1-k}=0$ using a CAS: $k=\frac{3}{2}$.

Therefore there is a potential point of inflection at $x=0$ when $k=\frac{3}{2}$.

- Use the 'triple derivative' test to check for a change in concavity:

$$f_{\frac{3}{2}}'''(0)=4.$$

Therefore there is a (stationary) point of inflection at $x=0$ when $k=\frac{3}{2}$.

Answer: $k=\frac{3}{2}$. **[A1]**

Evidence for check of change in concavity is required.

Question 6

a.

- Let C be the random variable "*Volume (ml) of coffee*".

$C \sim \text{Normal}\left(\mu_C = 240,\ \ \sigma_C = 8\right)$.

- Let the random variable $D = C_1 - C_2$

where C_1 and C_2 are independent copies of C.

$\Pr\left(|D| > 5\right)$ is required.

$\mu_D = \mu_{C_1} - \mu_{C_2} = 0$.

$\sigma_D^2 = \sigma_{C_1}^2 + (-1)^2 \sigma_{C_2}^2 = 2\sigma_C^2 = 2\left(8^2\right) = 128$

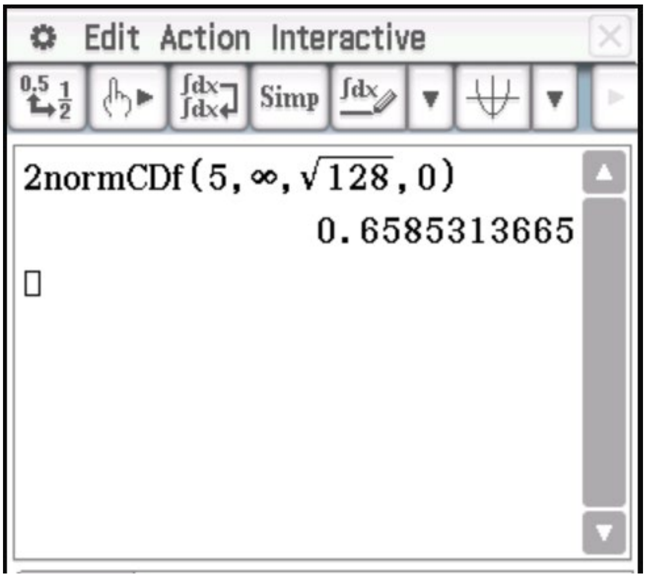
therefore $\sigma_D = \sqrt{128}$.

D follows a normal distribution since C_1 and C_2 are independent normal random variables:

$D \sim \text{Normal}\left(\mu_D = 0,\ \ \sigma_D = \sqrt{128}\right)$. **[A1]**

- From a CAS: $\Pr\left(|D| > 5\right) = 0.6585$.

Answer: 0.6585. **[A1]**



b.

- Let M be the random variable "*Volume (ml) of milk*".

$M \sim \text{Normal}\left(\mu_M = 10, \quad \sigma_M = 2\right)$.

- Let the random variable $T = C + M$.

$\Pr(T < 245)$ is required.

$\mu_T = \mu_C + \mu_M = 240 + 10 = 250$.

$\sigma_T^2 = \sigma_C^2 + \sigma_M^2 = 8^2 + 2^2 = 68$

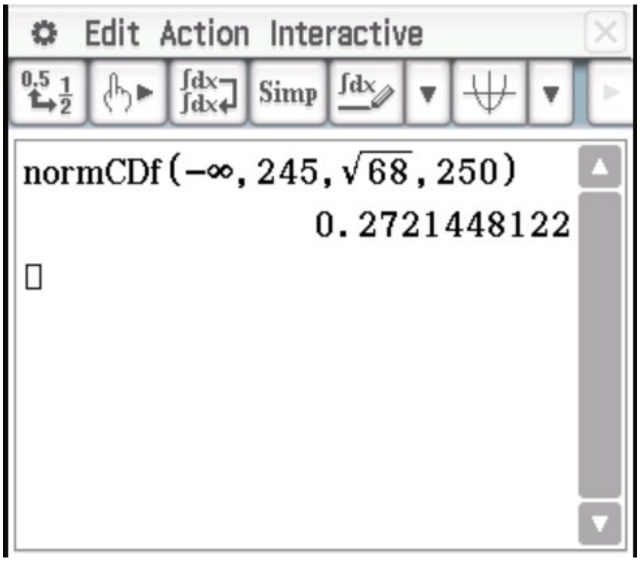
therefore $\sigma_T = \sqrt{68}$.

T follows a normal distribution since C and M are independent normal random variables:

$T \sim \text{Normal}\left(\mu_T = 250, \quad \sigma_T = \sqrt{68}\right)$. **[A1]**

- From a CAS: $\Pr(T < 245) = 0.2721$.

Answer: 0.2721. **[A1]**



c.

- Let the random variable $X = M_1 + M_2 + \ldots + M_n$

where M_1, M_2 …., M_n are independent copies of M.

The maximum value of n such that $\Pr(X \le 650) = 0.999$ is required.

$\mu_X = \mu_{M_1} + \mu_{M2} + \ldots + \mu_{M_n} = n\mu_M = 10n$.

$\sigma_X^2 = \sigma_{M_1}^2 + \sigma_{M_2}^2 + \ldots \sigma_{M_n}^2 = n\sigma_M^2 = n\left(2^2\right) = 4n$

therefore $\sigma_X = 2\sqrt{n}$.

X follows a normal distribution since M_1, M_2 …., M_n are independent normal random variables:

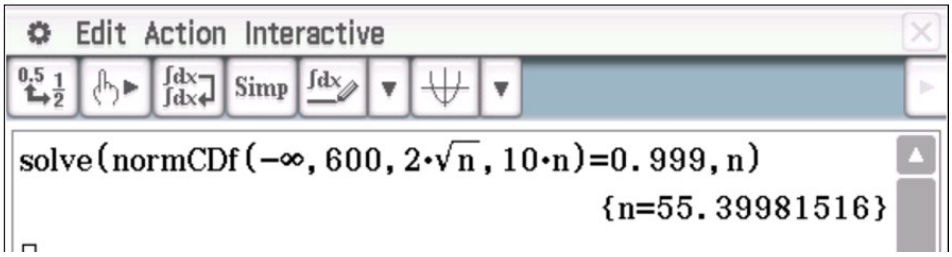
$X \sim \text{Normal}\left(\mu_X = 10n, \ \ \sigma_X = 2\sqrt{n}\right)$. **[A1]**

- Use a CAS to solve $\Pr(X \le 600) = 0.999$ for n: $n = 55.4$.

Either solve a probability distribution equation containing n or
use trial-and-error (substitute values of n) to narrow in on the answer.

Rounding **down** to the nearest whole number is required.

Answer: 55. **[A1]**



d. i.

Answer: $H_0: \ \mu = 240$.

$H_1: \ \mu < 240$.

[A1]

Both statements are required.

d. ii

$C \sim \text{Normal}\left(\mu_C = 240, \ \ \sigma_C = 8\right)$

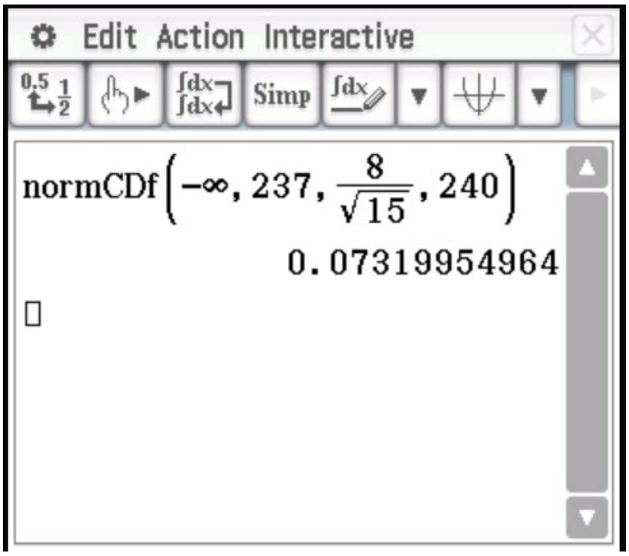
therefore the random variable $\overline{C} \sim \text{Normal}\left(\mu_{\overline{C}} = \mu_C = 240, \ \ \sigma_{\overline{C}} = \dfrac{\sigma_C}{\sqrt{n}} = \dfrac{8}{\sqrt{15}}\right)$.

Mean volume (ml) of a cup of coffee $= \dfrac{3555}{15} = 237$.

p value $= \Pr\left(\overline{C} < 237\right)$.

Use a CAS: $\Pr\left(\overline{C} < 237\right) = 0.0732$.

Answer: 0.0732. **[A1]**



d. iii.

No.

Level of significance $= 0.05$. p value $= 0.0732$.

p value > level of significance.

[A1]

d. iv.

Let $\overline{x_c}$ be the mean volume of a cup of coffee.

The smallest value of $\overline{x_c}$ such that $\Pr\left(\overline{C} < \overline{x_c}\right) < 0.05$ is required.

Use a CAS to find the value of $\overline{x_c}$ such that $\Pr\left(\overline{C} < \overline{x_c}\right) = 0.05$
(the 'inverse normal problem'):

$\overline{x_c} = 236.60$ (correct to two decimal places).

Smallest total amount of coffee (with no milk) served in a
sample of 15 cups for H_0 not to be rejected:

$15\left(\overline{x_c}\right) = 15(236.60) = 3549$.

Answer: 3,549. **[A1]**

Note: This answer is consistent with the answer to **part d. iii.**:

In **part iii.** H_0 is not rejected because *p* value > level of significance.

The answer to **part iv.** means that H_0 is not rejected if total amount of coffee served in this sample is greater than 3549 ml.
The **part d.** preamble says that the total is 3555 ml.
Therefore the answer to **part iv.** means that H_0 is not rejected, which is consistent with **part iii**.

END OF SOLUTIONS