The Mathematical Association of Victoria

Trial Exam 2023

SPECIALIST MATHEMATICS

Written Examination 1 - SOLUTIONS

Question 1

a.

Method 1:

$$\left(2\underset{\sim}{i}-\underset{\sim}{k}\right)\times\left(\underset{\sim}{j}+2\underset{\sim}{k}\right) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} \quad \text{(using the VCAA formula sheet)}$$

$$= \underset{\sim}{i}\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \qquad = \underset{\sim}{i}-4\underset{\sim}{j}+2\underset{\sim}{k}.$$

Method 2:

$$\left(2\underset{\sim}{i}-\underset{\sim}{k}\right)\times\left(\underset{\sim}{j}+2\underset{\sim}{k}\right) = 2\underset{\sim}{i}\times\underset{\sim}{j}+4\underset{\sim}{i}\times\underset{\sim}{k}-\underset{\sim}{k}\times\underset{\sim}{j}-\underset{\sim}{k}\times 2\underset{\sim}{k}$$

using the distributive (over addition) property of the cross product

$$= 2\underset{\sim}{i}\times\underset{\sim}{j}-4\underset{\sim}{k}\times\underset{\sim}{i}+\underset{\sim}{j}\times\underset{\sim}{k}-\underset{\sim}{k}\times 2\underset{\sim}{k}$$

using the anti-commutativity property of the cross product and
the associative law over scalar multiplication

$$= 2\underset{\sim}{k}-4\underset{\sim}{j}+\underset{\sim}{i}-\underset{\sim}{0}$$

using (by definition) $\underset{\sim}{i}\times\underset{\sim}{j}=\underset{\sim}{k}$, $\underset{\sim}{j}\times\underset{\sim}{k}=\underset{\sim}{i}$, $\underset{\sim}{k}\times\underset{\sim}{i}=\underset{\sim}{j}$ and $\underset{\sim}{i}\times\underset{\sim}{i}=\underset{\sim}{j}\times\underset{\sim}{j}=\underset{\sim}{k}\times\underset{\sim}{k}=\underset{\sim}{0}$

$$= \underset{\sim}{i}-4\underset{\sim}{j}+2\underset{\sim}{k}.$$

Marking scheme:

Method. **[M1]**

Answer: $\underset{\sim}{i}-4\underset{\sim}{j}+2\underset{\sim}{k}$. **[A1]**

Accept $\left(\underset{\sim}{j}+2\underset{\sim}{k}\right)\times\left(2\underset{\sim}{i}-\underset{\sim}{k}\right) = -\underset{\sim}{i}+4\underset{\sim}{j}-2\underset{\sim}{k}$.

b.

- The cartesian equation of a plane is given by $ax + by + cz = d$
where $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ is a vector normal to the plane.

- Substitute the answer from **part a.**:

$x - 4y + 2z = d$.

- Substitute the point $(3,\ 2,\ -1)$ lying in the plane:

$3 - 4(2) + 2(-1) = d \qquad \Rightarrow d = -7$.

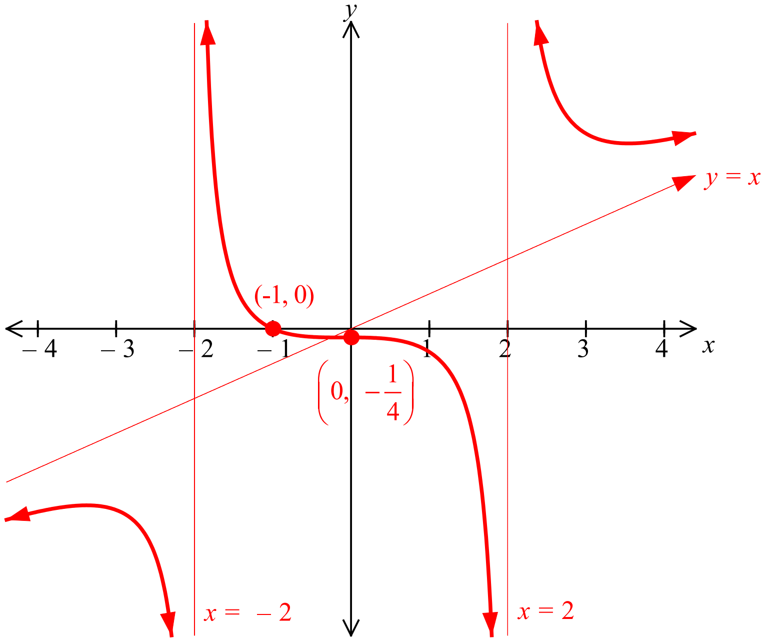
Answer: $x - 4y + 2z = -7$. **[H1]**

Accept all correct alternative cartesian forms.

Question 2

a.

Answer:



Marking scheme:

- Correct shape and vertical asymptotes $x = 2$ and $x = -2$. **[A1]**

- x-intercept at $(-1,\ 0)$ and y-intercept at $\left(0,\ -\frac{1}{4}\right)$. **[A1]**

A turning point at y-intercept must be shown(or at least suggested).

- Diagonal asymptote $y = x$. **[A1]**

Note:

The question only asks to label "any asymptotes with their equations and any axial intercepts with their coordinates." It does **not** ask students to calculate or label coordinates of turning points. Students who try to label coordinates of turning points are 'over-engaging' with the question. At best, over-engagement wastes time, at worst it wastes time **and** it results in the loss of marks if errors are made (VCAA penalises errors regardless of whether arise as part of over-engagement).

Calculations for the graph of $y = \dfrac{x^3+1}{x^2-4}$ (see Appendix 1 for further discussion)**:**

- Vertical asymptotes:

Solve $x^2 - 4 = 0$: $x = 2$ and $x = -2$.

- Approach towards vertical asymptotes:

$x = 2$: $\displaystyle\lim_{x \to 2^+} \frac{x^3+1}{x^2-4} = +\infty$. $\displaystyle\lim_{x \to 2^-} \frac{x^3+1}{x^2-4} = -\infty$.

$x = -2$: $\displaystyle\lim_{x \to -2^+} \frac{x^3+1}{x^2-4} = +\infty$. $\displaystyle\lim_{x \to -2^-} \frac{x^3+1}{x^2-4} = -\infty$.

- Horizontal and diagonal asymptotes:

$$\begin{array}{r|l} & x \\ x^2-4 & x^3 + 0x^2 + 0x + 1 \\ & -(x^3 \qquad -4x) \\ \hline & \qquad\qquad 4x+1 \end{array}$$

$$\Rightarrow y = \frac{x^3+1}{x^2-4} = x + \frac{4x+1}{x^2-4} \sim x.$$

Diagonal asymptote: $y = x$.

- Approach towards diagonal asymptote:

$x \to +\infty$: $y = x + \dfrac{4x+1}{x^2-4} \sim x + 0^+$

therefore the graph of $y = \dfrac{x^3+1}{x^2-4}$ approaches $y = x$ from above as $x \to +\infty$.

$x \to -\infty$: $y = x + \dfrac{4x+1}{x^2-4} \sim x + 0^-$

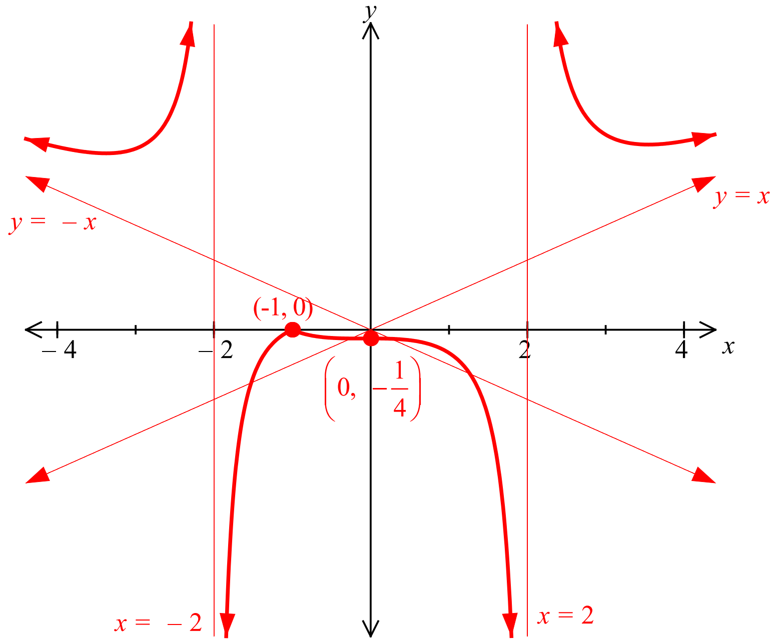
therefore the graph of $y = \dfrac{x^3+1}{x^2-4}$ approaches $y = x$ from below as $x \to -\infty$.

- The turning point at $x = 0$ is a maximum turning point because of the asymptotic behaviour for $x \to 2^-$.

- There is a minimum turning in the interval $(-1,\ 0)$ because the graph must pass through the point $(-1,\ 0)$.

b.

Answer:



[A1]

Marking scheme:

• Correct shape:

Reflection in the *x*-axis of the part of the graph of $y = \frac{x^3+1}{x^2-4}$ over $x < -1$.

• Correctly labelled features:

x-intercept at $(-1,\ 0)$, *y*-intercept at $\left(0,\ -\frac{1}{4}\right)$:

A minimum turning point at $x = 0$ and a maximum turning point lying in the interval $(-1,\ 0)$ must be shown.

x-intercept must be shown as a 'corner' (salient point).

Diagonal asymptotes: $y = x$ and $y = -x$.

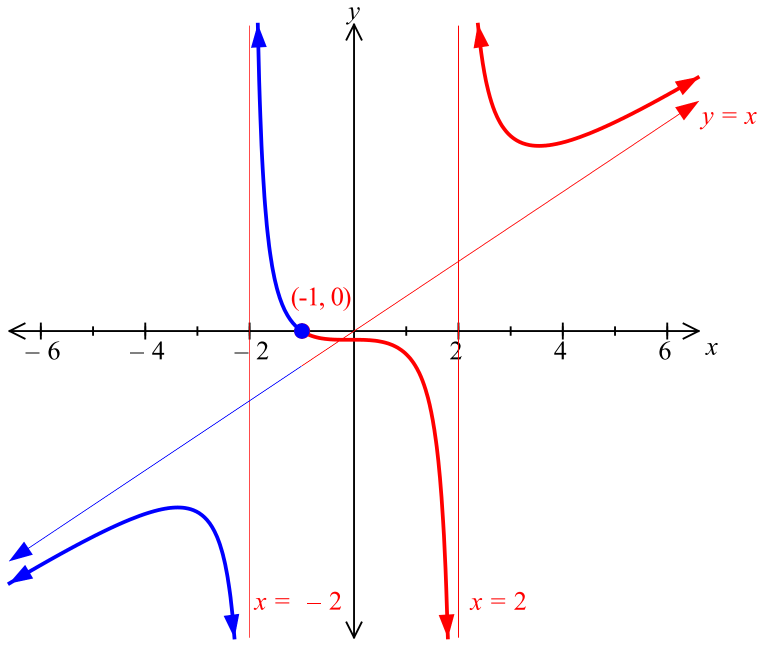
Vertical asymptotes: $x = 2$ and $x = -2$.

Solution:

$$|x^3+1| = \begin{cases} x^3+1, & x^3+1 \ge 0 \Rightarrow x \ge -1 \\ -\left(x^3+1\right), & x^3+1 < 0 \Rightarrow x < -1 \end{cases}$$

therefore $y = \dfrac{|x^3+1|}{x^2-4} = \begin{cases} \dfrac{x^3+1}{x^2-4}, & x \ge -1 \\ \dfrac{-\left(x^3+1\right)}{x^2-4} = -\left(\dfrac{x^3+1}{x^2-4}\right), & x < -1 \end{cases}$

Therefore inspect the graph of $y = \dfrac{x^3+1}{x^2-4}$ from **part a.**:



and reflect in the x-axis the part of this graph over $x < -1$.

Note: Any horizontal or diagonal asymptotes must also be reflected in the x-axis over $x < -1$.

Question 3

a.

Method 1: Polar form method.

- Let $z = r\text{cis}(\theta) \quad \Rightarrow z^2 = r^2\text{cis}(2\theta)$.
- $1 - i\sqrt{3} = 2\text{cis}\left(-\frac{\pi}{3} + 2n\pi\right), \quad n \in Z$.

The values of z such that $z^2 = 1 - i\sqrt{3}$ are required:

$$r^2\text{cis}(2\theta) = 2\text{cis}\left(-\frac{\pi}{3} + 2n\pi\right).$$

- Equate moduli: $\quad r^2 = 2 \Rightarrow r = \sqrt{2}$.
- Equate arguments: $\quad \text{cis}(2\theta) = \text{cis}\left(-\frac{\pi}{3} + 2n\pi\right)$

$$\Rightarrow 2\theta = -\frac{\pi}{3} + 2n\pi$$

$$\Rightarrow \theta = -\frac{\pi}{6} + n\pi.$$

- $z = \sqrt{2}\text{cis}\left(-\frac{\pi}{6}\right) \quad = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) \quad = \frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}$.
- $z = -\left(\frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}\right) \quad = -\frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}$.

Method 2: Cartesian form method.

- Let $z = a + ib \quad \Rightarrow z^2 = \left(a^2 - b^2\right) + 2abi$, $a, b \in R$.

- The values of z such that $z^2 = 1 - i\sqrt{3}$ are required:

$\left(a^2 - b^2\right) + 2abi = 1 - i\sqrt{3}$.

- Equate real parts: $a^2 - b^2 = 1$. .... (1)

- Equate imaginary parts: $2ab = -\sqrt{3} \quad \Rightarrow a = -\dfrac{\sqrt{3}}{2b}$. .... (2)

- Substitute equation (2) into equation (1):

$$\left(-\frac{\sqrt{3}}{2b}\right)^2 - b^2 = 1 \qquad \Rightarrow \frac{3}{4b^2} - b^2 = 1$$

$$\Rightarrow 4b^4 + 4b^2 - 3 = 0 \qquad \Rightarrow \left(2b^2 - 1\right)\left(2b^2 + 3\right) = 0.$$

Case 1: $2b^2 - 1 = 0 \qquad \Rightarrow b = \pm\dfrac{1}{\sqrt{2}} \qquad \Rightarrow a = \mp\dfrac{\sqrt{6}}{2}$.

Case 2: $2b^2 + 3 = 0$ is rejected because $b \in R$.

- Therefore $z = \dfrac{\sqrt{6}}{2} - \dfrac{i}{\sqrt{2}}$, $\quad z = -\dfrac{\sqrt{6}}{2} + \dfrac{i}{\sqrt{2}}$.

Marking scheme:

Method. **[M1]**

Answers: $z = \dfrac{\sqrt{6}}{2} - i\dfrac{\sqrt{2}}{2}$, $\quad z = -\dfrac{\sqrt{6}}{2} + i\dfrac{\sqrt{2}}{2}$. **[A1]**

Accept all correct alternative cartesian forms.

b.

$4z^3 + 4z^2 = -i\sqrt{3}z \quad \Rightarrow 4z^3 + 4z^2 + i\sqrt{3}z = 0 \quad \Rightarrow z\left(4z^2 + 4z + i\sqrt{3}\right) = 0$.

Case 1: $z = 0$.

Case 2: $4z^2 + 4z + i\sqrt{3} = 0 \quad \Rightarrow z = \dfrac{-1 \pm \sqrt{1 - i\sqrt{3}}}{2}$.

Substitute the answers from **part a.**:

Case 2a: $z = \dfrac{-1 + \frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}}{2} \quad = \dfrac{-2 + \sqrt{6} - i\sqrt{2}}{4}$.

Case 2b: $z = \dfrac{-1 - \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}}{2} \quad = \dfrac{-2 - \sqrt{6} + i\sqrt{2}}{4}$.

Marking scheme:

- $z = \dfrac{-1 \pm \sqrt{1 - i\sqrt{3}}}{2}$. **[M1]**

- **Answers:** $z = \dfrac{\sqrt{6}}{2} - i\dfrac{\sqrt{2}}{2}$, $\quad z = -\dfrac{\sqrt{6}}{2} + i\dfrac{\sqrt{2}}{2}$, $\quad z = 0$. **[H1]**

Question 4

- Let m be the total mass (in grams) of oranges in the sample.
- Let n be the number of oranges in the sample.
- Let $\bar{x}$ be the sample mean (in grams).

Then $m = n\bar{x}$.

- 95% confidence interval therefore the critical value is $z_c = 1.96$ (given in the question via $\Pr(-1.96 < Z < 1.95) = 0.95$).

Note:

The Study Design does not state which value to use. $z_c = 1.96$ is the conventional value used in most statistics textbooks.

From the given confidence interval:

$\bar{x} - 1.96\frac{5}{\sqrt{n}} = 198.6$. .... (1)

$\bar{x} + 1.96\frac{5}{\sqrt{n}} = 201.4$. .... (2)

Both equations **[M1]**

$(1)+(2)$: $2\bar{x} = 400 \quad \Rightarrow \bar{x} = 200$.

$(2)-(1)$: $2 \times 1.96\frac{5}{\sqrt{n}} = 2.8$ **[M1]**

$\Rightarrow \frac{9.8}{\sqrt{n}} = 1.4 \quad \Rightarrow \frac{9.8}{1.4} = \sqrt{n} \quad \Rightarrow \sqrt{n} = \frac{49}{7} = 7$

$\Rightarrow n = 49$.

- $m = n\bar{x} \quad = (49)(200) \quad = 9{,}800$.

Answer: 9,800. **[A1]**

Question 5

- Let $P(n)$ be the conjecture $3^{2n}+7$ is divisible by 8 $\forall n \in N$.

- Base case: Show the conjecture is true for a particular value of n.

The conjecture is true for $n=1$ since $3^{2(1)}+7=9+7=16$ is divisible by 8.
$P(1)$ is true.

- Inductive hypothesis:

ASSUME the conjecture is true for some $n=k$, that is, assume $P(k)$ is true.

Assume that $3^{2n}+7$ is divisible by 8 for some $k \in N$.

That is, $3^{2k}+7=8m$ where $m \in N$.

Structure: Conjecture, base case and inductive hypothesis. **[M1]**

- Show that $P(k) \Rightarrow P(k+1)$.

It is required to show $3^{2k}+7=8m$ implies $3^{2(k+1)}+7$ is divisible by 8:

$3^{2(k+1)}+7 \quad =3^{2k+2}+7 \quad =3^{2k}\times 3^2+7 \quad =9\left(3^{2k}\right)+7$

$=9\times\underline{(8m-7)}+7$ (using $P(k)$ **Note:** $3^{2k}+7=8m \Rightarrow 3^{2k}=8m-7$)

$=72m-63+7 \quad =72m-56 \quad =8(9m-7) \quad =8p$ where $p=9m-7\in N$.

Structure: $P(k)\Rightarrow P(k+1)$ and explicit use of the inductive hypothesis. **[M1]**

Therefore we have proved that if $P(k)$ is true then $P(k+1)$ is true $\forall k \in N$.

Since $P(1)$ is true and $P(k)\Rightarrow P(k+1)$, it follows from the axiom of mathematical induction that $P(n)$ is true $\forall n \in N$. ■

Structure: Clear conclusion. **[M1]**

Question from the cutting room floor:

b. Hence find the last two digits of $25\left(3^{2024}+7\right)$. 1 mark

Solution:

$25\left(3^{2024}+7\right)=25\left(3^{2(1012)}+7\right)$. From **part a.**: $3^{2(1012)}+7=8m$ where $m\in Z^+$.

Therefore $25\left(3^{2024}+7\right)=25(8m)=400m$ which has the obvious last two digits of 00.

Question 6

- $S = 2\pi \int_0^{\log_e(2)} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.

- Substitute $y = \frac{e^x + e^{-x}}{2} \quad \Rightarrow \frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$:

$$S = \pi \int_0^{\log_e(2)} \left(e^x + e^{-x}\right)\sqrt{1+\left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx$$ **[M1]**

$$= \pi \int_0^{\log_e(2)} \left(e^x + e^{-x}\right)\sqrt{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}\, dx \qquad = \pi \int_0^{\log_e(2)} \left(e^x + e^{-x}\right)\sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx$$

$$= \frac{\pi}{2} \int_0^{\log_e(2)} \left(e^x + e^{-x}\right)\left|e^x + e^{-x}\right|\, dx$$

$$= \frac{\pi}{2} \int_0^{\log_e(2)} \left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right)\, dx$$

since $\left|e^x + e^{-x}\right| = e^x + e^{-x}$ on the interval $x \in \left[0,\ \log_e(2)\right]$ **[M1]**

Integral and justification for dropping the modulus.

$$= \frac{\pi}{2} \int_0^{\log_e(2)} e^{2x} + 2 + e^{-2x}\, dx$$

$$= \frac{\pi}{2}\left|\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right|_0^{\log_e(2)} \qquad = \frac{\pi}{2}\left[\left(\frac{1}{2}e^{2\log_e(2)} + 2\log_e(2) - \frac{1}{2}e^{-2\log_e(2)}\right) - (0)\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2}e^{\log_e(4)} + 2\log_e(2) - \frac{1}{2}e^{\log_e\left(\frac{1}{4}\right)}\right] \qquad = \frac{\pi}{2}\left(\frac{1}{2}(4) + 2\log_e(2) - \frac{1}{2}\left(\frac{1}{4}\right)\right) \qquad = \frac{\pi}{2}\left(\frac{15}{8} + 2\log_e(2)\right).$$

Answer: $\pi\left(\frac{15}{16} + \log_e(2)\right)$. **[A1]**

Question 7

a.

- Domain: $-1 \le 2x - 1 \le 1 \quad \Rightarrow 0 \le 2x \le 2 \quad \Rightarrow 0 \le x \le 1$.

- Range: $y \in \left[\left(-\frac{\pi}{2}\right) + \frac{\pi}{2},\ \left(\frac{\pi}{2}\right) + \frac{\pi}{2}\right] \quad = [0,\ \pi]$.

Answer: Domain: $[0,\ 1]$. Range: $[0,\ \pi]$. **[A1]**

b.

- x-coordinate: Midpoint of domain.
- y-coordinate: Midpoint of range.

Answer: $\left(\frac{1}{2},\ \frac{\pi}{2}\right)$. **[H1]**

c.

Let $x = \sin^{-1}(2y-1) + \frac{\pi}{2}$ where $y = f^{-1}(x)$:

$x = \sin^{-1}(2y-1) + \frac{\pi}{2} \quad \Rightarrow x - \frac{\pi}{2} = \sin^{-1}(2y-1)$

$\Rightarrow \sin\left(x - \frac{\pi}{2}\right) = 2y - 1 \quad \Rightarrow \sin\left(x - \frac{\pi}{2}\right) + 1 = 2y$

$\Rightarrow y = \frac{1}{2}\sin\left(x - \frac{\pi}{2}\right) + \frac{1}{2}$.

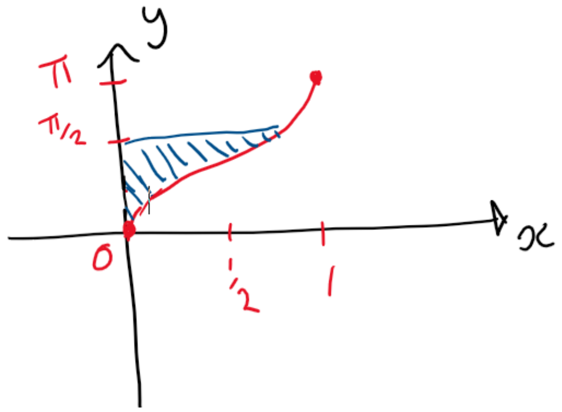
Answer: $f^{-1}(x) = \frac{1}{2}\sin\left(x - \frac{\pi}{2}\right) + \frac{1}{2}$. **[H1]**

Accept all correct alternative forms including $-\frac{1}{2}\cos(x) + \frac{1}{2}$.

d.

- Draw a simple graph of $y = \sin^{-1}(2x-1) + \frac{\pi}{2}$ (use the answers to **part a.**)

to see the area that is to be rotated:



- Required volume: $V = \pi \int_0^{\frac{\pi}{2}} x^2 dy$.

- Substitute $x = \frac{1}{2}\sin\left(y - \frac{\pi}{2}\right) + \frac{1}{2}$ (from **part a.**):

$$V = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \left(\sin\left(y - \frac{\pi}{2}\right) + 1\right)^2 dy.$$ **[M1]**

Method 1: Note that $\sin\left(y - \frac{\pi}{2}\right) = -\cos(y)$.

$$V = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} (1-\cos(y))^2 dy \qquad = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} 1 - 2\cos(y) + \cos^2(y)\ dy$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} 1 - 2\cos(u) + \frac{1+\cos(2u)}{2}\ du$$ **[M1]**

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{3}{2} - 2\cos(u) + \frac{1}{2}\cos(2u)\ du \qquad = \frac{\pi}{8}\int_0^{\frac{\pi}{2}} 3 - 4\cos(u) + \cos(2u)\ du$$

$$= \frac{\pi}{8}\left[3u - 4\sin(u) + \frac{1}{2}\sin(2u)\right]_0^{\frac{\pi}{2}} \qquad = \frac{\pi}{8}\left[3u - 4\sin(u) + \frac{1}{2}\sin(2u)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{8}\left\{\left(\frac{3\pi}{2} - 4\right) - (0)\right\} \qquad = \frac{3\pi^2}{16} - \frac{\pi}{2}.$$

Method 2: Substitute $u = y - \frac{\pi}{2}$ to simplify the calculations:

$$V = \frac{\pi}{4}\int_{-\frac{\pi}{2}}^{0} (\sin(u)+1)^2\, 2du \qquad = \frac{\pi}{4}\int_{-\frac{\pi}{2}}^{0} \sin^2(u) + 2\sin(u) + 1\ du$$

$$= \frac{\pi}{4}\int_{-\frac{\pi}{2}}^{0} \frac{1-\cos(2u)}{2} + 2\sin(u) + 1\ du$$ **[M1]**

$$= \frac{\pi}{8}\int_{-\frac{\pi}{2}}^{0} -\cos(2u) + 4\sin(u) + 3\ du \qquad = \frac{\pi}{8}\left[-\frac{1}{2}\sin(2u) - 4\cos(u) + 3u\right]_{-\frac{\pi}{2}}^{0}$$

$$= \frac{\pi}{8}\left\{(-4) - \left(-\frac{3\pi}{2}\right)\right\} \qquad = \frac{\pi}{8}\left\{\frac{3\pi}{2} - 4\right\} \qquad = \frac{3\pi^2}{16} - \frac{\pi}{2}.$$

Answer: $\frac{3\pi^2}{16} - \frac{\pi}{2}$ cubic units. **[A1]**

Accept all correct alternative forms including $\frac{3\pi^2 - 8\pi}{16}$.

Question 8

a.

- $\underset{\sim}{v}(t)=\cos(t)\underset{\sim}{i}+\left(\sin(2t)-1\right)\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{r}(t)=\sin(t)\underset{\sim}{i}+\left(-\frac{1}{2}\cos(2t)-t\right)\underset{\sim}{j}+\underset{\sim}{C}$. **[M1]**

- Substitute $\underset{\sim}{r}(0)=\underset{\sim}{i}+2\underset{\sim}{j}$:

$-\frac{1}{2}\underset{\sim}{j}+\underset{\sim}{C}=\underset{\sim}{i}+2\underset{\sim}{j} \qquad \Rightarrow \underset{\sim}{C}=\underset{\sim}{i}+\frac{5}{2}\underset{\sim}{j}$.

- Substitute $\underset{\sim}{C}=\underset{\sim}{i}+\frac{5}{2}\underset{\sim}{j}$:

$\underset{\sim}{r}(t)=\sin(t)\underset{\sim}{i}+\left(-\frac{1}{2}\cos(2t)-t\right)\underset{\sim}{j}+\underset{\sim}{i}+\frac{5}{2}\underset{\sim}{j}$

$=\left(1+\sin(t)\right)\underset{\sim}{i}+\left(-\frac{1}{2}\cos(2t)-t+\frac{5}{2}\right)\underset{\sim}{j}$

Answer: $\left(1+\sin(t)\right)\underset{\sim}{i}+\left(-\frac{1}{2}\cos(2t)-t+\frac{5}{2}\right)\underset{\sim}{j}$. **[A1]**

b.

- $\underset{\sim}{v}(t) = \cos(t)\,\underset{\sim}{i} + \left(\sin(2t) - 1\right)\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{a}(t) = -\sin(t)\,\underset{\sim}{i} + 2\cos(2t)\,\underset{\sim}{j}.$ **[A1]**

- $\underset{\sim}{v}(t) \cdot \underset{\sim}{a}(t) = -\frac{1}{2}$

$\Rightarrow \left(\cos(t)\,\underset{\sim}{i} + \left(\sin(2t) - 1\right)\underset{\sim}{j}\right) \cdot \left(-\sin(t)\,\underset{\sim}{i} + 2\cos(2t)\,\underset{\sim}{j}\right) = -\frac{1}{2}$

$\Rightarrow -\cos(t)\sin(t) + 2\left(\sin(2t) - 1\right)\cos(2t) = -\frac{1}{2}$ **[M1]**

$\Rightarrow \sin(2t) - 4\left(\sin(2t) - 1\right)\cos(2t) = 1$

$\Rightarrow \left(\sin(2t) - 1\right) - 4\left(\sin(2t) - 1\right)\cos(2t) = 0$

$\Rightarrow \left(\sin(2t) - 1\right)\left(1 - 4\cos(2t)\right) = 0, \quad t \geq 0.$ **[M1]**

Case 1: $\sin(2t) - 1 = 0 \qquad \Rightarrow \sin(2t) = 1 \qquad \Rightarrow t = \frac{\pi}{4}$

(since $t \geq 0$ and only the first time is required).

Case 2: $1 - 4\cos(2t) = 0 \qquad \Rightarrow \cos(2t) = \frac{1}{4} \qquad \Rightarrow t = \frac{1}{2}\cos^{-1}\left(\frac{1}{4}\right)$

(since $t \geq 0$ and only the first time is required).

- $t = \frac{1}{2}\cos^{-1}\left(\frac{1}{4}\right) \quad < \frac{1}{2}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad = \frac{\pi}{8} < \frac{\pi}{4}$

therefore the first time is $t = \frac{1}{2}\cos^{-1}\left(\frac{1}{4}\right)$.

Answer: $t = \frac{1}{2}\cos^{-1}\left(\frac{1}{4}\right)$. **[A1]**

Question 9

a.

Calculate $\int x \arctan\left(\frac{1}{x}\right) dx$ using integration by parts: $\int u\, dv = uv - \int v\, du$.

Let $u = \arctan\left(\frac{1}{x}\right) \quad \Rightarrow \frac{du}{dx} = \underbrace{\frac{1}{1+\left(\frac{1}{x}\right)^2} \times \left(-\frac{1}{x}\right)}_{\text{using the chain rule}} \quad = \frac{-1}{1+x^2}$ **[M1]**

$$\Rightarrow du = \frac{-dx}{1+x^2}.$$

Let $dv = xdx \quad \Rightarrow \frac{dv}{dx} = x \quad \Rightarrow v = \frac{1}{2}x^2.$

$$\int x \arctan\left(\frac{1}{x}\right) dx \quad = \underbrace{\arctan\left(\frac{1}{x}\right)}_{u} \times \underbrace{\frac{1}{2}x^2}_{v} - \int \underbrace{\frac{1}{2}x^2}_{v} \underbrace{\left(\frac{-dx}{x^2+1}\right)}_{du}$$ **[M1]**

$$= \frac{1}{2}x^2 \arctan\left(\frac{1}{x}\right) + \frac{1}{2}\int \frac{x^2}{x^2+1}\, dx$$

$$= \frac{1}{2}x^2 \arctan\left(\frac{1}{x}\right) + \frac{1}{2}\int 1 - \frac{1}{x^2+1}\, dx$$

$$= \frac{1}{2}x^2 \arctan\left(\frac{1}{x}\right) + \frac{x}{2} - \frac{1}{2}\arctan(x) + c. \qquad \ldots. (1)$$

Answer: $\frac{1}{2}x^2 \arctan\left(\frac{1}{x}\right) + \frac{x}{2} - \frac{1}{2}\arctan(x) + c$. **[A1]**

Since only *an* antiderivative is required, c can have any value including zero.

b.

- $v\dfrac{dv}{dx} = x\arctan\left(\dfrac{1}{x}\right)$

$\Rightarrow \int v\,dv = \int x\arctan\left(\dfrac{1}{x}\right)dx$

$\Rightarrow \dfrac{1}{2}v^2 = \int x\arctan\left(\dfrac{1}{x}\right)dx$. .... (2)

- Substitute (1) from **part a.** into (2):

$\dfrac{1}{2}v^2 = \dfrac{1}{2}x^2\arctan\left(\dfrac{1}{x}\right) + \dfrac{x}{2} - \dfrac{1}{2}\arctan(x) + c$. **[H1]**

- Substitute $v = 1$ when $x = 1$: $\dfrac{1}{2} = \dfrac{1}{2}\arctan(1) + \dfrac{1}{2} - \dfrac{1}{2}\arctan(1) + c \quad \Rightarrow c = 0$.

Substitute $c = 0$:

$\dfrac{1}{2}v^2 = \dfrac{1}{2}x^2\arctan\left(\dfrac{1}{x}\right) + \dfrac{x}{2} - \dfrac{1}{2}\arctan(x)$

$\Rightarrow v^2 = x^2\arctan\left(\dfrac{1}{x}\right) + x - \arctan(x)$.

- Substitute $x = \sqrt{3}$:

$v^2 = 3\arctan\left(\dfrac{1}{\sqrt{3}}\right) + \sqrt{3} - \arctan\left(\sqrt{3}\right) \quad = 3\left(\dfrac{\pi}{6}\right) + \sqrt{3} - \dfrac{\pi}{3} \quad = \sqrt{3} + \dfrac{\pi}{6}$

$\Rightarrow |v| = \sqrt{\sqrt{3} + \dfrac{\pi}{6}}$ ms^{-1}.

Answer: $\sqrt{\sqrt{3} + \dfrac{\pi}{6}}$ ms^{-1}. **[A1]**

Not consequential.

Question 10

$\sin(3\theta)=\sin(2\theta) \qquad \Rightarrow \sin(\theta+2\theta)=2\sin(\theta)\cos(\theta)$

$\Rightarrow \sin(\theta)\cos(2\theta)+\sin(2\theta)\cos(\theta)=2\sin(\theta)\cos(\theta)$ **[M1]**

$\Rightarrow \sin(\theta)\left(2\cos^2(\theta)-1\right)+2\sin(\theta)\cos^2(\theta)=2\sin(\theta)\cos(\theta)$

$\Rightarrow \sin(\theta)\left(2\cos^2(\theta)-1\right)+2\sin(\theta)\cos^2(\theta)-2\sin(\theta)\cos(\theta)=0$

$\Rightarrow \sin(\theta)\left(2\cos^2(\theta)-1+2\cos^2(\theta)-2\cos(\theta)\right)=0$

$\Rightarrow \sin(\theta)\left(4\cos^2(\theta)-2\cos(\theta)-1\right)=0$.

Case 1: $\sin(\theta)=0$

$\Rightarrow \cos(\theta)=\pm 1$. **[A1]**

Case 2: $4\cos^2(\theta)-2\cos(\theta)-1=0$

$\Rightarrow \cos(\theta)=\dfrac{2\pm\sqrt{20}}{8}=\dfrac{1\pm\sqrt{5}}{4}$. **[A1]**

See Appendix 2 for further discussion.

END OF SOLUTIONS

Appendix 1:

Discussion on diagonal asymptotes:

Definition:

A line $y = mx + c$ is a diagonal asymptote of the function $y = g(x)$ as $x \to +\infty$ if $\lim\limits_{x \to +\infty} \left(g(x) - [mx + c] \right) = 0$.

Similarly when $x \to -\infty$: $\lim\limits_{x \to -\infty} \left(g(x) - [mx + c] \right) = 0$.

Note: If $m = 0$ then the line is a horizontal asymptote.

When $g(x) = \dfrac{x^3 + 1}{x^2 - 4} = x + \dfrac{4x + 1}{x^2 - 4}$: $\lim\limits_{|x| \to +\infty} \left(g(x) - x \right) = \lim\limits_{|x| \to +\infty} \left(x + \dfrac{4x+1}{x^2 - 4} - x \right) = \lim\limits_{|x| \to +\infty} \left(\dfrac{4x+1}{x^2-4} \right) = 0$.

Therefore the line $y = x$ is a diagonal asymptote of $g(x) = \dfrac{x^3+1}{x^2-4}$ as $x \to \pm\infty$.

Note: It follows from the definition that

- $\lim\limits_{x \to +\infty} \dfrac{g(x)}{x} = m$. .... (1)
- $\lim\limits_{x \to +\infty} \left(g(x) - mx \right) = c$. .... (2)

Similarly when $x \to -\infty$:

- $\lim\limits_{x \to -\infty} \dfrac{g(x)}{x} = m$. .... (1')
- $\lim\limits_{x \to -\infty} \left(g(x) - mx \right) = c$. .... (2')

Proof of (1):

$$\lim_{x \to +\infty} \left(g(x) - [mx + c] \right) = 0 \quad \Rightarrow \lim_{x \to +\infty} \left(\frac{g(x) - [mx+c]}{x} \right) = 0 \quad \Rightarrow \lim_{x \to +\infty} \frac{g(x)}{x} = \lim_{x \to +\infty} \frac{mx+c}{x} \quad \Rightarrow \lim_{x \to +\infty} \frac{g(x)}{x} = \lim_{x \to +\infty} \frac{mx+c}{x}$$

$$\Rightarrow \lim_{x \to +\infty} \frac{g(x)}{x} = m + \lim_{x \to +\infty} \frac{c}{x} \quad \Rightarrow \lim_{x \to +\infty} \frac{g(x)}{x} = m.$$

Turning points:

Calculations to prove that $g(x) = \dfrac{x^3+1}{x^2-4}$ has four turning points including a turning point at $x = 0$.

Stationary points of $g(x) = \dfrac{x^3+1}{x^2-4}$ are found by solving

$$g'(x) = \frac{3x^2\left(x^2-4\right) - 2x\left(x^3+1\right)}{\left(x^2-4\right)^2} = \frac{x^4 - 12x^2 - 2x}{\left(x^2-4\right)^2} = 0:$$

$$x^4 - 12x^2 - 2x = 0 \qquad \Rightarrow x\left(x^3 - 12x - 2\right) = 0.$$

Case 1: $x = 0$.

Nature of stationary point:

Using the sign test could lead to the wrong conclusion because the interval defined by simple values of x to the left and to the right of $x = 0$ might contain another stationary point.

Using the double derivative test is the safest way of determining the nature of the stationary point at $x = 0$:

$$g''(x) = \frac{\left(4x^3 - 24x - 2\right)\left(x^2 - 4\right)^2 - 2\left(x^2 - 4\right)(2x)\left(x^4 - 12x^2 - 2x\right)}{\left(x^2 - 4\right)^4} \qquad \Rightarrow g''(0) = \frac{(-2)(-4)^2}{(-4)^4} = -\frac{2}{16} = -\frac{1}{8} < 0$$

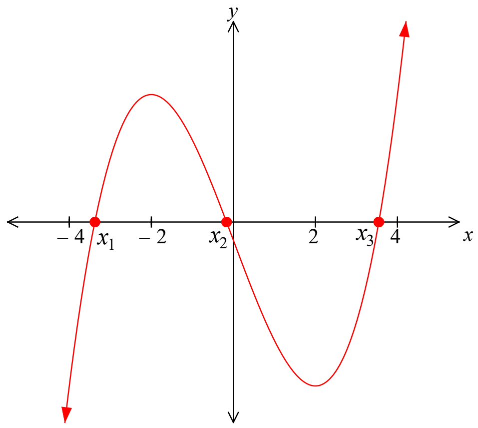
therefore there is a maximum turning point at $x = 0$.

Case 2: $x^3 - 12x - 2 = 0$. …. (1)

The rational root theorem shows that equation (1) has no rational solution so finding real solutions (there is at least one real solution) is challenging. However, the number of real solutions to equation (1) and their sign can be determined by considering a graph of $y = h(x) = x^3 - 12x - 2$:

$\frac{dy}{dx} = 3x^2 - 12 = 0 \quad \Rightarrow x = \pm 2$.

$x = 2 \Rightarrow y = -18$. $\quad x = -2 \Rightarrow y = 14$.



By inspection it can be seen that there are three x-intercepts: $x = x_1$, x_2, x_3.

Therefore there are three real solutions to equation (1): two negative solutions ($x = x_1$ and $x = x_2$) and one positive solution ($x = x_3$). Therefore as well as a maximum turning point at $x = 0$, $g(x)$ has stationary points at $x = x_1$, $x = x_2$ and $x = x_3$.

Furthermore, since $h\left(-\frac{1}{2}\right) = \frac{31}{8} > 0$, $h(0) = -2 < 0$ and $h(x) = x^3 - 12x - 2$ is continuous, it follows that $x_2 \in \left(-\frac{1}{2}, 0\right)$.

Nature of stationary points of $g(x)$ at $x = x_1$, $x = x_2$ and $x = x_3$:

$$g'(x) = \frac{x\left(x^3 - 12x - 2\right)}{\left(x^2 - 4\right)^2} = \frac{xh(x)}{+ve}.$$

Therefore the graph of $y = h(x)$ can be used to construct a sign test.

For example, to determine the nature of the stationary point of at $x = x_2$:

x	$x_2 - \varepsilon < 0$	x_2	$x_2 + \varepsilon < 0$
$h(x)$	$+ve$	0	$-ve$
$xh(x)$	$-ve$	0	$+ve$
$g'(x) = \dfrac{xh(x)}{+ve}$	< 0	0	> 0

ε is arbitrarily small and positive

Use the graph of $y = h(x)$ to get the sign of $h(x)$ on each side of x_2

Therefore $g(x)$ has a minimum turning point at $x = x_2$.

Appendix 2:

Question 10 - Further discussion:

The answers to **Question 10** allows the calculation of $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$.

Solve $\sin(3\theta) = \sin(2\theta)$ using the symmetry of the unit circle:

Case 1: $3\theta = 2\theta + 2n\pi$, $n \in Z$

$\Rightarrow \theta = 2n\pi$.

Case 2: $3\theta = (\pi - 2\theta) + 2n\pi$, $n \in Z$

$\Rightarrow 5\theta = (2n+1)\pi$ $\Rightarrow \theta = \frac{(2n+1)\pi}{5}$

From **Case 2** ($n = 0$): $\theta = \frac{\pi}{5}$ is a solution to $\sin(3\theta) = \sin(2\theta)$.

Therefore (from the answer to **Question 10**) the possible values of $\cos\left(\frac{\pi}{5}\right)$ are ± 1 and $\frac{1 \pm \sqrt{5}}{4}$.

But $0 < \cos\left(\frac{\pi}{5}\right) < 1$ therefore the values ± 1 and $\frac{1-\sqrt{5}}{4}$ are rejected. Therefore $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$.