YEAR 12 *Trial Exam Paper*

2023

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

Question 1

Worked solution

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\left(\sin(x)+\cos(x)\right)^2\,dx$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\left(\sin^2(x)+2\sin(x)\cos(x)+\cos^2(x)\right)dx$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\left(2\sin(x)\cos(x)+1\right)dx$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\left(\sin(2x)+1\right)dx$$

$$=\left[x-\frac{1}{2}\cos(2x)\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$=\left(\frac{\pi}{4}-\frac{1}{2}\cos\left(\frac{\pi}{2}\right)\right)-\left(\frac{\pi}{12}-\frac{1}{2}\cos\left(\frac{\pi}{6}\right)\right)$$

$$=\frac{\pi}{4}-0+\frac{\pi}{12}+\frac{\sqrt{3}}{4}$$

$$=\frac{\sqrt{3}}{4}+\frac{\pi}{6}$$

Mark allocation: 3 marks

- 1 mark for the correct simplification of integrand
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

Note: Any other correct equivalent integral and antiderivative should be given full marks.

Question 2a.

Worked solution

A vector that is perpendicular to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ can be determined using the vector cross product.

$$\underset{\sim}{a} \times \underset{\sim}{b} = (2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}) \times (\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$$

$$= \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -3 & 4 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix}\underset{\sim}{i} - \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix}\underset{\sim}{j} + \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}\underset{\sim}{k}$$

$$= (9-8)\underset{\sim}{i} - (-6-4)\underset{\sim}{j} + (4 - -3)\underset{\sim}{k}$$

$$= \underset{\sim}{i} + 10\underset{\sim}{j} + 7\underset{\sim}{k}$$

$$\left|\underset{\sim}{i} + 10\underset{\sim}{j} + 7\underset{\sim}{k}\right| = \sqrt{(1)^2 + (10)^2 + (7)^2} = \sqrt{150} = 5\sqrt{6}$$

Therefore a unit vector that is perpendicular to both vectors is given by

$$\frac{1}{5\sqrt{6}}(\underset{\sim}{i} + 10\underset{\sim}{j} + 7\underset{\sim}{k})$$

Mark allocation: 2 marks

- 1 mark for the correct cross product
- 1 mark for the correct answer

Question 2b.

Worked solution

From part **a**, a vector normal to the plane containing vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\underset{\sim}{n} = \underset{\sim}{i} + 10\underset{\sim}{j} + 7\underset{\sim}{k}$.

Vector $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$ can be taken as the position vector of a point in the plane.

The Cartesian equation of the plane can be found by solving for the constant k in $x + 10y + 7z = k$.

Using O (0, 0, 0):

$$0 + 10(0) + 7(0) = 0 = k$$

Therefore the Cartesian equation of the plane is $x + 10y + 7z = 0$..

Mark allocation: 2 marks

- 1 mark for choosing a correct point on the plane and vector normal to plane
- 1 mark for writing the correct Cartesian equation of the plane

Note: Any other point on the plane could be used to determine the value of k .

Question 2c.

Worked solution

The position vector of P is $\overrightarrow{OP} = -2\underset{\sim}{i} + 2\underset{\sim}{j} + 1\underset{\sim}{k}$.

The distance from P to the plane is given by:

$d = \left|\overrightarrow{OP} \cdot \hat{n}\right|$ where $\hat{n}$ is a unit vector perpendicular to the plane.

From part **a**, $\hat{n} = \frac{\sqrt{6}}{30}(\underset{\sim}{i} + 10\underset{\sim}{j} + 7\underset{\sim}{k})$

$$d = \left|\left(-2\underset{\sim}{i} + 2\underset{\sim}{j} + 1\underset{\sim}{k}\right) \cdot \frac{\sqrt{6}}{30}\left(\underset{\sim}{i} + 10\underset{\sim}{j} + 7\underset{\sim}{k}\right)\right| = \left|\frac{\sqrt{6}}{30}(-2 + 20 + 7)\right| = \frac{5}{\sqrt{6}}$$

Mark allocation: 2 marks

- 1 mark for the correct vector expression for the distance from the plane to P
- 1 mark for the correct answer for d

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Question 3a.

Worked solution

The given differential equation is separable and can be rearranged into the form

$$g(A)dA = h(t)dt$$

$$A^{-\frac{3}{2}}dA = t^{-2}dt$$

A general solution can then be obtained by integrating the left- and right-hand sides with reference to *A* and *t* respectively:

$$\int A^{-\frac{3}{2}}dA = \int t^{-2}dt$$

$$-2A^{-\frac{1}{2}} = -t^{-1} + C$$

A particular solution to the differential equation can then be determined by substituting the initial conditions into the general solution:

$$-2\left(\frac{1}{4}\right)^{-\frac{1}{2}} = -(1)^{-1} + C$$

$$-4 = -1 + C$$

$$C = -3$$

$$-2A^{-\frac{1}{2}} = -t^{-1} - 3$$

$$-2A^{-\frac{1}{2}} = -t^{-1} - 3$$

$$A^{-\frac{1}{2}} = \frac{1}{2t} + \frac{3}{2}$$

$$A^{-\frac{1}{2}} = \frac{1+3t}{2t}$$

$$A = \left(\frac{2t}{1+3t}\right)^2$$

$$A = \frac{4t^2}{(1+3t)^2}$$

Mark allocation: 3 marks

- 1 mark for setting up the correct integrals to solve the differential equation
- 1 mark for the correct general solution of the differential equation
- 1 mark for the correct particular solution of the differential equation

Question 3b.

Worked solution

As t approaches a very large number, $A(t)$ tends to $\frac{4t^2}{9t^2} = \frac{4}{9}$.

Alternatively, to find the limiting value, $A(t)$ can be expanded and then rearranged using long division into a form where the horizontal asymptote of $A(t)$ can be easily determined:

$$\begin{aligned}A(t) &= \frac{4t^2}{9t^2 - 6t + 1}\\ &= \frac{4}{9} - \frac{4}{9}\left(\frac{6t+1}{9t^2 - 6t + 1}\right)\end{aligned}$$

As t becomes large, the second term of $A(t)$ will approach zero.

Hence the limiting value of $A(t)$ is $\frac{4}{9}\,\text{m}^2$.

Mark allocation: 1 mark

- 1 mark for the correct limiting value

Tip

- *When given a rational function of the form* $f(x) = \frac{ax^2 + bx + c}{dx^2 + ex + f}$*, the quadratic term of the numerator and demoninator will become the dominant term as* x *tends to a very large value.*

Question 4

Worked solution

Start by making substitutions for z and $\overline{z}$ using $z = x + yi$ and $\overline{z} = x - yi$:

$$2(x+yi)^2 + (x-yi)^2 = 27 - 2\sqrt{10}i$$

$$2x^2 + 4xyi - 2y^2 + x^2 - 2xyi - y^2 = 27 - 2\sqrt{10}i$$

$$3x^2 - 3y^2 + 2xyi = 27 - 2\sqrt{10}i$$

Equating the real parts of the equation:

$3x^2 - 3y^2 = 27$

$x^2 - y^2 = 9$

Equating the imaginary parts of the equation:

$2xy = -2\sqrt{10}$

$xy = -\sqrt{10}$

Solving the two simultaneous equations for x and y:

$x^2y^2 = 10$

$x^2 = y^2 + 9$

$y^4 + 9y^2 - 10 = 0$

$(y^2 + 10)(y^2 - 1) = 0$

$y^2 = -10, y^2 = 1$

$y = \pm 1$

We disregard the $y = \pm\sqrt{10}i$ solutions as $y \in \mathbb{R}$.

When $y = 1$, $x = -\sqrt{10}$ and when $y = -1$, $x = \sqrt{10}$.

Therefore, the solutions are $z = -\sqrt{10} + i$ and $z = \sqrt{10} - i$..

Mark allocation: 4 marks

- 1 mark for the correct substitution for z and $\overline{z}$
- 1 mark for setting up correct equations relating x and y
- 2 marks: 1 mark for each correct solution

Tip

- *When solving equations over $\mathbb{C}$ that involve both z and its conjugate $\overline{z}$, make the substitution $z = x + yi$ and $\overline{z} = x - yi$, then consider both the real and imaginary parts of the equation to obtain the solution.*

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Question 5

Worked solution

When $x = 1$,

$$(1)^2 + (y-1)^3 = 9$$
$$(y-1)^3 = 8$$
$$y-1 = 2$$
$$y = 3$$

Using implicit differentiation on both sides of the relation that describes the curve gives:

$$2x + 3(y-x)^2 \times \left(\frac{dy}{dx} - 1\right) = 0$$

Point $(1, 3)$ can be substituted into the equation above to find the value of $\frac{dy}{dx}$ at that point:

$$2(1) + 3(3-1)^2\left(\frac{dy}{dx} - 1\right) = 0$$
$$\left(\frac{dy}{dx} - 1\right) = \frac{-2}{12}$$
$$\frac{dy}{dx} = -\frac{1}{6} + 1$$
$$\frac{dy}{dx} = \frac{5}{6}$$

The gradient of the line normal to the curve at point $(1, 3)$ is therefore $-\frac{6}{5}$.

The equation of the normal line at point $(1, 3)$ can be determined using point-slope form:

$$y - 3 = -\frac{6}{5}(x-1)$$
$$y = -\frac{6}{5}x + \frac{6}{5} + \frac{15}{5}$$
$$y = -\frac{6}{5}x + \frac{21}{5}$$

Mark allocation: 4 marks

- 1 mark for determining the correct y-coordinate of the relation when $x = 1$
- 1 mark for the correct implicit differentiation of both sides of the relation
- 1 mark for the correct gradient of the normal line
- 1 mark for the correct equation of the normal line

Tip

- *When determining equations of normal and tangents, using point-slope form of a line,* $y - y_1 = m(x - x_1)$*, is the most efficient approach.*

Question 6

Worked solution

The specified range of $y \in \left[0, \frac{\sqrt{3}}{9}\right]$ corresponds to a domain $x \in \left[0, \frac{1}{\sqrt{3}}\right]$.

The surface area of the volume of revolution is given by the integral:

$$S = 2\pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} x^3 \sqrt{1+\left(3x^2\right)^2}\,dx$$

$$S = 2\pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} x^3 \sqrt{1+9x^4}\,dx$$

The following substitution for the integrand and terminals can be made:

$$u = 1+9x^4$$

$$\frac{du}{dx} = 36x^3$$

$$\frac{1}{36}du = x^3 dx$$

$$x = \frac{1}{\sqrt{3}} \rightarrow u = 1+9(\frac{1}{\sqrt{3}})^4 = 2$$

$$x = 0 \rightarrow u = 1+0 = 1$$

The required integral then becomes:

$$S = \frac{2\pi}{36} \int_{u=1}^{u=2} u^{\frac{1}{2}} du$$

$$S = \frac{\pi}{18} \int_{u=1}^{u=2} u^{\frac{1}{2}} du$$

$$S = \frac{\pi}{18} \left[\frac{2}{3} u^{\frac{3}{2}}\right]_1^2$$

$$S = \frac{\pi}{18} \cdot \frac{2}{3} \left[(2)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]$$

$$S = \frac{\pi\left(2\sqrt{2}-1\right)}{27}$$

Mark allocation: 4 marks

- 1 mark for the correct integral expression to determine the surface area of the volume of rotation
- 1 mark for the correct u-substitution and change of terminals
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

Tip

- *When using a u-substitution to evaluate a definite integral, do not forget to change the terminals from values of x to values of u.*

Question 7a.

Worked solution

$W_n \sim N(25, 1^2)$

$S_n \sim N(5, 1.5^2)$

$C = W_1 + W_2 + W_3 + W_4 + W_5 + W_6 + W_7 + S_1 + S_2 + S_3 + S_4$

$E(C) = 7 \cdot E(W) + 4 \cdot E(S) = 7 \times 25 + 4 \times 5 = 195 \text{ mL}$

$Var(C) = 7 \cdot Var(W) + 4 \cdot Var(S) = 7 \times (1)^2 + 4 \times (1.5)^2 = 7 + 9 = 16$

$\sigma_C = \sqrt{16} = 4 \text{ mL}$

Mark allocation: 2 marks

- 1 mark for the correct value of the mean
- 1 mark for the correct value of the standard deviation

Question 7b.

Worked solution

$$\Pr(C > 202.84) = \Pr\left(Z > \frac{202.84 - 195}{4}\right) = \Pr\left(Z > \frac{7.84}{4}\right) = \Pr(Z > 1.96) \approx \frac{0.05}{2} = 0.025$$

Mark allocation: 1 mark

- 1 mark for correct value for the probability

Question 8a.

Worked solution

$\underset{\sim}{r}_A(3) = (2+2\times3)\underset{\sim}{i} + (9+3\times3)\underset{\sim}{j} + (15+6\times3)\underset{\sim}{k} = 8\underset{\sim}{i} + 18\underset{\sim}{j} + 33\underset{\sim}{k}$

$\underset{\sim}{r}_B(3) = (1+3)\underset{\sim}{i} + (2+4\times3)\underset{\sim}{j} + (2+8\times3)\underset{\sim}{k} = 4\underset{\sim}{i} + 14\underset{\sim}{j} + 26\underset{\sim}{k}$

$|\underset{\sim}{r}_B(3) - \underset{\sim}{r}_A(3)| = |(4-8)\underset{\sim}{i} + (14-18)\underset{\sim}{j} + (26-33)\underset{\sim}{k}| = |-4\underset{\sim}{i} - 4\underset{\sim}{j} - 7\underset{\sim}{k}|$

$= \sqrt{(-4)^2 + (-4)^2 + (-7)^2} = \sqrt{81} = 9 \text{ km}$

Mark allocation: 2 marks

- 1 mark for correct evaluation of the position vectors at t = 3.
- 1 mark for the correct answer

Question 8b.

Worked solution

The position vector of jet B describes a line with direction vector $\underset{\sim}{d} = \underset{\sim}{i} + 4\underset{\sim}{j} + 8\underset{\sim}{k}$.

The observation tower has position vector $\underset{\sim}{T} = 20\underset{\sim}{i} + 20\underset{\sim}{j}$.

A vector from T to a point on the linear path of jet B can be found by finding:

$\underset{\sim}{r}_{\text{B}}(t) - \underset{\sim}{T} = (t-19)\underset{\sim}{i} + (4t-18)\underset{\sim}{j} + (8t+2)\underset{\sim}{k}$

For the minimum distance from T to the path of jet B, the vector from T must make a right angle with the path.

$\left(\underset{\sim}{r}_{\text{B}}(t) - \underset{\sim}{T}\right) \cdot \underset{\sim}{d} = 1 \times (t-19) + 4 \times (4t-18) + 8 \times (8t+2)$

$= t - 19 + 16t - 72 + 64t + 16$

$= 81t - 75$

$\left(\underset{\sim}{r}_{\text{B}}(t) - \underset{\sim}{T}\right) \cdot \underset{\sim}{d} = 0$

$81t - 75 = 0$

$t = \dfrac{75}{81}$

$t = \dfrac{25}{27}$

Therefore jet B is closest to the observation tower at $t = \dfrac{25}{27}$ hours

Mark allocation: 2 marks

- 1 mark for setting up a vector equation to find the minimum distance
- 1 mark for the correct answer

Question 9

Worked Solution

Let $P(n)$ be the proposition that $4^n + 6n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

Base step Consider $P\ (1) \Rightarrow \text{LHS} = 4^1 + 6 \times 1 - 1 = 9$ which is divisible by 3.

Inductive step Suppose $P(k)$ is true for some k > 1. That is $4^k + 6(k) - 1 = 3m$ for some $m \in \mathbb{N}$. Consider $P(k + 1)$:

$$\begin{aligned} 4^{k+1} + 6(k+1) - 1 &= 4 \bullet 4^k + 6k + 5 \\ &= 4\underbrace{(3m - 6k + 1)}_{\text{as } 4^k = 3m - 6k + 1} + 6k + 5 \\ &= 12m - 18k + 9 \\ &= 3(4m - 6k + 3) \end{aligned}$$

which is divisible by 3.

So, $P(1)$ is true and $P(k)$ true implies $P(k + 1)$ is true. Therefore, by the principle of mathematical induction, $4^n + 6n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

Mark allocation: 3 marks

- 1 mark for showing that the proposition is true for $n = 1$
- 1 mark for writing the correct proposition for $n = k$
- 1 mark for showing that if the proposition is true for $n = k$, it is also true for $n = k + 1$

Question 10

Worked solution

The required area will be given by the integral $A = \int_0^{\pi} e^x \sin(x)dx.$

To evaluate A, integration by parts must be used twice, and the results combined to obtain an antiderivative.
For the first iteration of integration by parts:

$$u = e^x$$

$$du = e^x dx$$

$$dv = \sin(x)$$

$$v = \int \sin(x)dx = -\cos(x)$$

$$\int e^x \sin(x)dx = -e^x \cos(x) + \int e^x \cos(x)dx [1]$$

Integration by parts can be employed a second time to find $\int e^x \cos(x)dx$.

$$u = e^x$$

$$du = e^x dx$$

$$dv = \cos(x)$$

$$v = \int \cos(x)dx = \sin(x)$$

$$\int e^x \cos(x)dx = e^x \sin(x) - \int e^x \sin(x)dx \ [2]$$

Result [2] can be substituted into result [1] to find the required antiderivative:

$$\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx = -e^x \cos x + \left[e^x \sin x - \int e^x \sin x dx \right]$$

$$\int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$2\int e^x \sin x dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2}$$

This antiderivative can then be used to find the value of A:

$$A = \int_0^{\pi} e^x \sin x dx = \left[\frac{e^x(\sin x - \cos x)}{2} \right]_0^{\pi}$$

$$= \left(\frac{e^{\pi}\left(\sin(\pi) - \cos(\pi)\right)}{2} \right) - \left(\frac{e^0\left(\sin(0) - \cos(0)\right)}{2} \right)$$

$$= \left(\frac{e^{\pi}}{2} \right) - \left(-\frac{1}{2} \right) = \frac{e^{\pi} + 1}{2}$$

Mark allocation: 5 marks

- 1 mark for the correct integral expression for the required area
- 1 mark for the correct first application of integration by parts
- 1 mark for the correct second application of integration by parts
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

END OF WORKED SOLUTIONS