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SPECIALIST MATHS UNITS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2023

Section A – Multiple-choice answers

1.	D	**6.**	D	**11.**	A	**16.**	E
2.	A	**7.**	E	**12.**	E	**17.**	D
3.	A	**8.**	B	**13.**	A	**18.**	B
4.	B	**9.**	D	**14.**	A	**19.**	C
5.	B	**10.**	C	**15.**	C	**20.**	B

Section A - Multiple-choice solutions

Question 1

P is the statement 'a number is a multiple of 6'.
Q is the statement 'a number is a multiple of 3'.
Not P is the statement 'a number is not a multiple of 6'.
Not Q is the statement 'a number is not a multiple of 3'.
The contrapositive of $P \Rightarrow Q$ is the statement $(\text{not } Q) \Rightarrow (\text{not } P)$, that is,
'If a number is not a multiple of 3, then it is not a multiple of 6.'
The answer is D.

Question 2

<u>Method 1</u>
The pseudocode will go through the 'for' loop three times.
The first time:

```
c = a×b = 3×2 = 6
b = c = 6
a = a+1 = 4
print c
```
prints the number 6, so first number in the list is 6.

The second time:

```
c = a×b = 4×6 = 24
b = c = 24
a = a+1 = 5
print c
```
prints the number 24, so the second number in the list is 24.

The third time:

```
c = a×b = 5×24 = 120
b = c = 120
a = a+1 = 6
print c
```
prints the number 120, so the third number in the list is 120.

So the output of the pseudocode is 6, 24, 120.
The answer is A.

<u>Method 2</u> - using a table

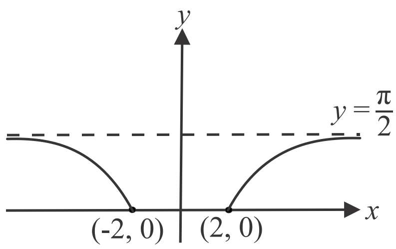
The pseudocode will go through the 'for' loop three times.

n	c	b	a
1	$3\times 2 = 6$	6	$3+1=4$
2	$4\times 6 = 24$	24	$4+1=5$
3	$5\times 24 = 120$	120	$5+1=6$

The three values of c are printed i.e. 6, 24, 120.
The answer is A.

Question 3

Do a quick sketch of the graph on the CAS.
This shows a domain of
$x \in (-\infty, -2] \cup [2, \infty)$.
This is equivalent to $R \setminus (-2, 2)$.



For the range, $f(2) = 0$ and $f(-2) = 0$.

Also, as $x \to \infty$, $\frac{2}{|x|} \to 0^+$, $f(x) \to \left(\frac{\pi}{2}\right)^-$

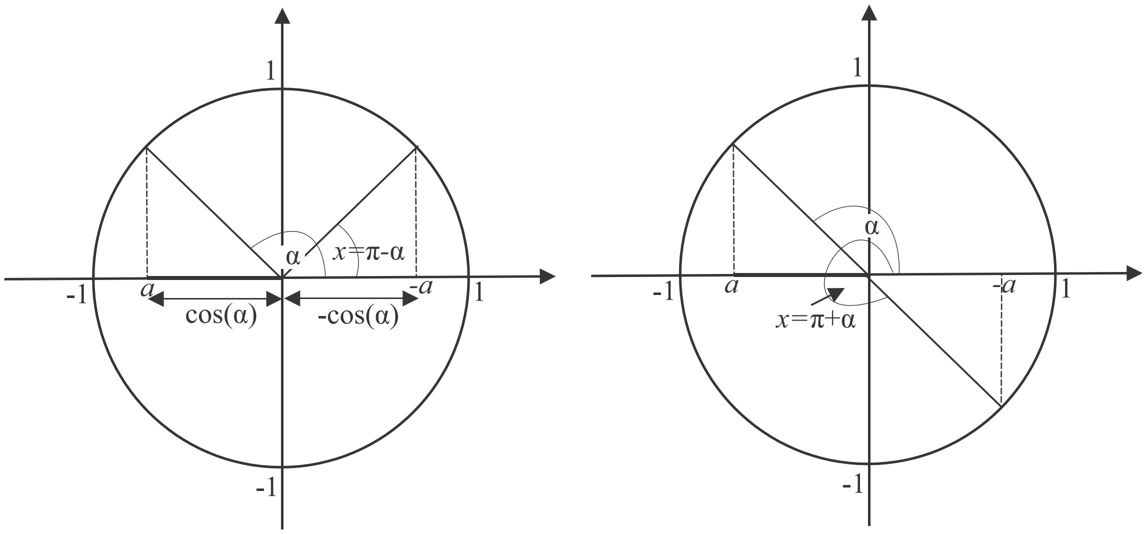
and as $x \to -\infty$, $\frac{2}{|x|} \to 0^+$, $f(x) \to \left(\frac{\pi}{2}\right)^-$.

range $= \left[0, \frac{\pi}{2}\right)$

The answer is A.

Question 4

If $\cos(\alpha)=a$, $\alpha\in\left(\frac{\pi}{2},\pi\right)$, then the angle α is in quadrant 2 and a must be a negative number.
Since $\cos(x)=-a$, then by symmetry, x must be a first or fourth quadrant angle, as shown in the diagrams below.



When x is a first quadrant angle then $x=\pi-\alpha$.
When x is a fourth quadrant angle then $x=\pi+\alpha$.

The answer is B.

Question 5

The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by $\dfrac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{\underset{\sim}{b}\bullet\underset{\sim}{b}}\underset{\sim}{b}$.

$\underset{\sim}{b}\bullet\underset{\sim}{b}=2^2+(-1)^2+2^2=9$, therefore the vector resolute is $\dfrac{\underset{\sim}{a}\bullet\underset{\sim}{b}}{9}(2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k})$.

So for the vector resolute to equal $2\underset{\sim}{b}$, we need $\underset{\sim}{a}\bullet\underset{\sim}{b}=18$.

If we let $\underset{\sim}{a}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$, then $\underset{\sim}{a}\bullet\underset{\sim}{b}=2x-y+2z$.

Look through the options to see which one has $2x-y+2z=18$.

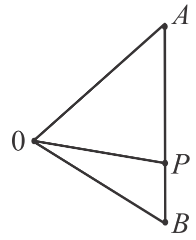
For $\underset{\sim}{a}=5\underset{\sim}{i}+4\underset{\sim}{j}+6\underset{\sim}{k}$, we have $2\times5-4+2\times6=18$.

The answer is B.

Question 6

<u>Method 1</u>

Draw a quick sketch of triangle *OAB* with point *P* on line *AB*.



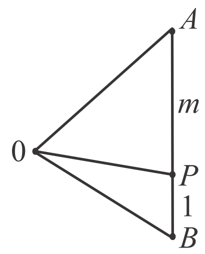
$$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$$
$$= -\underset{\sim}{a} + \frac{1}{5}\underset{\sim}{a} + \frac{4}{5}\underset{\sim}{b}$$
$$= -\frac{4}{5}\underset{\sim}{a} + \frac{4}{5}\underset{\sim}{b}$$
$$\overrightarrow{PB} = \overrightarrow{PO} + \overrightarrow{OB}$$
$$= -\left(\frac{1}{5}\underset{\sim}{a} + \frac{4}{5}\underset{\sim}{b}\right) + \underset{\sim}{b}$$
$$= -\frac{1}{5}\underset{\sim}{a} + \frac{1}{5}\underset{\sim}{b}$$

So $m = 4$.
The answer is D.

<u>Method 2</u>

Draw a quick sketch of triangle *OAB* with point *P* on line *AB* and using a scale of $\left|\overrightarrow{PB}\right| = 1$.

From the diagram,



$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$
$$= \overrightarrow{OA} + m\overrightarrow{PB}$$
$$= \overrightarrow{OA} + \frac{m}{m+1}\overrightarrow{AB}$$
$$= \underset{\sim}{a} + \frac{m}{m+1}(\underset{\sim}{b} - \underset{\sim}{a})$$
$$= \left(1 - \frac{m}{m+1}\right)\underset{\sim}{a} + \frac{m}{m+1}\underset{\sim}{b}$$

Equate $\underset{\sim}{a}$ and $\underset{\sim}{b}$ components.

Solve $1 - \frac{m}{m+1} = \frac{1}{5}$ and $\frac{m}{m+1} = \frac{4}{5}$ simultaneously.

So $m = 4$.
The answer is D.

Question 7

If $z = a + bi$, check each to see whether they are real or not for $a, b \in R$.

Option A – $\text{Re}(z) \times \text{Im}(z) = a \times b$ which is real.

Option B – $z + \overline{z} = a + bi + a - bi = 2a$ which is real.

Option C – $z\,\overline{z} = (a+bi)(a-bi) = a^2 + b^2$ which is real.

Option D – $\text{Im}(z) \times i^8 = bi^8 = b$ which is real.

Option E – $\text{Re}(\overline{z}) \times \overline{z} = a(a - bi) = a^2 - abi$ which has an imaginary component.

The answer is E.

Question 8

Method 1

Choose a convenient value of z for example $z = i$.

When $z = i,\ \theta = \frac{\pi}{2}$ and $\text{Arg}(z+1) = \text{Arg}(i+1) = \frac{\pi}{4} = \frac{\theta}{2}$.

The answer is B.

Method 2

If $z = \text{cis}(\theta)$, then $z+1 = \text{cis}(\theta)+1$

$$= \cos(\theta) + i\sin(\theta) + 1.$$

$$= \cos(\theta) + 1 + i\sin(\theta)$$

Therefore $\quad \text{Arg}(z+1) = \tan^{-1}\left(\frac{\sin(\theta)}{\cos(\theta)+1}\right).$

If we let $\quad \text{Arg}(z+1) = \alpha$, then $\tan\alpha = \frac{\sin(\theta)}{\cos(\theta)+1}$.

Using the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$, we have

$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right).$

Similarly, given the identity $\cos(2x) = 2\cos^2(x) - 1$, we have $\cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1$.

Therefore, $\tan\alpha = \dfrac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right) - 1 + 1}$

$$= \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \qquad \left(\cos\left(\frac{\theta}{2}\right) \neq 0\right)$$

$$\tan\alpha = \tan\left(\frac{\theta}{2}\right)$$

So $\alpha = \frac{\theta}{2}$.

The answer is B.

Question 9

$|z - 3 + i| = |z + 2 + 2i|$ is equivalent to $|z - (3 - i)| = |z - (-2 - 2i)|$.

This is a straight line which is the perpendicular bisector of the points $(3, -1)$ and $(-2, -2)$.

The answer is D.

Question 10

The angle between two planes is equal to the angle between the normal of each plane.

For $\prod_1$, $\underset{\sim}{n}_1 = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ and for $\prod_2$, $\underset{\sim}{n}_2 = \underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$.

Use the CAS function 'angle' to determine the angle between the two vectors $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$, or alternatively, use the dot product.

$$\underset{\sim}{n}_1 \bullet \underset{\sim}{n}_2 = |\underset{\sim}{n}_1||\underset{\sim}{n}_2|\cos(\theta)$$

$$2\times1+3\times5+1\times{}^-3 = \sqrt{2^2+3^2+1^2}\times\sqrt{1^2+5^2+(-3)^2}\cos(\theta)$$

$$\cos(\theta) = \frac{14}{\sqrt{14}\sqrt{35}}$$

$$\theta = 50.77° \text{ (correct to two decimal places)}$$

The answer is C.

Question 11

If a plane is perpendicular to a line, then a vector in the direction of the line is normal to the plane.
Hence, since $2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$ is the direction of the line, we can ascertain that $\underset{\sim}{n} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$.
Therefore the Cartesian equation of the plane is of the form $2x + 3y + z = k$ for $k \in R$.
Substitute the point $(-2,1,4)$ into this equation to find *k*.

$$2\times{}^-2 + 3\times1 + 4 = k$$

$$k = 3$$

Hence the equation of the plane is given by $2x + 3y + z = 3$.
The answer is A.

Question 12

Using CAS, $f''(x) = \dfrac{2a^2 - 6a + 4}{(x-a)^3}$.

A necessary condition for a point of inflection is $f''(x) = 0$. Solving this equation for *x* yields no solution. Solving $2a^2 - 6a + 4 = 0$ gives us $a = 1$ or $a = 2$.
However, since $f(x)$ can be written as $f(x) = \dfrac{(x-2)(x-1)}{x-a}$, if $a = 1$, $f(x) = x - 2$ and if $a = 2$, $f(x) = x - 1$. Both cases give us a linear equation, the graphs of which have a point of discontinuity, but clearly neither graph will have a point of inflection.
So there are no values of *a* for which the graph of *f* has a point of inflection, i.e. all values of *a* will give a graph with no point of inflection.
The graph of *f* will have no points of inflection for $a \in R$.
The answer is E.

Question 13

The gradient of the line joining points A and B is given by $\frac{y-3}{x-2}$.

Half of this gradient is therefore $\frac{1}{2}\times\frac{y-3}{x-2}=\frac{y-3}{2(x-2)}$.

The gradient of the tangent to a curve at $A(x, y)$ is $\frac{dy}{dx}$, and so $\frac{dy}{dx}=\frac{y-3}{2(x-2)}$ which leads us to $\frac{dy}{dx}-\frac{y-3}{2(x-2)}=0$.

The answer is A.

Question 14

Given $a=\sin^{-1}\left(\frac{x}{2}\right)$, then $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\sin^{-1}\left(\frac{x}{2}\right)$ $\qquad a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ from formula sheet

$$\frac{1}{2}v^2=\int\sin^{-1}\left(\frac{x}{2}\right)dx$$

$$\frac{1}{2}v^2=x\sin^{-1}\left(\frac{x}{2}\right)+\sqrt{4-x^2}+c \qquad \text{using CAS}$$

Given $v=2$ when $x=0$, $\quad 2=0+\sqrt{4}+c$

$$c=0$$

Therefore when $x=1$, $\quad \frac{v^2}{2}=\sin^{-1}\left(\frac{1}{2}\right)+\sqrt{3}$

$$\frac{v^2}{2}=2.2556...$$

Solving for v gives $v=\pm2.123...$, and since speed $=|v|$, then speed is 2.12 ms^{-1} (correct to 2 decimal places)

The answer is A.

Question 15

$$\int_1^3 (x-2)\sqrt{2x+1}\ dx$$

Using the substitution $u = 2x + 1$, $\frac{du}{dx} = 2$ and $x = \frac{u-1}{2}$.

This gives us $\int_1^3 \left(\frac{u-1}{2} - 2\right)\sqrt{u}\ \frac{du}{dx} \times \frac{1}{2}\ dx$ when $x = 1$, $u = 3$

when $x = 3$, $u = 7$

$$= \frac{1}{2}\int_3^7 \left(\frac{u-1}{2} - \frac{4}{2}\right)\sqrt{u}\ du$$

$$= \frac{1}{2}\int_3^7 \left(\frac{u-5}{2}\right)\sqrt{u}\ du$$

$$= \frac{1}{4}\int_3^7 (u-5)\sqrt{u}\ du$$

The answer is C.

Question 16

Using the trigonometric product-to-sum identity $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$

$$\int \sin\left(\frac{7x}{2}\right)\sin\left(\frac{3x}{2}\right)\ dx = \frac{1}{2}\int \cos\left(\frac{7x}{2} - \frac{3x}{2}\right) - \cos\left(\frac{7x}{2} + \frac{3x}{2}\right)\ dx$$

$$= \frac{1}{2}\int \cos(2x) - \cos(5x)\ dx$$

The answer is E.

Question 17

From the slope field, we can see that the gradient is undefined when $y = -1$.

This limits our possible answers to either A or D.

In quadrant 1, where $x > 0$ and $y > 0$, we can see that $\frac{dy}{dx} < 0$.

This is only possible for option D.

The answer is D.

Question 18

$\Pr(Y < 8) = \Pr((2X - 1) < 8)$

$= \Pr(2X < 9)$

$= \Pr\left(X < \frac{9}{2}\right)$

Using CAS, $\Pr\left(X < \frac{9}{2}\right) = \int_4^{\frac{9}{2}} \frac{12}{x^2}\, dx = \frac{1}{3} \approx 0.33$.

The answer is B.

Question 19

The confidence interval for μ (the population mean or actual mean) is given by

$$\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right).$$

This means that $z\frac{\sigma}{\sqrt{n}} = 1000$, and we are given $\sigma = 3000$ and $n = 30$.

Solving for z gives $z = \frac{\sqrt{30}}{3}$.

Using CAS, $\Pr\left(\frac{-\sqrt{30}}{3} < Z < \frac{\sqrt{30}}{3}\right) = 0.93211...$

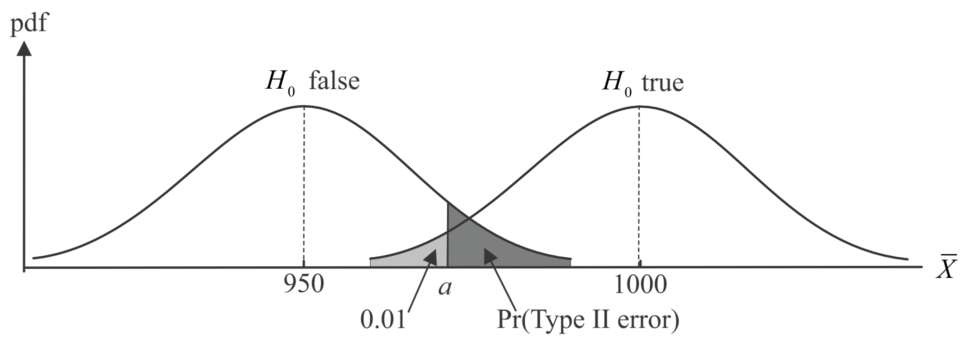
The closest answer is 93.2%.

The answer is C.

Question 20

$\Pr(\text{Type II error}) = \Pr(\text{Accept } H_0 \mid H_0 \text{ false}) = \Pr(\bar{X} > a \mid \mu = 950)$ where a is calculated from $\Pr(\bar{X} < a \mid \mu = 1000) = 0.01$.

For H_0 false, $\bar{X} \sim N\left(\mu = 950, \sigma = \frac{100}{\sqrt{50}}\right)$ and for H_0 true, $\bar{X} \sim N\left(\mu = 1000, \sigma = \frac{100}{\sqrt{50}}\right)$.



Use CAS to determine the value of a, such that $\Pr(\bar{X} < a \mid \mu = 1000) = 0.01$ (inverse normal)

$a = 967.1005$ (correct to four decimal places)

$\Pr(\text{Type II error}) = \Pr(\bar{X} > 967.1005 \mid \mu = 950) = 0.11329...$ using CAS

The closest answer is 0.1133.

The answer is B.

SECTION B

Question 1 (9 marks)

a. $f(x)=\dfrac{6x^2+4}{x^2+2}$

$$\begin{array}{r|l} & 6 \\ \hline x^2+2\,) & 6x^2+4 \\ & 6x^2+12 \\ \hline & -8 \end{array}$$

So $f(x)=6-\dfrac{8}{x^2+2}$.

The equation of the asymptote is $y=6$. **(1 mark)**

b. Solve $f'(x)=0$ for x.

$x=0$

$f(0)=2$

The stationary point occurs at $(0,2)$. **(1 mark)**

c. Solve $f''(x)=0$ for x.

$$x=\pm\frac{\sqrt{6}}{3}$$

Now check for a sign change of $f''(x)$ bearing in mind that $\dfrac{\sqrt{6}}{3}=0.8164...$

(1 mark) attempting a sign change check

$f''(-1)=-\dfrac{16}{27}<0$

$f''(0)=4>0$

$f''(1)=-\dfrac{16}{27}<0$

Since $f\left(-\dfrac{\sqrt{6}}{3}\right)=f\left(\dfrac{\sqrt{6}}{3}\right)=3$, the points of inflection occur at

$\left(-\dfrac{\sqrt{6}}{3},3\right)$ and $\left(\dfrac{\sqrt{6}}{3},3\right)$. **(1 mark)**

Use CAS to graph *f* and confirm the exact answers that you have found.

d. Solve $50\pi=\pi\displaystyle\int_0^a\left(\frac{6x^2+4}{x^2+2}\right)^2dx$ for a. **(1 mark)**

$a=3.056$ (correct to 3 decimal places) **(1 mark)**

e. **i.** $g(x)=\dfrac{6x^2+4}{x^2+b}$

$g'(x)=\dfrac{4x(3b-2)}{\left(x^2+b\right)^2}$

The stationary point occurs when $x=0$.
If the stationary point is a local minimum, then $g'(x)>0$ for $x>0$.
For that to happen $3b-2>0$

$$b>\frac{2}{3}$$

(1 mark)

ii. $g''(x)=\dfrac{-4(3x^2-b)(3b-2)}{\left(x^2+b\right)^3}=0$

Note that $g(x)=\dfrac{6x^2+4}{x^2+b}$

$=6+\dfrac{4-6b}{x^2+b}$

If $3b-2=0$ there is no point of inflection because $g(x)=6$.

If $3x^2-b=0$

$x^2=\dfrac{b}{3}$

$x=\pm\dfrac{\sqrt{3b}}{3}$ **(1 mark)**

If the two points of inflection are 3 units apart then

solve $\dfrac{\sqrt{3b}}{3}\times 2=3$ for b.

$b=6.75$ **(1 mark)**

Question 2 (11 marks)

a. $z_1 = \sqrt{3} - i$ is a solution so $\sqrt{3} + i$ is also a solution. (Conjugate root theorem)

(1 mark)

$$\begin{aligned} P(z) &= \left(z - \sqrt{3} - i\right)\left(z - \sqrt{3} + i\right)\left(z + c\right) \\ &= \left[\left(z - \sqrt{3}\right)^2 - i^2\right](z + c) \\ &= \left(z^2 - 2\sqrt{3}\,z + 4\right)(z + c) \end{aligned}$$

Since $P(z) = z^3 + az^2 + bz + 12$, then $4c = 12$ so $c = 3$. **(1 mark)**

$$\begin{aligned} P(z) &= \left(z^2 - 2\sqrt{3}\,z + 4\right)(z + 3) \\ &= z^3 + \left(3 - 2\sqrt{3}\right)z^2 + \left(4 - 6\sqrt{3}\right)z + 12 \end{aligned}$$

Equating coefficients of z^2 and z gives

$a = 3 - 2\sqrt{3}$ and $b = 4 - 6\sqrt{3}$. **(1 mark)**

b. $z_1 = 2\text{cis}\left(-\frac{\pi}{6}\right)$ using CAS. **(1 mark)**

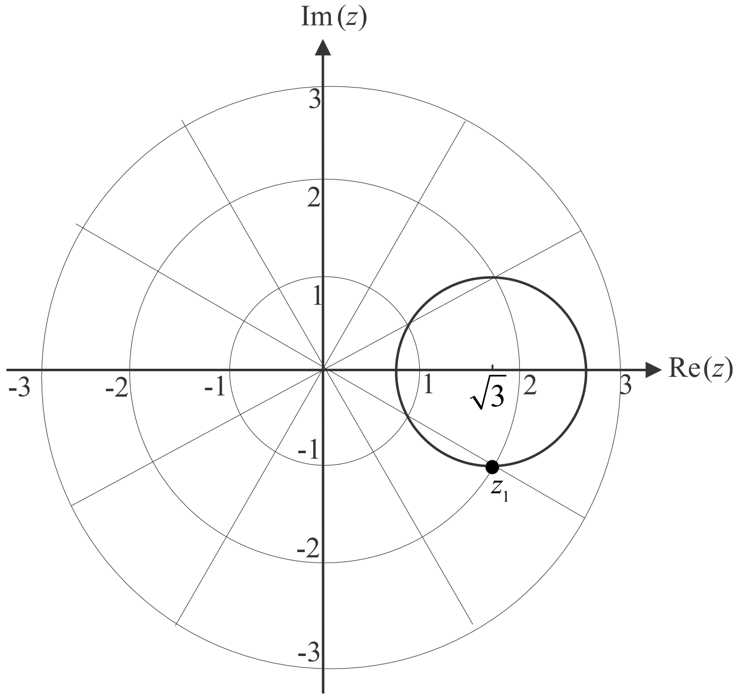
c.

$$\begin{aligned} z^6 + w &= 0 \\ \left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^6 + w &= 0 \\ 2^6\text{cis}\left(-\frac{\pi}{6} \times 6\right) + w &= 0 \qquad \text{(De Moivre)} \\ 64\text{cis}(-\pi) + w &= 0 \\ -64 + w &= 0 \\ w &= 64 \quad \text{as required} \end{aligned}$$

(1 mark)

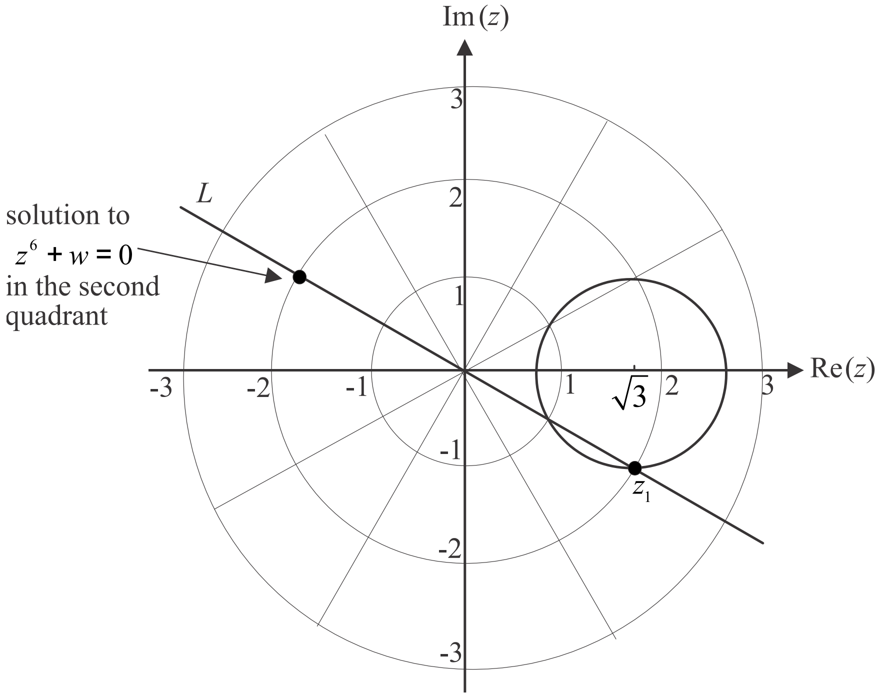
d. $\left|z-\sqrt{3}\right|=1$ is a circle with centre at $\left(\sqrt{3},0\right)$ and radius of 1 unit.

It passes through z_1, i.e. the point $\left(\sqrt{3},-1\right)$.



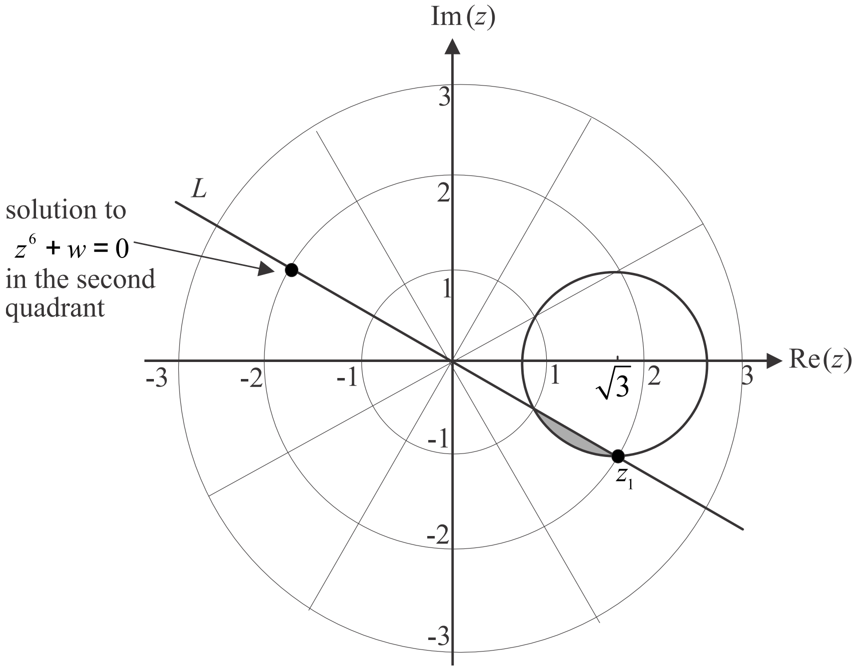
(1 mark) for correct circle **(1 mark)** for z_1

e. There are six solutions to $z^6+w=0$, i.e. to $z^6=-64$. They are evenly spaced (with spacing $\frac{2\pi}{6}$) around the circle with centre at the origin and with radius of 2 units. Since z_1 is one of them, then the one in the second quadrant will be located at $2\text{cis}\left(\frac{5\pi}{6}\right)$ as shown below.



(1 mark)

f. The minor segment is shaded in the diagram below.



Line L passes through z_1 at $\left(\sqrt{3}, -1\right)$ and the origin.

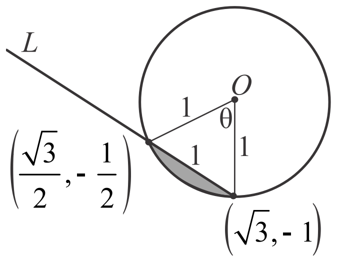
The equation of line L is $y = -\frac{1}{\sqrt{3}}x$.

The equation of the circle is $\left(x - \sqrt{3}\right)^2 + y^2 = 1$.

L and the circle intersect at $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(\sqrt{3}, -1\right)$. **(1 mark)**

The distance between these two points is $\sqrt{\left(\sqrt{3} - \frac{\sqrt{3}}{2}\right)^2 + \left(-1 - -\frac{1}{2}\right)^2} = 1$.

So the triangle is equilateral and all of the internal angles are $\frac{\pi}{3}$. **(1 mark)**



area of segment = area of sector − area of triangle

$$= \left(\frac{\pi}{3} \div 2\pi\right) \times \pi r^2 - \frac{1}{2}bc\sin\left(\frac{\pi}{3}\right) \quad \text{where } r = 1 \text{ and } b = 1,\ c = 1$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{4} \text{ square units}$$

(1 mark)

Question 3 (9 marks)

a. $\underset{\sim}{r}(s,t) = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k} + s(3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}) + t(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k})$ **(1 mark)**

b. From part **a.**, $\underset{\sim}{u} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$

$\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$

A vector normal to the plane is $\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v}$ (using the cross product)

$= (-4+12)\underset{\sim}{i} - (-6+4)\underset{\sim}{j} + (9-2)\underset{\sim}{k}$

$= 8\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k}$ **(1 mark)**

The Cartesian equation is therefore given by $8x + 2y + 7z = k$.

Since the point $(2,-1,3)$ lies on Π_1, substitute it into this equation.

$16 - 2 + 21 = k$

$k = 35$

The Cartesian equation is $8x + 2y + 7z = 35$ as required. **(1 mark)**

c. $\underset{\sim}{r}(\lambda) = (2+\lambda)\underset{\sim}{i} + (1-5\lambda)\underset{\sim}{j} + (\lambda-3)\underset{\sim}{k}$

Since the plane has Cartesian equation $8x + 2y + 7z = 35$,

solve $8(2+\lambda) + 2(1-5\lambda) + 7(\lambda-3) = 35$ for λ using CAS, so $\lambda = \frac{38}{5}$. **(1 mark)**

So $\underset{\sim}{r}\left(\frac{38}{5}\right) = \frac{48}{5}\underset{\sim}{i} - 37\underset{\sim}{j} + \frac{23}{5}\underset{\sim}{k}$ and the line intersects with the plane at the point

$\left(\frac{48}{5}, -37, \frac{23}{5}\right)$. **(1mark)**

d. Let θ be the angle between the line and a normal to Π_1.

A vector parallel to the line is $\underset{\sim}{d} = \underset{\sim}{i} - 5\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{n} = 8\underset{\sim}{i} + 2\underset{\sim}{j} + 7\underset{\sim}{k}$. **(1 mark)**

$\cos(\theta) = \frac{\underset{\sim}{d} \bullet \underset{\sim}{n}}{|\underset{\sim}{d}||\underset{\sim}{n}|} = 0.0889...$ (using the dot product and CAS)

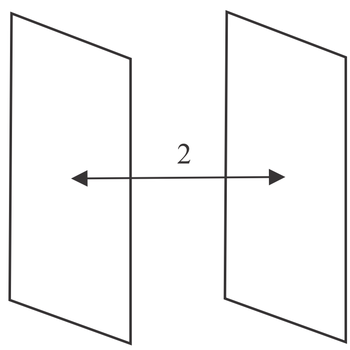
$\theta = 84.8962...°$

The angle between the line and the plane is

$90° - 84.8962...° = 5.1037...°$

$= 5.10°$ (correct to 2 decimal places) **(1 mark)**

e. Distance of Π_1 from origin is $\frac{k}{|\underset{\sim}{n}|} = \frac{35}{3\sqrt{13}}$.



Π_2 needs to be $\frac{35}{3\sqrt{13}} \pm 2$ i.e. $\frac{35 \pm 6\sqrt{13}}{3\sqrt{13}}$

from origin so $k = 35 \pm 6\sqrt{13}$. **(1 mark)**

The required plane is either $8x + 2y + 7z = 35 + 6\sqrt{13}$

or $8x + 2y + 7z = 35 - 6\sqrt{13}$. **(1 mark)**

Question 4 (10 marks)

a. Using Euler's method,

$t_0 = 0, \quad P_0 = 20$

$t_1 = 1, \quad P_1 = 20 + 1 \times \frac{dP}{dt}$ at the point $(0, 20)$

$= 20.99$

$t_2 = 2, \quad P_2 = 20.99 + 1 \times \frac{dP}{dt}$ at the point $(1, 20.99)$

$= 22.028...$

$= 22$ (to the nearest whole number)

(1 mark) for 20.99 **(1 mark)** for 22

b. $\frac{dP}{dt} = 0.05P\left(1 - \frac{P}{2000}\right) \quad$ where $P(0) = 20$

$= \frac{P}{20}\left(\frac{2000 - P}{2000}\right)$

$\frac{dt}{dP} = \frac{40\,000}{P(2000 - P)}$

$t = 40\,000\int \frac{1}{P(2000 - P)}\,dP$

Let $\frac{1}{P(2000 - P)} = \frac{A}{P} + \frac{B}{2000 - P}$

$= \frac{A(2000 - P) + BP}{P(2000 - P)}$

True iff $1 = A(2000 - P) + BP$

Put $P = 0, \quad 1 = 2000A \quad$ so $A = \frac{1}{2000}$

Put $P = 2000, \quad 1 = 2000B \quad$ so $B = \frac{1}{2000}$

So $t = \frac{40\,000}{2000}\int \frac{1}{P} + \frac{1}{2000 - P}\,dP$ **(1 mark)**

$= 20\left(\log_e(P) - \log_e(2000 - P)\right) + c \qquad 0 < P < 2000$

$= 20\log_e\left(\frac{P}{2000 - P}\right) + c$ **(1 mark)**

Since $P(0) = 20$,

$0 = 20\log_e\left(\frac{20}{2000 - 20}\right) + c$

$c = -20\log_e\left(\frac{1}{99}\right) = 20\log_e(99)$

So $t = 20\log_e\left(\frac{P}{2000 - P}\right) + 20\log_e(99)$

$t = 20\log_e\left(\frac{99P}{2000 - P}\right)$

$$e^{\frac{t}{20}} = \frac{99P}{2000-P}$$

$$e^{\frac{t}{20}}(2000-P) = 99P$$

$$2000e^{\frac{t}{20}} = P(99+e^{\frac{t}{20}})$$

$$P = \frac{2000e^{\frac{t}{20}}}{99+e^{\frac{t}{20}}} \quad \text{as required}$$

(1 mark)

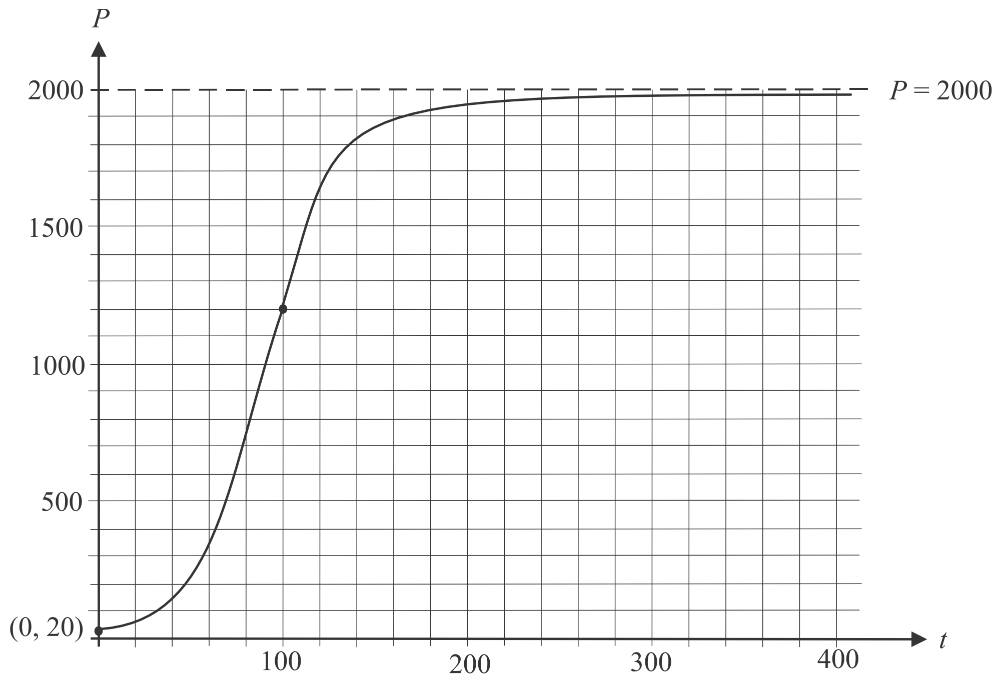
c. $P = \dfrac{2000e^5}{99+e^5}$

$= 1199.719...$

$= 1200$ (to the nearest whole number)

(1 mark)

d.



(1 mark) correct shape through (100,1200)
(1 mark) correct asymptote and y-intercept

e. Solve $\dfrac{d^2P}{dt^2} = 0$ for t. **(1 mark)**

$t = 91.9023...$

$= 92$ days (to the nearest whole number)

(1 mark)

Question 5 (10 marks)

a. **i.**

$$x = 10 - 2t$$
$$2t = 10 - x$$
$$t = \frac{10 - x}{2}$$

Substitute this into $y = -t^2 + 6t - 8$

$$y = -\frac{x^2}{4} + 2x - 3$$
$$= \frac{-(x-2)(x-6)}{4}$$

(1 mark)

ii. When $t = 0,\quad x = 10$

When $t = 10,\quad x = -10$

domain is $x \in [-10, 10]$ **(1 mark)**

b. distance $= \int_0^{10} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ (formula sheet)

$$= \int_0^{10} \sqrt{(-2)^2 + (-2t+6)^2}\, dt$$ **(1 mark)**

$= 63.45$ metres (correct to 2 decimal places) **(1 mark)**

c. distance from origin $D = \sqrt{(10-2t)^2 + \left(-t^2 + 6t - 8\right)^2}$

solve $\frac{dD}{dt} = 0$ for t to find when the rabbit is closest to O

$t = 4.378...$ **(1 mark)**

$\underset{\sim}{v}(t) = -2\underset{\sim}{i} + (-2t+6)\underset{\sim}{j}$

$$\left|\underset{\sim}{v}(4.378...)\right| = \left|-2\underset{\sim}{i} + (-2 \times 4.378... + 6)\underset{\sim}{j}\right|$$

$= 3.41\,\text{ms}^{-1}$ (correct to 2 decimal places) **(1 mark)**

d. $\underset{\sim}{r}(4) = 2\underset{\sim}{i} + 0\underset{\sim}{j}$

$\underset{\sim}{f}(4) = (10 - 16a)\underset{\sim}{i} + (4b - 10)\underset{\sim}{j}$

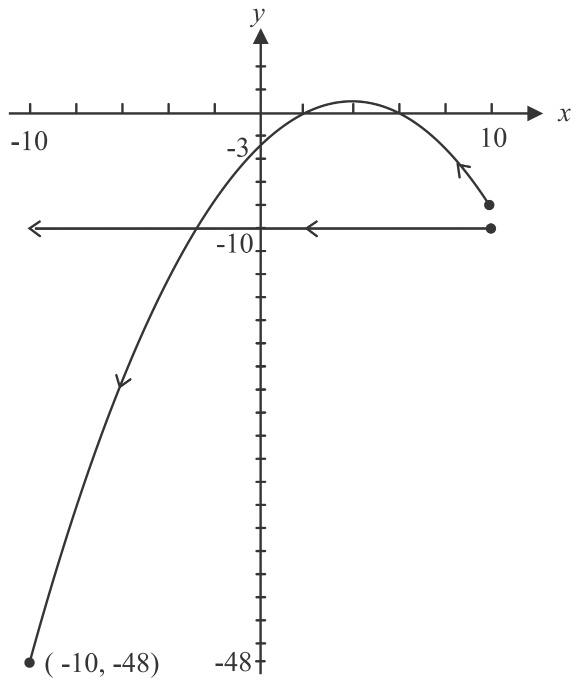
We require $10 - 16a = 2$ AND $4b - 10 = 0$

$a = \frac{1}{2}$ $\qquad$ $b = \frac{5}{2}$

(1 mark)

e. The path of the rabbit (from part **a.**) is parabolic and is shown below. Its starting point is $(10,-8)$ and its endpoint is $(-10,-48)$.

The path of the fox when $b=0$ is a straight line with equation $y=-10$ and is also shown below. Its starting point is $(10,-10)$ and its endpoint will depend on the value of *a*.



We need the path of the fox to continue sufficiently far to the left so as to meet the path of the rabbit.

The paths intersect when $-\frac{x^2}{4}+2x-3=-10$

$$x=-2\left(\sqrt{11}-2\right) \quad \text{or} \quad x=2\left(\sqrt{11}+2\right)$$ **(1 mark)**

Reject this last answer because $2\left(\sqrt{11}+2\right)>10$ which is to the right of the starting point of both animals.

The *x*-coordinate of the fox when $t=10$ is $10-100a$.

Therefore it is required that $10-100a\le -2\left(\sqrt{11}-2\right)$. **(1 mark)**

$$\text{Therefore } a\ge \frac{\sqrt{11}+3}{50}.$$ **(1 mark)**

If you have time, you can check a couple of values of *a*.

For example, when $a=1$ and $t=10$, the end of the fox's path will be at $(-90,-10)$ which will mean that the paths intersect.

For example, when $a=0$ and $t=10$, the end of the fox's path will be at $(10,-10)$ which will mean that the paths won't intersect.

Question 6 (11 marks)

a. Let $\bar{X}$ represent the distribution of the mean height of random samples of 20 sprays.

$$\bar{X} \sim N\left(75, \left(\frac{4}{\sqrt{20}}\right)^2\right)$$ **(1 mark)**

$\Pr(73 < \bar{X} < 77) = 0.97465...$

$= 0.975$ (correct to 3 decimal places) **(1 mark)**

i.e. $\text{normCdf}\left(73, 77, 75, \frac{4}{\sqrt{20}}\right)$

b. $Y \sim \text{Bi}(4, 0.97465...)$

$\Pr(Y > 2) = 0.99627...$

$= 0.996$ (correct to 3 decimal places) **(1 mark)**

i.e. $\text{binomCdf}(4, 0.97465..., 3, 4)$

c. Let S represent the height of a spray where $S \sim \text{N}(75, 4^2)$.

Let C represent the height of a cap where $C \sim \text{N}(10, 3^2)$.

S and C are independent random variables.

The random variable we require is $S + C_1 + C_2$ where C_1 and C_2 are independent and identically distributed 'clones' of the random variable C.

$$\begin{aligned} &\text{E}(S + C_1 + C_2) \\ &= \text{E}(S) + \text{E}(C_1) + \text{E}(C_2) \\ &= 75 + 10 + 10 \\ &= 95 \end{aligned}$$

$$\begin{aligned} &\text{Var}(S + C_1 + C_2) \\ &= 1^2\,\text{Var}(S) + 1^2\,\text{Var}(C_1) + 1^2\,\text{Var}(C_2) \\ &= 16 + 9 + 9 \\ &= 34 \end{aligned}$$

So the mean is 95 mm and the standard deviation is $\sqrt{34}$ mm.

(1 mark) for mean **(1 mark)** for standard deviation

d. $(S + C_1 + C_2) \sim \text{N}(95, 34)$

$\Pr((S + C_1 + C_2) > 110) = 0.00504...$

$= 0.005$ (correct to 3 decimal places) **(1 mark)**

e. $H_0 : \mu = 75$

$H_1 : \mu \neq 75$ **(1 mark)**

f. $p = 2\Pr(\bar{W} > 76 \mid \mu = 75)$ where $\bar{W} \sim \text{N}\left(75, \left(\frac{4}{\sqrt{60}}\right)^2\right)$

$= 0.05280...$ using CAS

$= 0.053$ (correct to 3 decimal places)

(1 mark)

g. Since $p > 0.05$ there is **insufficient evidence** to suggest that the machine has not been properly serviced.

(1 mark)

h. Use CAS (inverse normal) to find *w* where

$\Pr(\overline{W} < w) = 0.975$ and $\overline{W} \sim \mathrm{N}\left(75, \left(\frac{4}{\sqrt{60}}\right)^2\right)$.

$w = 76.01212...$

The largest value of the mean height of the sample of 60 sprays for H_0 **not** to be rejected is 76.012 (correct to 3 decimal places).

(1 mark)

i. The confidence interval, found using CAS, (zInterval 4,76,60,0.98) is $(74.8, 77.2)$ where endpoints are correct to one decimal place.

(1 mark)