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SPECIALIST MATHEMATICS UNITS 3 & 4

TRIAL EXAMINATION 1

2023

Reading Time: 15 minutes
Writing time: 1 hour

Instructions to students

This exam consists of 10 questions.
All questions should be answered in the spaces provided.
There is a total of 40 marks available.
The marks allocated to each of the questions are indicated throughout.
Students may **not** bring any notes or calculators into the exam.
Where more than one mark is allocated to a question, appropriate working must be shown.
An exact answer is required to a question unless otherwise specified.
Unless otherwise indicated, diagrams in this exam are not drawn to scale.
Formula sheets can be found at the end of this exam.

Question 1 (3 marks)

Prove by mathematical induction that the number $6^n + 4$ is divisible by 5, for all $n \in N$.

Question 2 (3 marks)

Find the solution to the differential equation $\frac{dy}{dx} = \frac{\sin^2(x)}{\cos(y)}$, given $y(0) = \frac{\pi}{6}$.

Give your answer as an expression for *y* in terms of *x*.

Question 3 (4 marks)

Wholegrain sourdough loaves of bread are baked for a large supermarket chain.
The weight of these loaves is normally distributed with a mean of 850 g and a standard deviation of 10 g.

a. Find the probability that the **total** weight of four randomly selected loaves is less than 3360 g. Use $\Pr(-2 < Z < 2) = 0.95$ and give your answer correct to three decimal places. 2 marks

b. The loaves are placed on trays which hold 16 loaves.
Find the probability that the mean weight of a loaf on a randomly selected tray is more than 855 g. Give your answer correct to three decimal places. 2 marks

Question 4 (3 marks)

Find the gradient of the curve $x\sec(2y)+\log_e(x)+\frac{6}{\pi}y^2=2+\frac{\pi}{6}$ at the point $\left(1,\frac{\pi}{6}\right)$.

Express your answer in the form $\frac{a-2a\sqrt{a}}{b}$, where $a,b\in N$.

Question 5 (4 marks)

Let $z = \frac{(1+i)^6}{\left(\sqrt{3}-i\right)^4}$.

a. Write z in the form $a+bi$, where a and b are real constants. 3 marks

b. Let $z^n = d$, where $n \in Z$ and $d \in R$.
Find all possible values of n. 1 mark

Question 6 (5 marks)

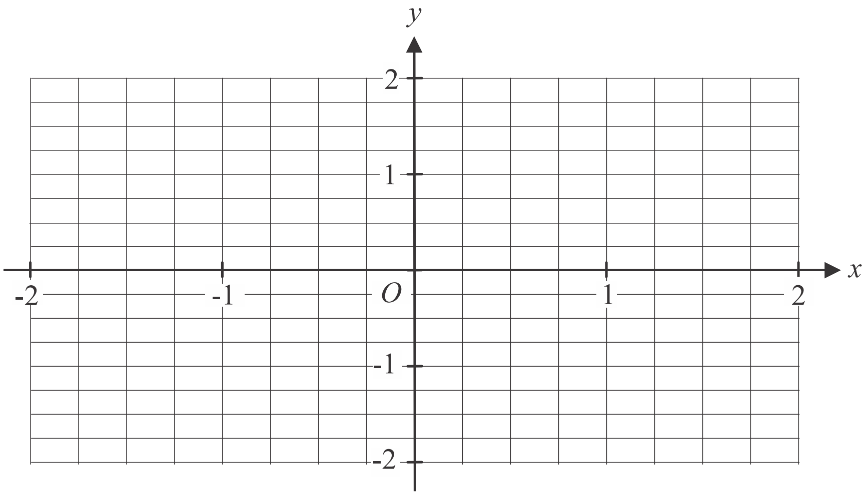
A particle moves so that its velocity at time t is given by

$$\underset{\sim}{v}(t) = 2\sin(4t)\underset{\sim}{i} + 4\cos(4t)\underset{\sim}{j} \qquad \text{for } 0 \le t \le \frac{\pi}{4}.$$

a. Given that $\underset{\sim}{r}(0) = -\frac{1}{2}\underset{\sim}{i}$, find the position vector $\underset{\sim}{r}(t)$ of the particle. 2 marks

b. Show that the Cartesian equation of the path followed by the particle is given by $4x^2 + y^2 = 1$. 1 mark

c. Sketch the path followed by the particle on the set of axes below. Indicate the particle's starting point and label any axis intercepts with their coordinates. 2 marks



Question 7 (5 marks)

Find the surface area obtained by rotating the curve $y=\dfrac{x^2-1}{2}$, $0\le x\le 3$, about the *y*-axis.

Express your answer in the form $\dfrac{2\pi}{a}(b\sqrt{b}-1)$, where $a,b\in N$.

Question 8 (3 marks)

Find the value of $\lambda \in R$, such that the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{c} = -2\underset{\sim}{i} + \lambda\underset{\sim}{j} + \underset{\sim}{k}$ are linearly dependent.

Question 9 (5 marks)

a. Find the coordinates of the point of intersection of the lines

$$\underset{\sim}{r}_1(s) = \underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k} + s\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$$

$$\underset{\sim}{r}_2(t) = 3\underset{\sim}{i} + \underset{\sim}{j} - 21\underset{\sim}{k} + t\left(\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k}\right)$$

3 marks

b. Find the shortest distance from the point of intersection found in part **a.** to the plane Π with equation $2x + y + 2z = 4$. 2 marks

Question 10 (5 marks)

a. Evaluate $\int_0^1 x e^{2x} dx$. Give your answer in the form $\frac{1}{a}e^2 + \frac{1}{a}$ where $a \in Z$. 2 marks

b. Evaluate $\int_0^1 \frac{5x-3}{(x+1)(x^2+3)} dx$. Give your answer in the form $\frac{\sqrt{b}\,\pi}{c} - \log_e(b)$, where $b, c \in Z$ 3 marks

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\text{cis}(\theta)$	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \text{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$	de Moivre's theorem	$z^n = r^n \text{cis}(n\theta)$

Data analysis, probability and statistics

for independent random variables $X_1, X_2 \ldots X_n$	$\text{E}(aX_1 + b) = a\text{E}(X_1) + b$ $\text{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1\text{E}(X_1) + a_2\text{E}(X_2) + \ldots + a_n\text{E}(X_n)$	
	$\text{Var}(aX_1 + b) = a^2\text{Var}(X_1)$ $\text{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + \ldots + a_n^2\text{Var}(X_n)$	
for independent identically distributed variables $X_1, X_2 \ldots X_n$	$\text{E}(X_1 + X_2 + \ldots + X_n) = n\mu$	
	$\text{Var}(X_1 + X_2 + \ldots + X_n) = n\sigma^2$	
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$	
distribution of sample mean $\bar{X}$	mean	$\text{E}(\bar{X}) = \mu$
	variance	$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \ne -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \log_e \lvert x\rvert + c$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\cot(ax)\right) = -a\text{cosec}^2(ax)$	$\int \text{cosec}^2(ax)\,dx = -\frac{1}{a}\cot(ax) + c$
$\frac{d}{dx}\left(\sec(ax)\right) = a\sec(ax)\tan(ax)$	$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + c$
$\frac{d}{dx}(\text{cosec}(ax)) = -a\text{cosec}(ax)\cot(ax)$	$\int \text{cosec}(ax)\cot(ax)\,dx = -\frac{1}{a}\text{cosec}(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(ax)\right) = \frac{a}{\sqrt{1-(ax)^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\cos^{-1}(ax)\right) = \frac{-a}{\sqrt{1-(ax)^2}}$	$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\tan^{-1}(ax)\right) = \frac{a}{1+(ax)^2}$	$\int \frac{a}{a^2 + x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \ne -1$
	$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e \lvert ax+b\rvert + c$

Calculus - continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$
Euler's method	If $\frac{dy}{dx} = f(x, y), x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about y-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about x-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about y-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u + v)t$

Vectors in two and three dimensions

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 . \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \quad y(t) = y_1 + y_2t \quad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s,t) = \underset{\sim}{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2$ $= (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s,t) = x_0 + x_1s + x_2t,\ \ y(s,t) = y_0 + y_1s + y_2t,\ \ z(s,t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$