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SPECIALIST MATHS UNITS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2023

Question 1 (3 marks)

Let $P(n)$ be the proposition that for each natural number n, $6^n + 4$ is divisible by 5.

<u>Step 1</u> Show $P(1)$ is true.
$6^1 + 4 = 10$ which is divisible by 5

(1 mark)

<u>Step 2</u> Assume that $P(k)$ is true for $k \in N$.
That is, assume that $6^k + 4 = 5m, \quad m \in N$

(1 mark)

<u>Step 3</u> Prove that $P(k)$ implies that $P(k+1)$ is true.

$$\begin{aligned}
&6^{k+1} + 4 \\
&= 6^k \times 6^1 + 4 \\
&= 6^k(5+1) + 4 \\
&= 5 \times 6^k + 1 \times 6^k + 4 \\
&= 5 \times 6^k + 5m \qquad \text{using } P(k) \\
&= 5(6^k + m), \qquad 6^k, m \in N \text{ so } 6^k + m \in N
\end{aligned}$$

It follows that $P(k)$ implies that $P(k+1)$ is true.
Using the principle of mathematical induction, it therefore follows that $P(n)$ is true for all natural numbers n.

(1 mark)

Question 2 (3 marks)

$$\frac{dy}{dx} = \frac{\sin^2(x)}{\cos(y)}$$

$$\int \cos(y)dy = \int \sin^2(x)dx$$ **(1 mark)**

$$\begin{aligned}
\sin(y) + c_1 &= \int \frac{1}{2}(1 - \cos(2x))dx \\
&= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c_2 \\
\sin(y) &= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c \quad \text{where } c = c_2 - c_1
\end{aligned}$$

(1 mark)

$$y(0) = \frac{\pi}{6}$$

$$c = \frac{1}{2}$$

So $y = \sin^{-1}\left(\frac{1}{2}x - \frac{1}{4}\sin(2x) + \frac{1}{2}\right)$

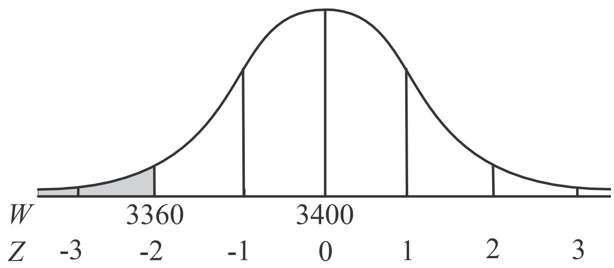
(1 mark)

Question 3 (4 marks)

a. Let X represent the weight of a loaf, where $\mu = 850$ and $\sigma = 10$.
Let W represent the **total** weight of four loaves.
$W = (X_1 + X_2 + X_3 + X_4)$ where $X_1,\ X_2,\ X_3$ and X_4 are four identically distributed variables.

$\mathrm{E}(W) = 4 \times 850 = 3400$
$\mathrm{Var}(W) = 4 \times 10^2 = 400$
(formula sheet)
$\mathrm{sd}(W) = \sqrt{400} = 20$

(1 mark)



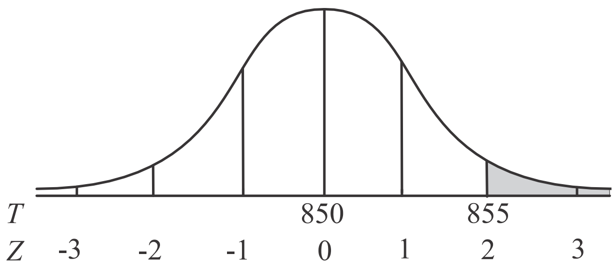
$W \sim N(3400, 20^2)$

$$\begin{aligned}\Pr(W < 3360) &= \Pr(Z < -2)\\ &= \frac{1-0.95}{2} \quad \text{(given } \Pr(-2 < Z < 2) = 0.95\text{)}\\ &= 0.025\end{aligned}$$

(1 mark)

b. Let T represent the distribution of the sample mean of samples of 16 loaves.
$\mathrm{E}(T) = 850$ and $\mathrm{sd}(T) = \dfrac{10}{\sqrt{16}} = 2.5$ **(1 mark)**

$T \sim N(850,\ 2.5^2)$



$$\begin{aligned}\Pr(T > 855) &= \Pr(Z > 2)\\ &= \frac{0.05}{2} \quad \text{(given that } \Pr(-2 < Z < 2) = 0.95 \text{ from part } \mathbf{a.}\text{)}\\ &= 0.025\end{aligned}$$

(1 mark)

Question 4 (3 marks)

$$x\sec(2y)+\log_e(x)+\frac{6}{\pi}y^2=2+\frac{\pi}{6}$$

$$x(2\sec(2y)\tan(2y))\frac{dy}{dx}+\sec(2y)+\frac{1}{x}+\frac{12}{\pi}y\frac{dy}{dx}=0$$

Note that the derivative of $\sec(2y)$ is on the formula sheet.

(1 mark) an attempt to use the product rule
(1 mark) all correct

Substitute in $\left(1,\frac{\pi}{6}\right)$.

$$1\times 2\sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)\frac{dy}{dx}+\sec\left(\frac{\pi}{3}\right)+1+\frac{12}{\pi}\times\frac{\pi}{6}\frac{dy}{dx}=0$$

$$2\times 2\times\sqrt{3}\frac{dy}{dx}+2+1+2\frac{dy}{dx}=0$$

$$\left(4\sqrt{3}+2\right)\frac{dy}{dx}=-3$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{-3}{4\sqrt{3}+2}\\&=\frac{-3}{4\sqrt{3}+2}\times\frac{4\sqrt{3}-2}{4\sqrt{3}-2}\\&=\frac{6-12\sqrt{3}}{48-4}\\&=\frac{2\left(3-6\sqrt{3}\right)}{44}\\&=\frac{3-6\sqrt{3}}{22}\end{aligned}$$

(1 mark)

Question 5 (4 marks)

a. $$z = \frac{(1+i)^6}{\left(\sqrt{3}-i\right)^4}$$

For $1+i$, $r=\sqrt{2}$ and $\theta = \tan^{-1}(1) = \frac{\pi}{4}$ (first quadrant angle).

For $\sqrt{3}-i$, $r=\sqrt{4}$ and $\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ (fourth quadrant angle).

$$z = \frac{\left(\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\right)^6}{\left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^4}$$ **(1 mark)**

$$= \frac{8\text{cis}\left(\frac{3\pi}{2}\right)}{16\text{cis}\left(-\frac{2\pi}{3}\right)}$$ (De Moivre) **(1 mark)**

$$= \frac{1}{2}\text{cis}\left(\frac{3\pi}{2}+\frac{2\pi}{3}\right)$$

$$= \frac{1}{2}\text{cis}\left(\frac{13\pi}{6}\right)$$

$$= \frac{1}{2}\text{cis}\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{2}\cos\left(\frac{\pi}{6}\right) + \frac{1}{2}\times i\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{4}+\frac{1}{4}i$$ **(1 mark)**

b. $$z^n = \left[\frac{1}{2}\text{cis}\left(\frac{\pi}{6}\right)\right]^n$$

$$= \frac{1}{2^n}\left(\cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right)\right)$$

For z^n to be real, then we require $\sin\left(\frac{n\pi}{6}\right)=0$, so $\frac{n\pi}{6} = \ldots -\pi,\ 0,\ \pi,\ 2\pi,\ 3\pi, \ldots$

and $n = \ldots -6,\ 0,\ 6,\ 12,\ 18, \ldots$

So $n = 6k,\ k \in Z$. **(1 mark)**

Question 6 (5 marks)

a. $\underset{\sim}{v}(t) = 2\sin(4t)\underset{\sim}{i} + 4\cos(4t)\underset{\sim}{j} \qquad 0 \le t \le \frac{\pi}{4}$

$\underset{\sim}{r}(t) = -\frac{1}{2}\cos(4t)\underset{\sim}{i} + \sin(4t)\underset{\sim}{j} + \underset{\sim}{c}$ **(1 mark)**

$\underset{\sim}{r}(0) = -\frac{1}{2}\underset{\sim}{i}$

$-\frac{1}{2}\underset{\sim}{i} = -\frac{1}{2}\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{c}$ so $\underset{\sim}{c} = \underset{\sim}{0}$

$\underset{\sim}{r}(t) = -\frac{1}{2}\cos(4t)\underset{\sim}{i} + \sin(4t)\underset{\sim}{j}$ **(1 mark)**

b. $x = -\frac{1}{2}\cos(4t) \qquad y = \sin(4t)$

$-2x = \cos(4t) \qquad y = \sin(4t)$

$4x^2 = \cos^2(4t) \qquad y^2 = \sin^2(4t)$

$4x^2 + y^2 = 1$ as required **(1 mark)**

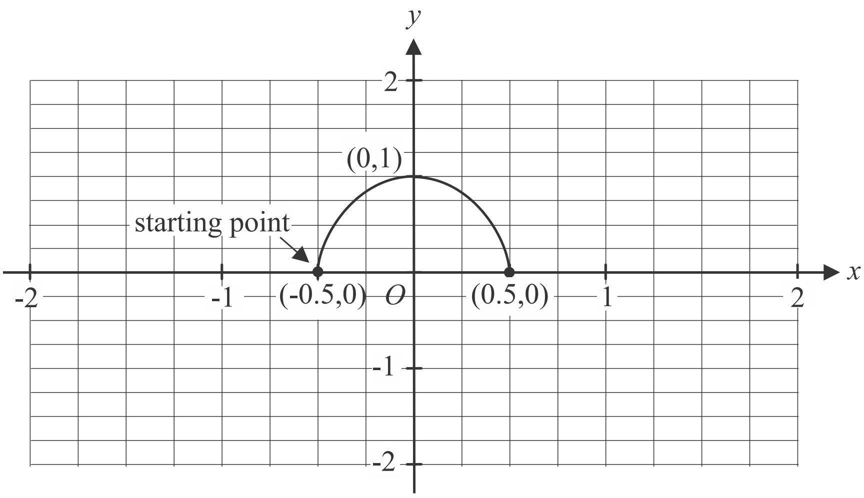
c. We need to consider the restrictions on the domain of the original velocity function i.e. $0 \le t \le \frac{\pi}{4}$.

$\underset{\sim}{r}(0) = -\frac{1}{2}\cos(0)\underset{\sim}{i} + \sin(0)\underset{\sim}{j}$ so starting point is at $\left(-\frac{1}{2}, 0\right)$.

$\underset{\sim}{r}\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(4 \times \frac{\pi}{4}\right)\underset{\sim}{i} + \sin\left(4 \times \frac{\pi}{4}\right)\underset{\sim}{j}$ so endpoint is at $\left(\frac{1}{2}, 0\right)$.

Note that $\sin(4t) \ge 0$ for $0 \le t \le \frac{\pi}{4}$, so $y \ge 0$.

The y-axis intercept is at (0,1). The graph is symmetrical about the y axis.



(1 mark) correct shape
(1 mark) correct starting point and axis intercepts

Question 7 (5 marks)

$$y = \frac{x^2 - 1}{2}$$

surface area $= \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ (formula sheet) **(1 mark)**

$x = 0, \quad y = -\frac{1}{2}$ and $2y + 1 = x^2$

$x = 3, \quad y = 4$

$x = \sqrt{2y+1}$ as $x > 0$

$$\frac{dx}{dy} = \frac{1}{2}(2y+1)^{-\frac{1}{2}} \times 2$$

$$= \frac{1}{\sqrt{2y+1}}$$

surface area $= 2\pi \int_{-\frac{1}{2}}^{4} \sqrt{2y+1}\sqrt{1 + \frac{1}{2y+1}}\, dy$ **(1 mark)**

$$= 2\pi \int_{-\frac{1}{2}}^{4} \sqrt{2y+1}\sqrt{\frac{2y+1}{2y+1} + \frac{1}{2y+1}}\, dy$$

$$= 2\pi \int_{-\frac{1}{2}}^{4} \sqrt{2y+1}\sqrt{\frac{2y+2}{2y+1}}\, dy$$

$= 2\pi \int_{-\frac{1}{2}}^{4} (2y+2)^{\frac{1}{2}}\, dy$ **(1 mark)**

$= 2\pi \left[\frac{(2y+2)^{\frac{3}{2}}}{\frac{3}{2} \times 2} \right]_{-\frac{1}{2}}^{4}$ **(1 mark)**

$$= \frac{2\pi}{3} \left[(2y+2)^{\frac{3}{2}} \right]_{-\frac{1}{2}}^{4}$$

$$= \frac{2\pi}{3} \left(10^{\frac{3}{2}} - 1^{\frac{3}{2}} \right)$$

$= \frac{2\pi}{3}\left(10\sqrt{10} - 1\right)$ square units **(1 mark)**

Question 8 (3 marks)
The three vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent if there exist real numbers *m* and *n* such that, for example, $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.
Equating the components for $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$, we have

$$-2 = 2m + 3n \qquad (1)$$
$$\lambda = -m + 3n \qquad (2)$$
$$1 = 3m - 2n \qquad (3)$$

(1 mark)

$(1)\times 3$ gives $\quad -6 = 6m + 9n \qquad (4)$
$(3)\times 2$ gives $\quad 2 = 6m - 4n \qquad (5)$

$(4)-(5)$ gives $-8 = 13n$

$$n = -\frac{8}{13}$$ **(1 mark)**

Substitute this value into (1).

$$-2 = 2m + 3\times -\frac{8}{13}$$
$$-\frac{2}{13} = 2m$$
$$m = -\frac{1}{13}$$

Substitute this value into (2).

$$\lambda = \frac{1}{13} + 3\times -\frac{8}{13}$$
$$= \frac{1}{13} - \frac{24}{13}$$
$$= -\frac{23}{13}$$

So the three vectors are linearly dependent when $\lambda = -\frac{23}{13}$. **(1 mark)**

Question 9 (5 marks)

a. At the point of intersection, $\underset{\sim}{r}_1(s) = \underset{\sim}{r}_2(t)$.

$$\underset{\sim}{r}_1(s) = \underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k} + s(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$$
$$\underset{\sim}{r}_2(t) = 3\underset{\sim}{i} + \underset{\sim}{j} - 21\underset{\sim}{k} + t(\underset{\sim}{i} + 3\underset{\sim}{j} + 8\underset{\sim}{k})$$

Equating $\underset{\sim}{i}$ components gives $1 + 2s = 3 + t$ so $2s - t = 2$ (1)
Equating $\underset{\sim}{j}$ components gives $3 + 2s = 1 + 3t$ so $2s - 3t = -2$ (2) **(1 mark)**

$(1)-(2)$ gives $2t = 4$

$t = 2$ and $s = 2$ **(1 mark)**

So $\underset{\sim}{r}_1(2) = \underset{\sim}{r}_2(2) = 5\underset{\sim}{i} + 7\underset{\sim}{j} - 5\underset{\sim}{k}$ and the point of intersection is $(5, 7, -5)$. **(1 mark)**

If you have time, check that these values are consistent with the $\underset{\sim}{k}$ components.
$-3 - s = -21 + 8t$ becomes $-3 - 2 = -21 + 16$ which is true

b. distance $= \left|\overrightarrow{PQ} \cdot \hat{\underset{\sim}{n}}\right|$, $P = (5, 7, -5)$, Q is any point on Π, for example $Q(2, 0, 0)$, and

$\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{n}}{|\underset{\sim}{n}|}$ where $\hat{\underset{\sim}{n}}$ is a unit vector normal to the plane.

$$\overrightarrow{PQ} \cdot \hat{\underset{\sim}{n}} = \left(-3\underset{\sim}{i} - 7\underset{\sim}{j} + 5\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right) \times \frac{1}{3}$$ **(1 mark)**

$$\left|\overrightarrow{PQ} \cdot \hat{\underset{\sim}{n}}\right| = \left|(-6 - 7 + 10) \times \frac{1}{3}\right|$$
$$= \left|-3 \times \frac{1}{3}\right|$$
$$= 1 \text{ unit}$$ **(1 mark)**

Question 10 (5 marks)

a. $\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$ **(1 mark)**

let $u = x$ $\quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}e^{2x}$

So $\int_0^1 xe^{2x}dx$

$= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \left[\frac{1}{4}e^{2x}\right]_0^1$

$= \left(\frac{1}{2}e^2 - 0\right) - \left(\frac{1}{4}e^2 - \frac{1}{4}e^0\right)$

$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$

$= \frac{1}{4}e^2 + \frac{1}{4}$ **(1 mark)**

b. Let $\frac{5x-3}{(x+1)(x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3}$

$= \frac{A(x^2+3)+(Bx+C)(x+1)}{(x+1)(x^2+3)}$

True iff $\quad 5x-3 = A(x^2+3)+(Bx+C)(x+1)$

Put $x=-1$, $\quad -8 = 4A \quad A = -2$

Put $x = 0$, $\quad -3 = 3A + C$

$-3 = -6 + C \quad C = 3$

Put $x = 1$, $\quad 2 = 4A + (B+C)\times 2$

$2 = -8 + 2B + 6 \quad B = 2$

So $\int_0^1 \frac{5x-3}{(x+1)(x^2+3)}dx$

$= \int_0^1 \left(\frac{-2}{x+1} + \frac{2x+3}{x^2+3}\right)dx$ **(1 mark)**

$= \int_0^1 \left(\frac{-2}{x+1} + \frac{2x}{x^2+3} + \frac{3}{x^2+3}\right)dx$

$= \left[-2\log_e|x+1| + \log_e|x^2+3| + \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$ **(1 mark)**

$= \left(-2\log_e(2) + \log_e(4) + \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) - \left(0 + \log_e(3) + 0\right)$

$= -\log_e(2^2) + \log_e(2^2) + \frac{\sqrt{3}\pi}{6} - \log_e(3)$

$= \frac{\sqrt{3}\pi}{6} - \log_e(3)$ **(1 mark)**