2022 VCAA Specialist Maths Exams

Exam 1 - Short Answer Questions

Question 1 (3 marks)

Consider the equation $p(z) = z^2 + 6iz - 25,\ z \in \mathbb{C}$.

a. Express $p(z)$ in the form $p(z) = (z + ai)^2 + b$, where $a, b \in \mathbb{R}$. 1 mark

This is just an exercise in completing the square - with nice simple numbers too!

$$p(z) = \left[z^2 + 6iz\right] - 25 = \left[(z + 3i)^2 - (3i)^2\right] - 25$$

$$\boxed{p(z) = (z + 3i)^2 - 16\text{ , i.e., } a = 3,\ b = -16}$$

b. Hence, or otherwise, find the solutions of the equation $p(z) = 0$. 2 marks

"Hence" method:

$$0 = (z + 3i)^2 - 16 = (z + 3i - 4)(z + 3i + 4)$$

$$\Longrightarrow \boxed{z = 4 - 3i \text{ or } z = -4 - 3i}$$

"Otherwise" -- use quadratic equation:

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1)(-25)}}{2 \times 1} = \frac{-6i \pm \sqrt{100 - 36}}{2} = -3i \pm 4$$

Question 2 (3 marks)

Solve the differential equation $\dfrac{dy}{dx} = -x\sqrt{4 - y^2}$ given that $y(2) = 0$.

Give your answer in the form $y = f(x)$.

Rearrange DE to get $x = \dfrac{-1}{\sqrt{4 - y^2}}\dfrac{dy}{dx}$

Integrate from $x = 2$ to some x: $\displaystyle\int_2^x x\,dx = \int_2^x \frac{-1}{\sqrt{4 - y^2}}\frac{dy}{dx}dx$

$$\Longrightarrow \left[\frac{1}{2}x^2\right]_2^x = \frac{1}{2}x^2 - 2 = -\int_0^y \frac{1}{\sqrt{4 - y^2}}dy = -\arcsin\left(\frac{y}{2}\right) + 0$$

$$\Longrightarrow \boxed{y = 2\sin\left(2 - \frac{1}{2}x^2\right)}$$

Check:

$$y(2) = 2\sin(2 - 2) = 0$$

$$\frac{dy}{dx} = 2\left(-\tfrac{1}{2}\right)(2x)\cos\left(2 - \tfrac{1}{2}x^2\right) = -2x\sqrt{1 - \sin^2\left(2 - \tfrac{1}{2}x^2\right)}$$

$$= -2x\sqrt{1 - 2^{-2}y^2} = -x\sqrt{4 - y^2}$$ ☺

Question 3 (4 marks)

The time taken by a coffee machine to dispense a cup of coffee varies normally with a mean of 10 seconds and a standard deviation of 1.5 seconds.

a. Find the probability that more than 34 seconds is needed to dispense a total of four cups of coffee. Give your answer to two decimal places. 2 marks

T = "random variable for the time to dispense one cup of coffee"

$T \sim \text{N}(\mu = 10,\ \sigma = 1.5)$

Let X = "the random variable for the time to dispense 4 cups of coffee"

i.e., $X = T_1 + T_2 + T_3 + T_4$. We can calculate its mean and variance:

$$E(X) = E(T_1 + T_2 + T_3 + T_4) = E(T_1) + E(T_2) + E(T_3) + E(T_4) = 4E(T)$$

$$\text{Var}(X) = \text{Var}(T_1 + T_2 + T_3 + T_4) = \text{Var}(T_1) + \text{Var}(T_2) + \text{Var}(T_3) + \text{Var}(T_4) = 4\text{Var}(T)$$

$\implies \mu_X = 40$ and $\sigma_X = 2\sigma_T = 3$

$X \sim \text{N}(\mu = 40,\ \sigma = 3)$

$\Pr(X > 34) = \Pr(X > \mu_X - 2\sigma_X) = \Pr(Z > -2)$, where Z is a standard normal deviate

From the 68-95-99.7 rule, it follows that

$\Pr(Z > -2) \approx 0.95 + 0.025 = 0.975$

$\boxed{\Pr X > 34) \approx 0.98}$

b. The machine is to be serviced. After it is serviced, it is expected that the mean time taken to dispense a cup of coffee will be reduced, but that the standard deviation will remain the same. Following the service, the mean time taken to dispense 25 cups of coffee is found to be 9 seconds.

Find a 95% confidence interval for the mean time that the machine takes to dispense a cup of coffee following the service. Give your answer in seconds, correct to one decimal place. 2 marks

$n = 25,\ \hat{y} = 9$

$$CI = \hat{y} \pm z_* \sigma_{\hat{Y}} = 9 \pm 1.96\,\frac{1.5}{\sqrt{25}} \approx 9 \pm 2 \times \frac{3}{10} = 9 \pm \frac{3}{5}$$

$\boxed{CI = (8.4,\ 9.6)}$

Question 4 (4 marks)

Find $\displaystyle\int \frac{3x^2 + 4x + 12}{x\left(x^2 + 4\right)} dx$

Easy way is to carefully split up the numerator

$$\boxed{\int \frac{3\left(x^2 + 4\right) + 4x}{x\left(x^2 + 4\right)} dx = \int \frac{3}{x} + \frac{4}{x^2 + 4} dx = 3\ln|x| + 2\arctan\left(\frac{x}{2}\right) + c,\quad c \text{ is a constant}}$$

Default approach would be to use partial fractions

$$\frac{3x^2+4x+12}{x\left(x^2+4\right)} = \frac{A}{x} + \frac{B+Cx}{x^2+4} = \frac{A\left(x^2+4\right)+x(B+Cx)}{x\left(x^2+4\right)} = \frac{(A+C)x^2+Bx+4A}{x\left(x^2+4\right)}$$

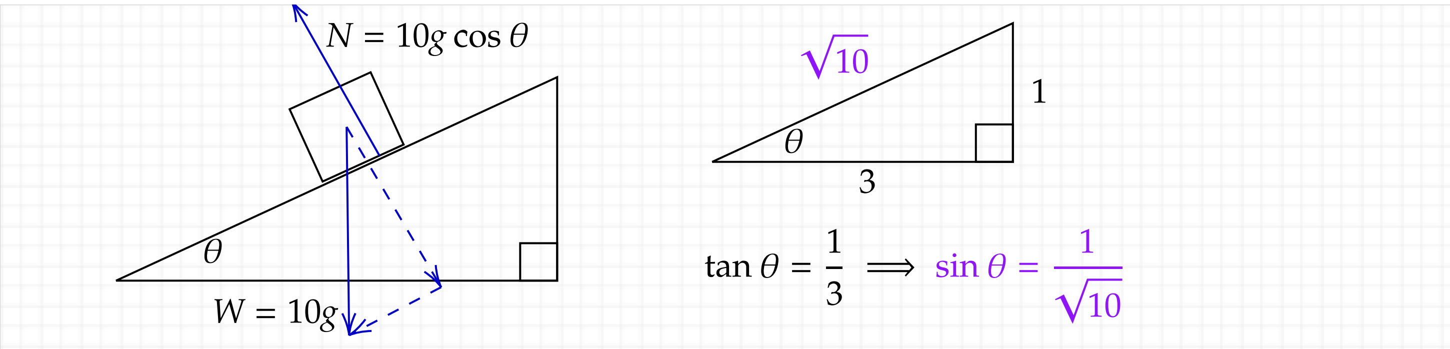
Comparing coefficients of x^n in the numerator:

$A=3,\ B=4,\ C=0$

$$\implies \int \frac{3x^2+4x+12}{x\left(x^2+4\right)}dx = \int \frac{3}{x} + \frac{4}{x^2+4}dx = 3\ln|x| + 2\arctan\left(\frac{x}{2}\right) + c,\quad c \text{ is a constant}$$

Question 5 (3 marks)

A body of mass 10 kg, which is initially at rest, slides down a smooth inclined plane, as showing in the diagram below. The plane is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{1}{3}$.



a. Find the speed of the body after it has been in motion for two seconds. 2 marks

$$F_{net} = 10g\sin\theta = 10g \times \frac{1}{\sqrt{10}} = \sqrt{10}\,g$$

$a = \frac{F_{net}}{m} = \frac{g}{\sqrt{10}}$. This is a constant acceleration, so can use SUVAT with $u=0,\ t=2,\ v=?$

$$v = u + at = 0 + \frac{g}{\sqrt{10}} \times 2 \implies \boxed{v = \frac{\sqrt{10}g}{5}}$$

b. After the body has been in motion for two seconds, a constant braking force, R Newtons, is applied to the body parallel to the plane, so that the body has a constant velocity.
Find the value [magnitude?] of R 1 mark

The braking force exactly cancels the net force (component of the weight down the plane) found above, $R = \sqrt{10}g$ up the plane (against the motion).

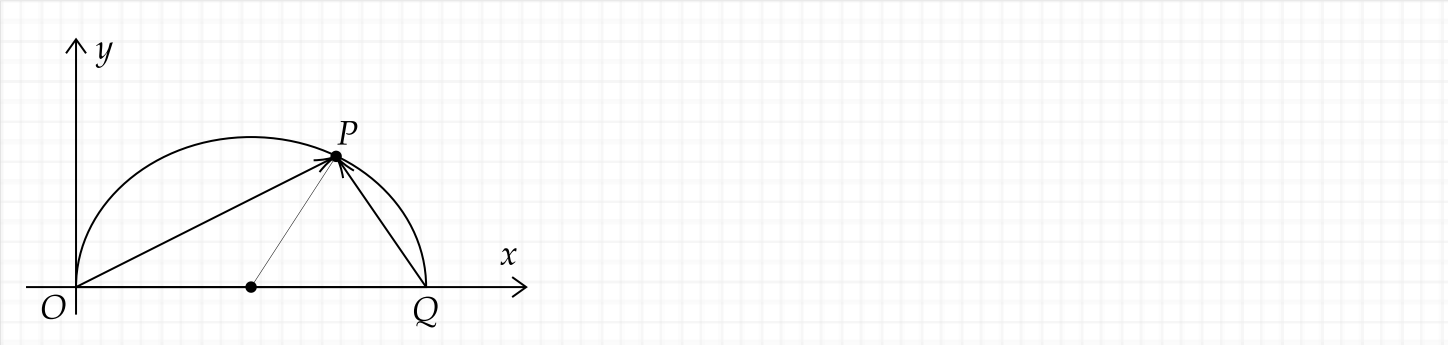
Question 6 (6 marks)

a. Find the cosine of the acute angle between the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ 2 marks

Let θ be the angle between **a** and **b**

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\ |\mathbf{b}|} = \frac{2-6+12}{\sqrt{4+9+36}\sqrt{1+4+4}} = \frac{8}{7\times 3} \implies \boxed{\cos\theta = \frac{8}{21}}$$

b. OPQ is a semicircle of radius a with equation $y = \sqrt{a^2-(x-a)^2}$.
$P(x, y)$ is a point on the semicircle OPQ, as shown below



i. Express the vectors $\overrightarrow{OP}$ and $\overrightarrow{QP}$ in terms of $a,\ x,\ y,\ \mathbf{i},\ \mathbf{j}$ — 1 mark

$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$
$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP} = -2a\mathbf{i} + x\mathbf{i} + y\mathbf{j} = (x-2a)\mathbf{i} + y\mathbf{j}$

ii. Hence, use the dot product to determine whether $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{QP}$ — 3 marks

$\overrightarrow{OP}\cdot\overrightarrow{QP} = x(x-2a) + y^2 = x^2 - 2ax + y^2$
But, x, y are on the circle, so $y^2 = a^2 - (x-a)^2 = 2ax - x^2$
$\overrightarrow{OP}\cdot\overrightarrow{QP} = x^2 - 2ax + \left(2ax - x^2\right) = 0$
So, the angle between $\overrightarrow{OP}$ and $\overrightarrow{QP}$ is $90°$ and the two vectors are perpendicular.
(This is just a vector proof of Thales' theorem)

Question 7 (3 marks)

A curve has equation $x\cos(x+y) = \dfrac{\pi}{48}$.

Find the gradient of the curve at the point $\left(\dfrac{\pi}{24}, \dfrac{7\pi}{24}\right)$.

Give your answer in the form $\dfrac{a\sqrt{b}-\pi}{\pi}$, where $a, b \in \mathbb{Z}$.

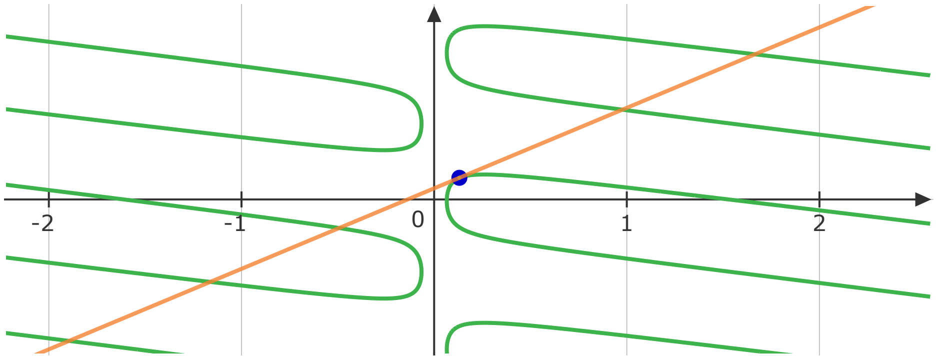
Differentiate both sides of the equation: $0 = \dfrac{d}{dx}(x\cos(x+y)) = \cos(x+y) + x\left(1+\dfrac{dy}{dx}\right)(-\sin(x+y))$

$$\implies 0 = \cos(x+y) - x\sin(x+y)\left(1+\frac{dy}{dx}\right)$$

$$\implies \frac{dy}{dx} = \frac{\cos(x+y)}{x\sin(x+y)} - 1 = \frac{1}{x}\cot(x+y) - 1$$

At the given point: $\dfrac{dy}{dx} = \dfrac{24}{\pi}\cot\left(\dfrac{\pi}{3}\right) - 1 = \dfrac{24}{\pi}\dfrac{1}{\sqrt{3}} - 1 \implies \boxed{\dfrac{dy}{dx} = \dfrac{8\sqrt{3}-\pi}{\pi}}$

It is actually a pretty nice periodic curve - pictured below, including the tangent at the given point



Question 8 (4 marks)

A body moves in a straight line so that when its displacement from a fixed origin O is x meters, its acceleration, a, is $-4x$ ms^{-2}. The body accelerates from rest and its velocity, v, is equal to -2 ms^{-1} as it passes through the origin. The body then comes to rest again.
Find v in terms of x for this interval.

$a = -4x = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$\implies c - 2x^2 = \dfrac{1}{2}v^2$, c is a constant

When $x = 0,\ v = -2$, so $c = 2$

$\implies v^2 = 4\left(1 - x^2\right)$

$\implies \boxed{v = -2\sqrt{1-x^2}}$

Note that the velocity is negative during the specified integral.

This is just a mass on a spring being released from a positive displacement until it reaches its largest negative displacement. The phase space is just the ellipse given by $v(x)$ above.

Question 9 (4 marks)

Given that $f'(x) = \dfrac{\cos(2x)}{\sin^3(2x)}$ and $f\left(\dfrac{\pi}{8}\right) = \dfrac{3}{4}$, find $f(x)$

$f(x) - f\left(\dfrac{\pi}{8}\right) = \displaystyle\int_{\frac{\pi}{8}}^{x} f'(x)dx = \int_{\frac{\pi}{8}}^{x} \dfrac{\cos(2x)}{\sin^3(2x)}dx$, let $u = \sin(2x),\ du = 2\cos 2x\, dx$

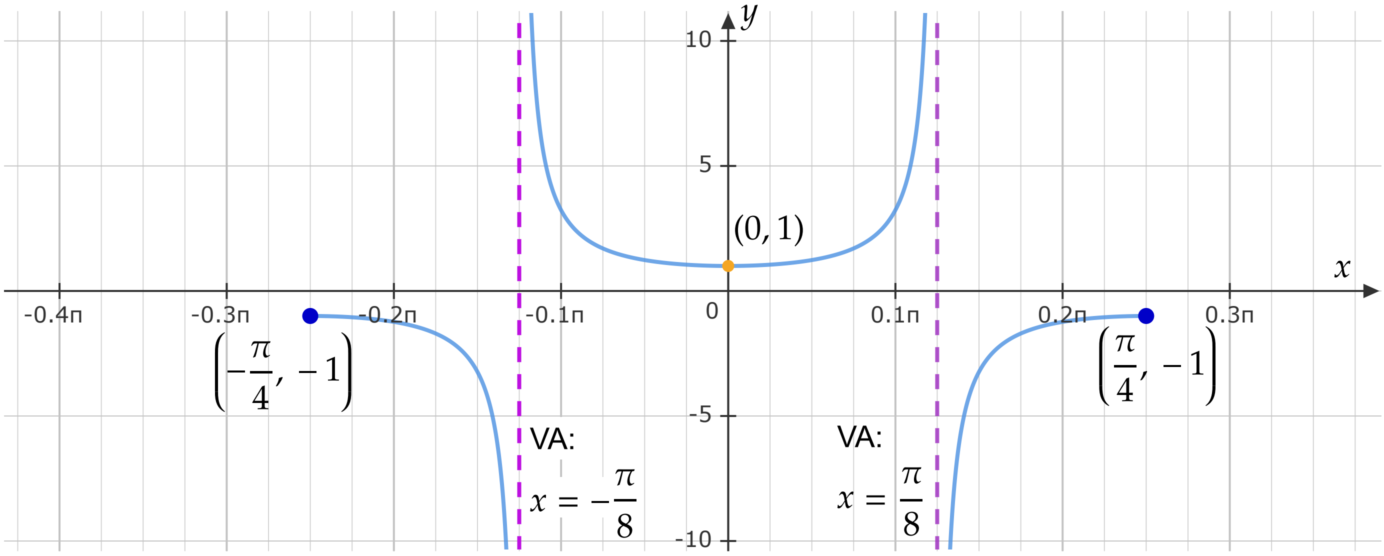
$\implies f(x) = \dfrac{3}{4} + \dfrac{1}{2}\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\sin(2x)} \dfrac{1}{u^3}du = \dfrac{3}{4} - \dfrac{1}{4}\left[u^{-2}\right]_{\frac{1}{\sqrt{2}}}^{\sin 2x} = \dfrac{3}{4} - \dfrac{1}{4}\left(\dfrac{1}{\sin^2 2x} - \dfrac{1}{1/2}\right)$

$\implies \boxed{f(x) = \dfrac{1}{4}\left(5 - \csc^2 2x\right)}$ or equivalent expression

Question 10 (6 marks)

Let $f(x) = \sec(4x)$

a. Sketch the graph of f for $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ on the set of axes below. Label any asymptotes with their equations and label any turning points and endpoints with their coordinates. 3 marks



b. The graph of $y = f(x)$, for $x \in \left[-\frac{\pi}{24}, \frac{\pi}{48}\right]$ is rotated about the x-axis to form a solid of revolution.

Find the volume of this solid. Give your answer in the form $\frac{\left(a-\sqrt{b}\right)\pi}{c}$, where $a, b, c \in \mathbb{R}$ 3 marks

$$V = \int_{-\frac{\pi}{24}}^{\frac{\pi}{48}} \pi\, y^2 dx = \int_{-\frac{\pi}{24}}^{\frac{\pi}{48}} \pi \sec(4x)^2\, dx = \left[\frac{\pi}{4}\tan(4x)\right]_{-\frac{\pi}{24}}^{\frac{\pi}{48}} = \frac{\pi}{4}\left(\tan\left(\frac{\pi}{12}\right) - \tan\left(-\frac{\pi}{6}\right)\right)$$

Need $\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ and $\tan\left(\frac{\pi}{12}\right)$, which is a bit more annoying...

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1} = \frac{\left(\sqrt{3}-1\right)^2}{2} = 2-\sqrt{3}$$

$$\Longrightarrow \boxed{V = \frac{\pi}{4}\left(\left(2-\sqrt{3}\right) + \frac{\sqrt{3}}{3}\right) = \frac{\left(3-\sqrt{3}\right)\pi}{6}}$$

could have any real a and b you want and $V = \frac{\left(a-\sqrt{b}\right)\pi}{c}$, with $c = 6\,\frac{a-\sqrt{b}}{3-\sqrt{3}}$...

Exam 2 - Multiple Choice Questions

Question 1 (B)

By hand: $|2x-1|-|x-3| = (2x-1)+(x-3) = 3x-4$ given $\frac{1}{2} \le x \le 3$

CAS: abs$(2x-1)$ − abs$(x-3)$ | $\frac{1}{2} \le x \le 3 \;\rightarrow\; 3x-4$

Question 2 (E)

$$1-\frac{4\sin^2 x}{\tan^2 x+1} = 1-\frac{4\sin^2 x}{\sec^2 x} = 1-4\sin^2 x\cos^2 x = 1-\sin^2 2x = \cos^2 2x$$

CAS: tExpand$\left(1-\frac{4\sin(x)^2}{\tan(x)^2+1}-\cos(2x)^2\right) \;\rightarrow\; 0$

Question 3 (E)

$$y=\frac{x^2+2x+c}{x^2-4}=\frac{x^2-4+2x+c+4}{x^2-4}=1+\frac{2x+c+4}{(x-2)(x+2)}$$

This has the HA $y=1$ and at least one VA at either $x=2$ or $x=-2$ or both

Question 4 (B)

$p(z)=(z-a)(z-b)(z-c)$ expands to a cubic with real coefficients.
Given $a, b, c \in \mathbb{C}$ AND Re$(a) \ne 0$, Re$(b) \ne 0$, Re$(c) \ne 0$ and Im$(b)=0$.

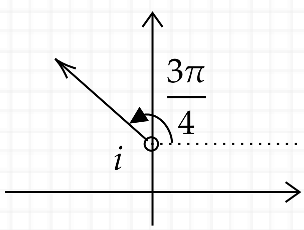
So, b is a real root and

either $\overline{a}=c$	OR $a, c \in \mathbb{R}$.
This gives the answer as B	OR no correct answer

As the question asks which is "necessarily" true, there is no correct answer...
This is similar to Q2.a.i in Exam 2 in 2021

Question 5 (A)

Just draw a picture!

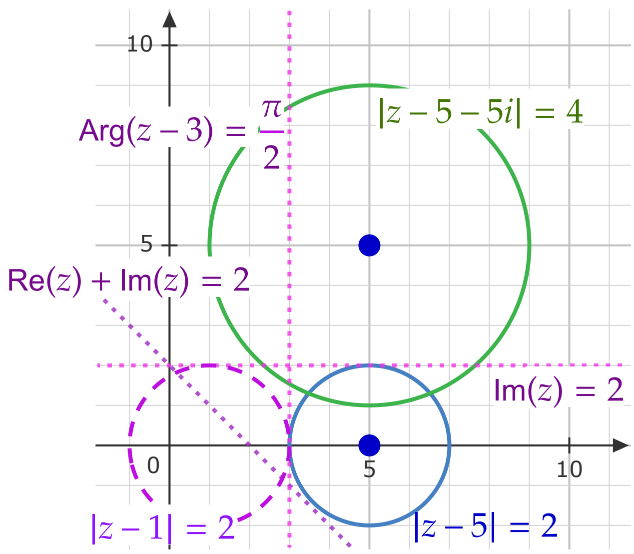


$$\text{Arg}(z-i)=\text{Arg}(z-(0+i))=\frac{3\pi}{4}$$

$$\implies y-1=\tan\left(\frac{3\pi}{4}\right)(x-0),\; x<0$$

$$\implies y=1-x,\; x<0$$

Question 6 (E)



$|z-5| = 2$ is a circle of radius 2 centred at $5 + 0i$.
The circle $|z-5-5i| = 4$ intersects it twice.

Notes - don't need to draw the others on, but can!

$\text{Arg}(z-3) = \frac{\pi}{2} \implies x = 3$ is tangential to $|z-5| = 2$

$|z-1| = 2$ just touches $|z-5| = 2$ once at $x = 3$

$\text{Im}(z) = 2 \implies y = 2$ is tangential to $|z-5| = 2$

$\text{Re}(z) + \text{Im}(z) = 2 \implies y = 2 - x$ does not intersect $|z-5| = 2$

Question 7 (D)

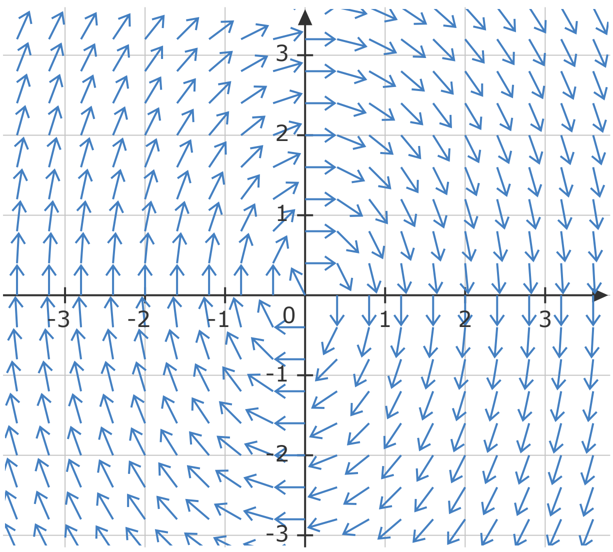
$$\int_0^{\ln 2} \frac{1}{1+e^x} dx,$$

let $u = 1 + e^x \implies du = e^x dx = (u-1)dx.$

$$x = 0 \implies y = 2,\ x = \ln 2 \implies x = 3$$

$$\int_2^3 \frac{1}{u\,(u-1)} du = \int_2^3 \frac{1}{u-1} - \frac{1}{u} du$$

Question 8 (C)



Imagine the vector field to the left is a gradient field...

It corresponds to the DE $\frac{dy}{dx} = \frac{-2x}{y}$

Note it can't be $\frac{dy}{dx} = \frac{2x}{y}$ or $\frac{dy}{dx} = \frac{x^2}{2}$ or $\frac{dy}{dx} = \frac{y^2}{2} + x^2$ as the slope in Q1 (on the line $y = x$) is always negative; the ellips shape of the field explains the 2nd two distractors.

The choice was then between $\frac{dy}{dx} = \frac{-2x}{y}$ and $\frac{dy}{dx} = \frac{-x}{2y}$ but on the line $y = x$ the vetor field is more like -2 than $-\frac{1}{2}$, which we can judge as the scale is 1:1

Question 9 (B)

Using Euler's method to approximate the solution to $\dfrac{dy}{dx} = 2x^2$

starting at $x_0 = 1,\ y_0 = 2$ and getting $y_2 = 2.976$ for some step size h.

Can solve this by hand:

Euler's method (forward difference) for this DE is $y_{n+1} = y_n + 2x_n^2\, h$.

$x_0 = 1, \qquad y_0 = 2$

$x_1 = 1 + h, \quad y_1 = 2 + 2(1)^2 h = 2 + 2h$

$x_2 = 1 + 2h, \ y_2 = (2 + 2h) + 2(1 + h)^2 h = 2 + 4h + 4h^2 + 2h^3$

If $h = 0.1$, then this gives $y_2 = 2.442$ (the first distractor)

If $h = 0.2$, this gives $y_2 = 2 + 0.8 + 0.16 + 0.016 = 2.9676$, so done!

Check: If $h = 0.3$, then $y_2 = 3.2 + 4h^2 + 2h^3$ ... too big & sim for $h = 0.4$

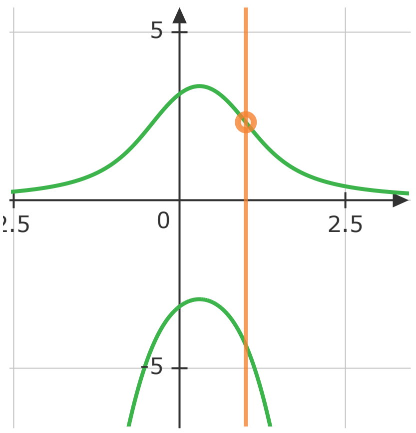
Can also solve using the CAS

$steps(h){:=}\text{euler}(2\cdot x^2, x, y, \{1, 1+2\cdot h\}, 2, h)$ — *Done*

$steps(0.1)$ — $\begin{bmatrix} 1. & 1.1 & 1.2 \\ 2. & 2.2 & 2.442 \end{bmatrix}$

$steps(0.2)$ — $\begin{bmatrix} 1. & 1.2 & 1.4 \\ 2. & 2.4 & 2.976 \end{bmatrix}$

Question 10* (E)



Want to find when the tangent to the curve $5x^2y - 3xy + y^2 = 10$ at $(1, m)$ has negative gradient.

First note that there are two points on the curve when

$x = 1{:}\ \ y = -1 \pm \sqrt{11}$.

At both of those points the gradient is negative → E

I fell into this trap 1st time through : Differentiate both sides:

$0 = 10xy + 5x^2y' - 3y - 3xy' + 2yy'$

$\quad = y'(5x^2 - 3x + 2y) + (10xy - 3y)$

$\implies y' = \dfrac{3y - 10xy}{5x^2 - 3x + 2y} \overset{(1,m)}{\longrightarrow} \dfrac{3m - 10m}{5 - 3 + 2m} = \dfrac{-7m}{2(1+m)} < 0$

$\implies m > 0$ or $m < -1 \implies m \in \mathbb{R} \setminus [-1, 0] \ \rightarrow$ A

$eqn{:=}5\cdot x^2\cdot y - 3\cdot x\cdot y + y^2 = 10$	$5\cdot x^2\cdot y - 3\cdot x\cdot y + y^2 = 10$
$\text{impDif}(eqn, x, y)$	$\dfrac{-(10\cdot x - 3)\cdot y}{5\cdot x^2 - 3\cdot x + 2\cdot y}$
$\text{solve}(eqn \text{ and } x=1, y)$	$y = -(\sqrt{11}+1) \text{ and } x=1 \text{ or } y = \sqrt{11}-1 \text{ and } x=1$
$\text{impDif}(eqn, x, y)\vert x=1 \text{ and } y=-(\sqrt{11}+1)$	-4.55528970602
$\dfrac{7\cdot\sqrt{11}}{22} - \dfrac{7}{2}$	-2.44471029398
$\text{impDif}(eqn, x, y)\vert x=1 \text{ and } y=m$	$\dfrac{-7\cdot m}{2\cdot(m+1)}$
$\text{solve}\left(\dfrac{-7\cdot m}{2\cdot(m+1)} < 0, m\right)$	$m < -1 \text{ or } m > 0$

Question 11 (A)

$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + p\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - q\mathbf{k}$, $\mathbf{c} = -3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$

These are linearly dependent when $\exists\, p, q$ st $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$

$$\begin{cases} 2\alpha + \beta = -3 \\ -3\alpha + 2\beta = 2 \\ \alpha p - \beta q = 5 \end{cases} \implies \begin{cases} \alpha = -8/7 \\ \beta = -5/7 \\ 8p - 5q = -35 \end{cases} \implies 8p = 5q - 35$$

Note: Can also just use the determinant of the matrix of vectors - probably the fastest route

$$\text{solve}(2\cdot a + b = -3 \text{ and } -3\cdot a + 2\cdot b = 2, a, b)$$

$$a = \frac{-8}{7} \text{ and } b = \frac{-5}{7}$$

$$\det\left(\begin{bmatrix} 2 & -3 & p \\ 1 & 2 & -q \\ -3 & 2 & 5 \end{bmatrix}\right) = 0 \qquad 8\cdot p - 5\cdot (q-7) = 0$$

Question 12 (A)

$\mathbf{u}(x) = -\csc(x)\mathbf{i} + \sqrt{3}\mathbf{j},\ v(x) = \cos(x)\mathbf{i} + \mathbf{j}$ then

$\mathbf{u}(x) \cdot \mathbf{v}(x) = -\csc(x)\cos(x) + \sqrt{3} \times 10 = 0$

$$\implies \cot(x) = \sqrt{3} \implies \tan(x) = \frac{1}{\sqrt{3}} \implies x = \frac{\pi}{6} + n\pi,\ n \in \mathbb{Z}$$

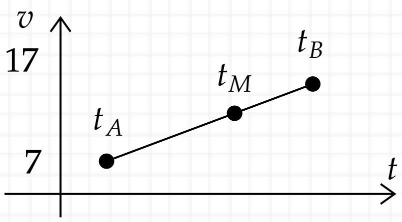
Question 13 (B)

$\ddot{r}(t) = \sin(t)i + 2\cos(t)j,\ t \geqslant 0$. Given $\dot{r}(0) = 2i + j$, then

$$\dot{r}(t) = \dot{r}(0) + \int_0^t \ddot{r}(\tau)d\tau = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} \sin\tau \\ 2\cos\tau \end{bmatrix} d\tau = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -\cos\tau \\ 2\sin\tau \end{bmatrix}_0^t$$

$$\dot{r}(t) = \begin{bmatrix} 2 - (\cos t - 1) \\ 1 + 2\sin t \end{bmatrix} = \begin{bmatrix} 3 - \cos t \\ 1 + 2\sin t \end{bmatrix}$$

Question 14* (D)



Note: Originally I goofed on this and chose C

$a = const$, at points A and B, $v_A = u = 7,\ v_B = v = 17$

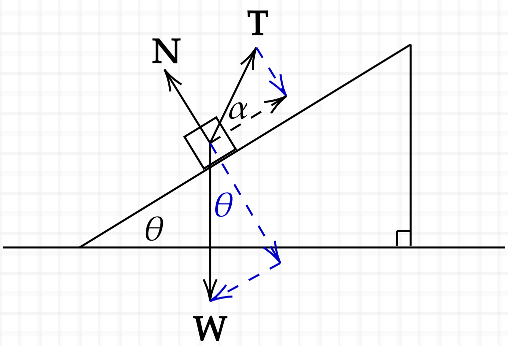
Because it is constantly accelarating, the midpoint is not at the half-time.

$$v^2 = u^2 + 2as \implies s = \frac{v^2 - u^2}{2a} = \frac{17^2 - 7^2}{2a} = \frac{120}{a},\quad \frac{1}{2}s = \frac{60}{a}$$

At midpoint: $v_M^2 = u^2 + 2a\left(\frac{1}{2}s\right) = 7^2 + 2 \times 60 = 169$

$$v_M = \sqrt{169} = 13$$

Question 15 (E)



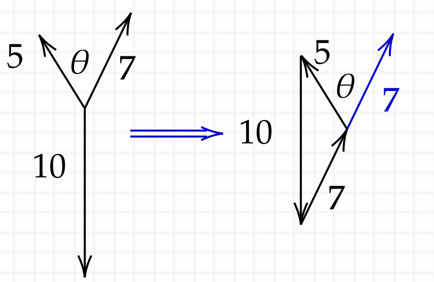
$F_{net} = \mathbf{N} + \mathbf{T} + \mathbf{W} = m\mathbf{a}$

This always holds true and is option E!

But the earlier distractors in this question were distracting!

Question 16 (A)

Three coplanar forces of magnitudes 5N, 7N and 10N are in equilibrium.



$10^2 = 5^2 + 7^2 - 2 \times 5 \times 7\cos(\pi - \theta)$

$100 = 25 + 49 + 70\cos(\theta)$ → **A**

Question 17 (C)

$m = 7$ kg, $a =$ const, $u = 3$ ms^{-1}, $s = 30$ m, $t = 6$ s

Change of momentum is: $\Delta p = m\Delta v$,

SUVAT without a: $s = \dfrac{u+v}{2}t \implies v = \dfrac{2s}{t} - u = 10 - 3 = 7$

$\Delta p = m(v - u) = 7 \times (7 - 3) = 28$ → **C**

Question 18 (B)

Random variable for time to travel to school is: $T \sim \mathrm{N}(30,\ 2.5)$

want $\Pr(|T_1 - T_2| > 6) = \Pr(T_1 - T_2 > 6) + \Pr(T_1 - T_2 < -6)$

So we need the random variable $X = T_1 - T_2$ which we know to be normal with

$E(X) = E(T_1 - T_2) = E(T_1) - E(T_2) = 0$,

$\text{Var}(X) = \text{Var}(T_1 - T_2) = 1^2\text{Var}(T_1) + (-1)^2\text{Var}(T_2) = 2 \times 2.5^2 = 12.5$

So $X = T_1 - T_2 \sim \mathrm{N}\left(0,\ \sqrt{5}\right)$ and we want $\Pr(|X| > 6) = 2\Pr(X > 6) \approx 0.089676$ → **B**

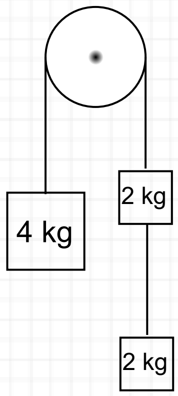
CAS: `2*normCdf(6,∞,0,sqrt(12.5)) → 0.089685961282976`

Question 19 (D)

No sample mean, so can't find a confidence interval... but, let's play along
Cost of producing an item of mass m is $C = 0.3m + 0.5$
Random variable for masses is $M \sim \mathrm{N}(7,\ 0.1)$
100 items are produced,
the "*95% confidence interval for the average cost per item is closest to*"
Read as "*95% of the time the average cost will be will be in the range...*"

So let's find the range:
$M \sim \mathrm{N}(7,\ 0.1) \implies C \sim \mathrm{N}(2.6,\ 0.03)$
$\left[\ \because E(C) = 0.3\,E(M) + 0.5 = 2.1 + 0.5 = 2.6,\ \text{Var}(C) = 0.3^2\text{Var}(M) = 0.3^2 0.1^2 = 0.03^2\right]$
95% of the time $\widehat{C}$ is in the range $E(\widehat{C}) \pm z_* \sigma_{\widehat{C}} = 2.6 \pm 1.96\,\dfrac{0.03}{\sqrt{100}} \approx (2.594,\ 2.606)$

Question 20 (C)



A 4kg mass is almost balanced by two 2kg masses
Where the labeled masses are normally distributed with the parameters

Labeled mass (kg)	**Mean (kg)**	**Std Dev (kg)**
2	1.980	0.015
4	3.940	0.002

If the 4kg mass moves up (assuming ideal pully, massless & inextensible string etc)
then $m_2 + m_{2'} - m_4 > 0$, so define the random variable $X = M_2 + M_{2'} - M_4$
$E(X) = 2 \times 1.980 - 3.940 = -0.02,$
$\text{Var}(X) = \text{Var}(M_2) + \text{Var}(M_2) + \text{Var}(M_4) \approx 0.000454$
$\Rightarrow X \sim \mathrm{N}(-0.02,\ 0.0213)$ and $\Pr(X > 0) \approx 0.8261$

Exam 2 - Extended Response Questions

Question 1 (11 marks)

Consider the family of function f with the rule $f(x) = \dfrac{x^2}{x-k}$, where $k \in \mathbb{R}\backslash\{0\}$.

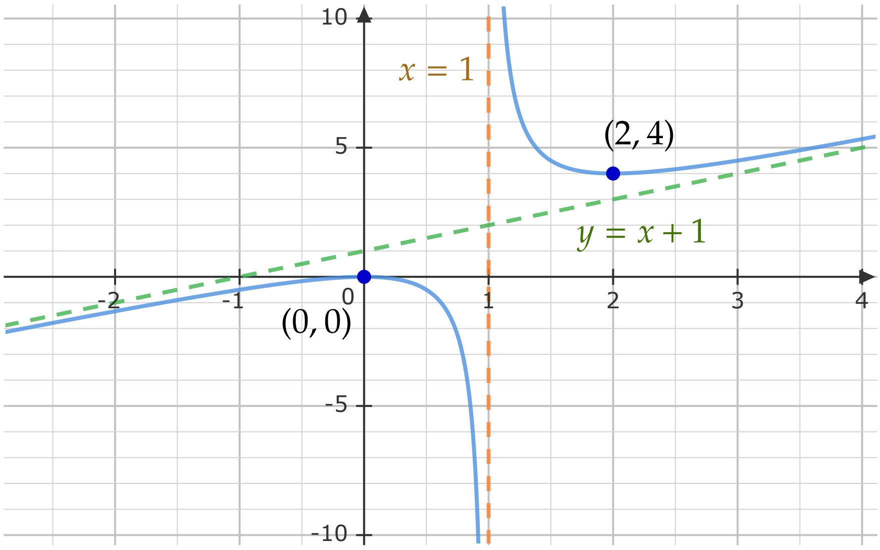
a. Write down the equations of the two asymptotes of the graph of f when $k = 1$ — 2 marks

$f(x) = \dfrac{x^2}{x-k} = \dfrac{x^2 - k^2 + k^2}{x-k} = x + k + \dfrac{k^2}{x-k}$ (note: can just use `expand` on the CAS to do this)

Vertical Asymptote: $x = k \quad \rightarrow x = 1$

Oblique Asymptote: $y = x + k \ \rightarrow y = x + 1$

b. Sketch the graph of $y = f(x)$ for $k = 1$ below:



$f(x) = x + 1 + \dfrac{1}{x-1}$

To find turning points:

$0 = f'(x) = 1 - \dfrac{1}{(x-1)^2}$

$\implies x = 1 \pm 1 = 0, 2$

$\implies f(x) = 0, 4$ resp

c. **i.** Find, in terms of k, the equations of any asymptote of $y = f(x)$

Vertical Asymptote: $x = k$

Oblique Asymptote: $y = x + k$

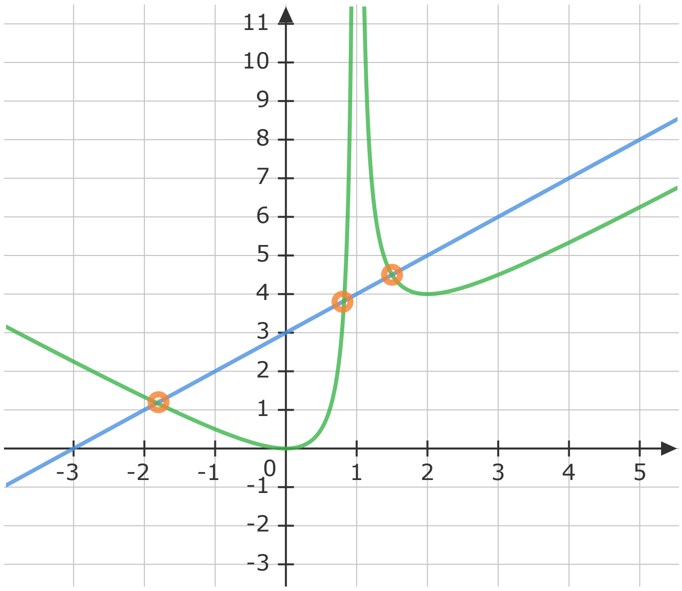
ii. Find the distance between the two turning points of the graph $y = f(x)$ in terms of k

Find TPs: $0 = f'(x) = 1 - \dfrac{k^2}{(x-k)^2} \implies TP@\ (0,0),\ (2k,\ 4k)$

Distance: $\sqrt{(2k-0)^2 + (4k-0)^2} = 2k\sqrt{1+4} = 2\sqrt{5}\,k$

d. Now consider $h(x0 = x + 3$ and $g(x) = \left|\dfrac{x^2}{x-1}\right|$.

The region bounded by the curves is rotated about the x-axis to form a volume



There are three intersection points, $x = \frac{-1 \pm \sqrt{7}}{2}, \frac{3}{2}$

CAS: solve$(h(x) = g(x), x)$

Only the left two bound a finite area

Volume of rotation $V = \int_{\frac{-1-\sqrt{7}}{2}}^{\frac{-1+}{2}} \pi\left((x+3)^2 - \left(\frac{x^2}{x-1}\right)^2\right)dx$

Evaluate to get

$V = 2\pi\left(4\ln\left(\sqrt{7}+3\right) - 2\ln(2) + \sqrt{7}\right) \approx 51.42$ units^2

Question 2 (9 marks)

Complex numbers $u = a + i,\ v = b - \sqrt{2}i$ with $a, b \in \mathbb{R}$

a. **i.** Given that $uv = \left(\sqrt{2}+\sqrt{6}\right) + \left(\sqrt{2}-\sqrt{6}\right)i,$ *show that* $a^2 + \left(1-\sqrt{3}\right)a - \sqrt{3} = 0$

$$uv = (a+i)\left(b-\sqrt{2}i\right) = ab + \sqrt{2} + \left(b - \sqrt{2}a\right)i \overset{?}{=} \left(\sqrt{2}+\sqrt{6}\right) + \left(\sqrt{2}-\sqrt{6}\right)i$$

Re: $ab + \sqrt{2} = \sqrt{2} + \sqrt{6} \implies ab = \sqrt{6}$

Im: $b - \sqrt{2}a = \sqrt{2} - \sqrt{6} \implies b = \sqrt{2} - \sqrt{6} + \sqrt{2}a = \sqrt{2}\left(1 - \sqrt{3} + a\right)$

$\implies ab = a \times \sqrt{2}\left(1-\sqrt{3}+a\right) = \sqrt{2}\left(a^2 + \left(1-\sqrt{3}\right)a\right) = \sqrt{6}$

$\implies a^2 + \left(1-\sqrt{3}\right)a - \sqrt{3} = 0$

ii. One set of possible values for a and b is $a = \sqrt{3},\ b = \sqrt{2}.$
Hence or otherwise, what are the other values?

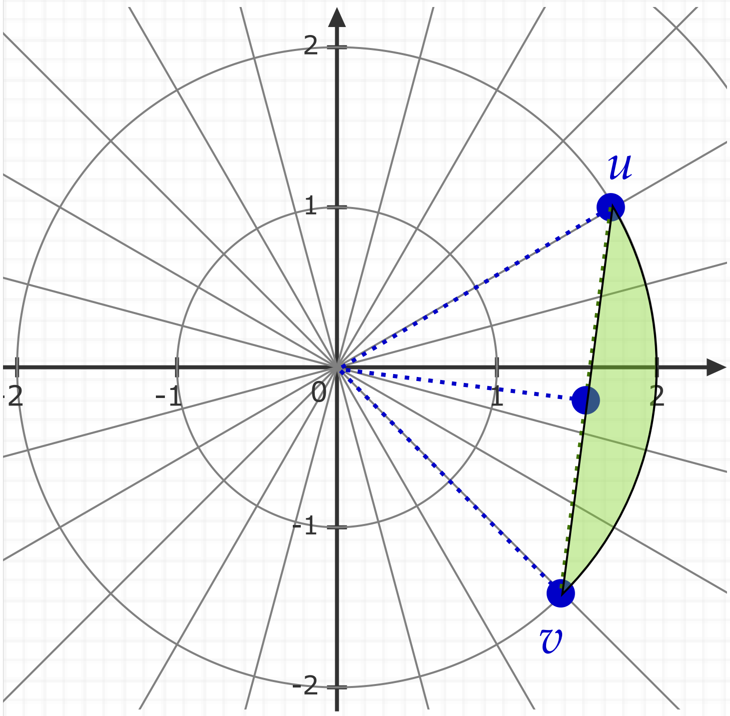
Option 1: Just been told that $\left(a - \sqrt{3}\right)$ is a factor of $a^2 + \left(1-\sqrt{3}\right)a - \sqrt{3},$
so $a^2 + \left(1-\sqrt{3}\right)a - \sqrt{3} = \left(a-\sqrt{3}\right)(a+1) = 0 \implies a = -1$ and $b = \sqrt{6}/a = -\sqrt{6}$

Could also just use the CAS

$u{:=}a+\boldsymbol{i}$	$a+\boldsymbol{i}$
$v{:=}b-\sqrt{2}\cdot\boldsymbol{i}$	$b-\sqrt{2}\cdot\boldsymbol{i}$
$\text{solve}\left(u\cdot v=\sqrt{2}+\sqrt{6}+\left(\sqrt{2}-\sqrt{6}\right)\cdot\boldsymbol{i},a,b\right)$	
	$a=-1$ and $b=-\sqrt{6}$ or $a=\sqrt{3}$ and $b=\sqrt{2}$

b. Plot the points $u =$ and v below

c. The ray $\operatorname{Arg}(z) = \theta$ passes through the midpoint of the line interval joining u and v.
Find θ and plot it in the diagram



c. - Option 1: Calculate

$$m = \frac{u+v}{2} = \frac{\sqrt{3}+\sqrt{2}}{2} - \frac{1-\sqrt{2}}{2}i$$

$$\implies \tan\theta = \frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}$$

$$\implies \theta = \frac{-\pi}{24}$$ (note: CAS does not give this nicely)

On the TI Nspire: $\frac{\tan^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right)}{\pi}$ ▶approxFraction(5.E−14)

Option 2: Geometry

The triangle $0,\ u = 2\text{cis}\left(\frac{\pi}{6}\right),\ v = 2\text{cis}\left(-\frac{\pi}{4}\right)$ is isoceles,

so the midpoint of u, v will divide the triangle in half.

Thus half the total angle is, $\frac{1}{2}\left(\frac{\pi}{6} - \frac{-\pi}{4}\right) = \frac{5\pi}{24}$

Adding this to $\frac{-\pi}{4}$ gives $\theta = -\frac{\pi}{24}$

d. The line segment joining u and v defines a minor segment with the circle $|z| = 2$. Find its area to 2 d.p.

$$A = \text{sector} - \text{triangle} = \frac{1}{2}\left(\frac{5\pi}{12}\right)(2)^2 - \frac{1}{2}2\times 2\times\sin\left(\frac{5\pi}{12}\right) = 2\left(\frac{5\pi}{12} - \sin\left(\frac{5\pi}{12}\right)\right) \approx 0.69 \text{ units}^2$$

Question 3 (10 marks)

A particle moves in a straight line, with distance from the origin x meters after time t seconds.

Motion satisfies $\frac{dx}{dt} = \frac{2e^{-x}}{1+4t^2}$ where $x = 0$ when $t = 0$.

a. **i.** Express the DE in the from $\int g(x)dx = \int f(t)dt$

$$\int e^x dx = \int \frac{2}{1+4t^2}dt$$

ii. Hence, show that $x = \log_e\left(\tan^{-1}(2t) + 1\right)$

$$\int e^x dx = e^x = \int \frac{2}{1+4t^2}dt = \arctan(2t) + C$$

When $t = 0,\ x = 0$: $\implies e^0 = 1 = \arctan(2\times 0) + C = C$

$\implies e^x = \arctan(2t) + 1$

$\implies x = \ln(\arctan(2t) + 1)$

b. The graph of $x(t) = \ln(\arctan(2t) + 1)$

As $t \to \infty,\ \arctan(t) \to \frac{\pi}{2}$, so $x(t) \to x = \ln\left(\frac{\pi}{2} + 1\right)$ is a horizontal asymptote

As $t \to -\infty$, $\arctan(t) \to -\dfrac{\pi}{2}$, so $x(t) \to \ln\left(-\dfrac{\pi}{2}+1\right)$ which is not real.

But we're restricting $t \geqslant 0$ anyway!

c. The speed of the particle when $t = 3$ is:

$$\left.\frac{dx}{dt}\right|_{t=3} \approx 0.022469 \approx 0.02 \text{ m/s}$$

Two seconds after the first particle departs O a second one follows with the equation $x = \log_e\left(\tan^{-1}(3t-6)+1\right)$

d. Verify that the particles are the same distance from O when $t = 6$

This comes down to noting that when $t = 6$: $2t = 3t - 6 = 12$,

so both particles are at $x = \log_e\left(\tan^{-1}(12)+1\right)$ units from O

e. The ratio of the speed of the first to the speed of the second particle when $t = 6$ is

$$\frac{x_1'(6)}{x_2'(6)} = \frac{2}{3}$$

there isn't much enlightenment in doing the derivatives by hand.

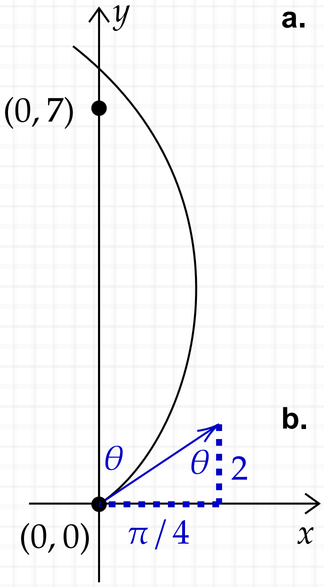
Just a couple of chain rules with expressions that happen to cancel

when $t = 6$ for the reason noted above.

Question 4 (11 marks)

This question is very CAS heavy - it can somewhat be done without the CAS, but it's all to 2 or 3 decimal places in the end.

A ball is hit in minigolf along the path $\mathbf{r}(t) = \dfrac{1}{2}\sin\left(\dfrac{\pi t}{4}\right)\mathbf{i} + 2t\,\mathbf{j}$ for $t \in [0, 5]$. It starts at the origin $(0, 0)$ and curves under a string wind and missing the hole which is at $(0, 7)$.



a. Find $\theta°$ correct to one decmal place

$$\dot{\mathbf{r}}(t) = \frac{\pi}{8}\cos\left(\frac{\pi t}{4}\right)\mathbf{i} + 2\,\mathbf{j}$$

Intial velocity: $\dot{\mathbf{r}}(0) = \frac{\pi}{8}\mathbf{i} + 2\mathbf{j}$

$$\theta = \arctan\left(\frac{\pi/8}{2}\right) = \arctan\left(\frac{\pi}{16}\right) \approx 11.1°$$

$r(t) := \begin{bmatrix}\frac{1}{2}\cdot\sin\left(\frac{\pi\cdot t}{4}\right) & 2\cdot t\end{bmatrix}$ *Done*

$v(t) := \frac{d}{dt}(r(t))$ *Done*

$v(0)$ $\quad \begin{bmatrix}\frac{\pi}{8} & 2\end{bmatrix}$

$\tan^{-1}\left(\frac{\pi}{16}\right)\cdot\frac{180}{\pi}$ $\quad 11.1086805752$

b. The speed at O is $\sqrt{\left(\frac{\pi}{8}\right)^2 + (2)^2} = \frac{1}{8}\sqrt{\pi^2 + 16^2} \approx 2.04$

And max/min speeds occur when $\frac{d}{dt}|v(t)| = 0 \Rightarrow t = 0, 2,\ 4$

This gives local maximum speeds of 2.038m/s at $t = 0$ and 4

And a minimum speed of 4m/s when $t = 2$

$\text{norm}(v(0))$ $\quad 2.038188550$

$\text{solve}\left(\frac{d}{dt}(\text{norm}(v(t)))=0,t\right)|0\le t\le 5$ $\quad t=0$ or $t=2$ or t

$\text{norm}(v(0))$ $\quad 2.038188550$

$\text{norm}(v(2))$

Note: $0 = \frac{dv}{dt} \Longrightarrow 0 = \frac{dv^2}{dt} = 2v\cdot a$, can find max & min speeds at $v \perp a$.

Also, as $v = \frac{\pi}{8}\cos\left(\frac{\pi t}{4}\right)\mathbf{i} + 2\mathbf{j}$, the length of this is min when cos $= 0 \Longrightarrow t = 2$

and greatest speed is when the x component is maximal $\Rightarrow t = 0,\ 4$

c. Minimum distance from the hole is when $\frac{d}{dt}|\mathbf{r} - 7\mathbf{j}| = 0$

$$\Longrightarrow \frac{\pi}{8}\sin\left(\frac{\pi t}{4}\right)\cos\left(\frac{\pi t}{4}\right) + 8t - 28 = \frac{\pi}{16}\sin\left(\frac{\pi t}{2}\right) + 8t - 28 = 0$$

A transcendental equation, an approximate solution is $t \approx 3.51689$

and the corresponding minim distance is 0.188 m

⚠ $\text{solve}\left(\frac{d}{dt}(\text{norm}(r(t)-[0\ \ 7]))=0,t\right)|0\le t\le 5$ $\quad t=3.516888556$

$\text{fMin}(\text{norm}(r(t)-[0\ \ 7]),t)|0\le t\le 5$ $\quad t=3.516888427$

$\text{norm}(r(t)-[0\ \ 7])|t=3.5168885568233$ $\quad 0.18825278825$

d. Total distance travelled in 1st 4 seconds is

$$\int_0^4 |\dot{\mathbf{r}}(t)|dt \approx 8.0765576744 \approx 8.077 \text{ m}$$

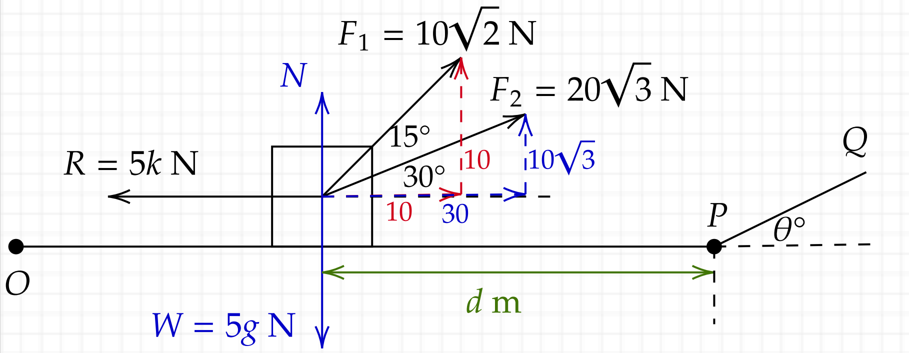
$\int_0^4 \text{norm}(v(t))\,dt$ $\quad 8.0765576744$

Question 5 (10 marks)

A particle of mass 5 kg is moving on a level surface.

It has the 3 labelled forces $R,\ F_1,\ F_2$ and a weight and normal reaction force acting on it.

At point O it is moving to the right at $0.5\ \mathrm{ms}^{-1}$



a. The horizontal forces don't cancel and $F_{net} = 10 + 30 - 5k = 40 - 5k = 5a$

Solve for acceleration: $a = \dfrac{40 - 5k}{5} = 8 - k\ \mathrm{ms}^{-2}$ as required.

b. After 5 seconds the mass reaches P (assume it has negligible size) and has speed $2\ \mathrm{ms}^{-1}$

The change in momentum is $\Delta p = m\Delta v = 5(2 - 0.5) = \dfrac{15}{2}\ \mathrm{kg\ m/s}$

c. We also know that $\Delta v = \displaystyle\int_0^5 a\, dt = \int_0^5 (8 - k)\, dt = 40 - 5k$

Comparing momentum changes gives $40 - 5k = \dfrac{3}{2} \implies k = \dfrac{77}{10} = 7.7$

d. Distance OP: Const acceleration, so SUVAT:

$$d = \frac{u + v}{2}t = \frac{0.5 + 2}{2} \times 5 = \frac{15}{4}\ \mathrm{m}$$

e. When the mass passes P the forces F_1 and F_2 disappear and the mass loses a bit of momentum and now has speed $1.95\ \mathrm{ms}^{-1}$.

The resistance force is still (very coincidentally) 38.5 N acting down the slope.

The mass comes to rest 0.2 m up the slope from P. Find θ to one decimal place.

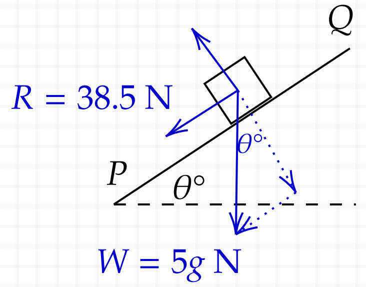
Up the plane: $F_{net} = -38.5 - 5g\sin\theta = ma \implies a = -7.7 - g\sin\theta$, a constant, so SUVAT

$u = 1.95\ \mathrm{ms}^{-1}$, $v = 0\ \mathrm{ms}^{-1}$, $s = 0.2\ \mathrm{m}$, need a, don't need t

$v^2 = u^2 + 2as \implies a = -7.7 - g\sin\theta = -\dfrac{1.95^2}{2 \times 0.2}$

$\implies g\sin\theta = -7.7 + \dfrac{1.95^2}{2 \times 0.2} = 1.80625$

$\implies \theta = \sin^{-1}(1.80625/g) \approx \sin^{-1}(0.184311) = 0.18537\ \mathrm{rad} \approx 10.6°$



Question 6 (9 marks)

External supplier claims aluminium soft-drink cans are normally distributed with a mean of 15 g and standard deviation 0.25 g. A random sample of 64 empty cans is found to have a mean mass of 14.94 g (assume same σ)

Setting up a one-tailed test with 5% significance level

a. H_0: $\mu = 15$ g
H_1: $\mu < 15$ g

b. p-value: $p = \Pr(\overline{X} \leqslant 14.94 | H_0)$, assuming H_0 means that $\overline{X} = \mathrm{N}\left(15, \dfrac{0.25}{\sqrt{64}}\right) = \mathrm{N}(15, 0.03125)$

$p = \Pr(\overline{X} \leqslant 14.95) \approx 0.0274$ (CAS: `normCdf(-∞,14.94,15,0.03125)→0.027428881`)

c. This p-value is less than the significance level $0.0274 < 0.05$, so the null-hypothesis can be rejected.
I.e., it does **not** support the supplier's claim

d. The smallest mean mass for the sample of 64 cans that would not allow us to reject H_0 satisfies $\Pr(\overline{X} \leqslant m_0) = 0.05$, where still $\overline{X} = \mathrm{N}(\mu_{\overline{X}} = 15, \sigma_{\overline{X}} = 0.03125)$.

Define standard normal variable: $Z = \dfrac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}}$ then we require $\Pr(Z \leqslant z_0) \leqslant 0.05$ where

$$z_0 = \frac{m_0 - \mu_{\overline{X}}}{\sigma_{\overline{X}}}$$

Use inverse normal CDF to find $z_0 \gtrsim -1.96$ then $\dfrac{m_0 - \mu_{\overline{X}}}{\sigma_{\overline{X}}} \geqslant -1.96 \implies m_0 \geqslant 14.93875 \approx 14.94$

[Note we rounded up, which is correct as that is the smallest avg mass to 2dp that won't reject H_0.
Also that this is just a one-tailed test. If m_0 gets too big, then we should probably reject H_0 then too!]

After the cans are filled, they are weighed. It is known that the weights are normally distributed with mean mass 406g and standard deviation 5g.

e. Probability that two randomly selected cans differ by **no more than** 3g.
Cans have this distribution of mass: $M \sim \mathrm{N}(406,\ 5)$. Let M_1 and M_2 draw from the same distribution.
Want: $\Pr(|M_1 - M_2| < 3) = \Pr(-3 < M_1 - M_2 < 3)$
Let $D = M_1 - M_2$, know $E(D) = 0$, $\mathrm{Var}(D) = 2\mathrm{Var}(M) = 50$
So want: $\Pr(-3 < D < 3) \approx 0.3286$ (CAS: `normCdf(-3,3,0,√(50))→0.32862669`)
Probability is 32.9%

f. 1mL of soft drink has a mass of 1.04 g (slightly denser than water).
Assume the cans have mean mass of 15g and standard deviation of 0.25g.

The probability that a randomly selected can of soft drink has less than 375mL is...

Know mass of filled can is distributed as $M \sim N(406,\ 5)$

and mass of empty can is distributed as $X = N(15, 0.25)$

So, diving in and ignoring the rocks, the volume of soft drink in the can should/might be a random variable V

such that $V = \dfrac{M - X}{1.04}$ (motivated by the total mass being the sum of the can and the soft-drink: $m = x + 1.04v$),

Can calculate: $E(V) = \dfrac{E(M) - E(X)}{1.04} \approx 375.9615$,

and assuming the variables are independent $\sigma_V = \sqrt{\text{Var}(V)} = \sqrt{\dfrac{\text{Var}(M) + \text{Var}(X)}{1.04^2}} \approx 4.813698$

$V \sim N(375.96,\ 4.8137)$

$\Pr(V < 375) \approx 0.42083789 \approx 42.1\%$