VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2021

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | Letter | |

SPECIALIST MATHEMATICS

Written examination 1

Tuesday 1 June 2021

Reading time: 2.00 pm to 2.15 pm (15 minutes)
Writing time: 2.15 pm to 3.15 pm (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 12 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

THIS PAGE IS BLANK

Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (4 marks)

Consider the function f with rule $f(x) = \frac{x^2}{2} + \sin(x)$.

a. Find all values of x for which the second derivative is equal to zero. 2 marks

b. Explain whether the graph of f has any points of inflection. 2 marks

Question 2 (5 marks)

An object of mass 10 kg on a horizontal plane is acted upon by a constant force of magnitude P newtons and a horizontal resistance force opposing the motion that has a magnitude equal to one quarter of the magnitude of the normal reaction force.

As a result, the object accelerates horizontally at 2 ms^{-2}.

a. Find the value of P in terms of g if P acts horizontally. 2 marks

b. Find the value of P in terms of g if P acts at an angle of 30° upwards from the horizontal. 3 marks

DO NOT WRITE IN THIS AREA

Question 3 (3 marks)

Let X be a binomially distributed random variable with $n = 4$ and $p = \frac{1}{2}$. Let Y be a binomially distributed random variable with $n = 6$ and $p = \frac{1}{2}$. X and Y are independent random variables.

Let Z be the random variable defined by $Z = 2X + 3Y$.

a. Find the mean of Z. 1 mark

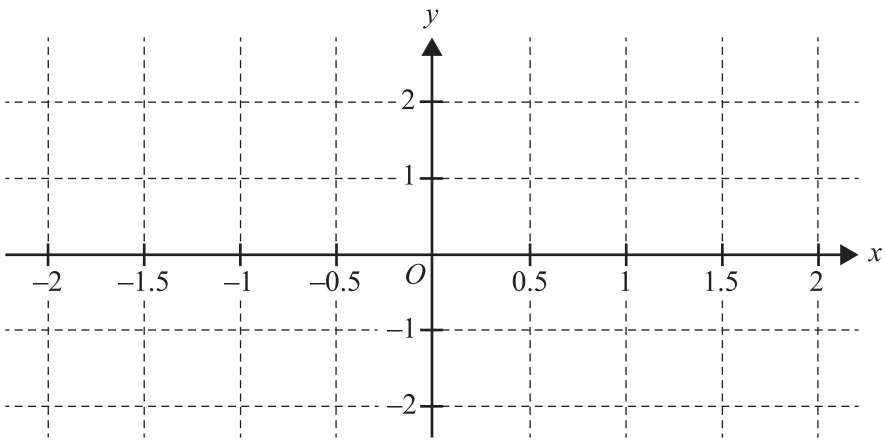
b. Find the standard deviation of Z. 2 marks

DO NOT WRITE IN THIS AREA

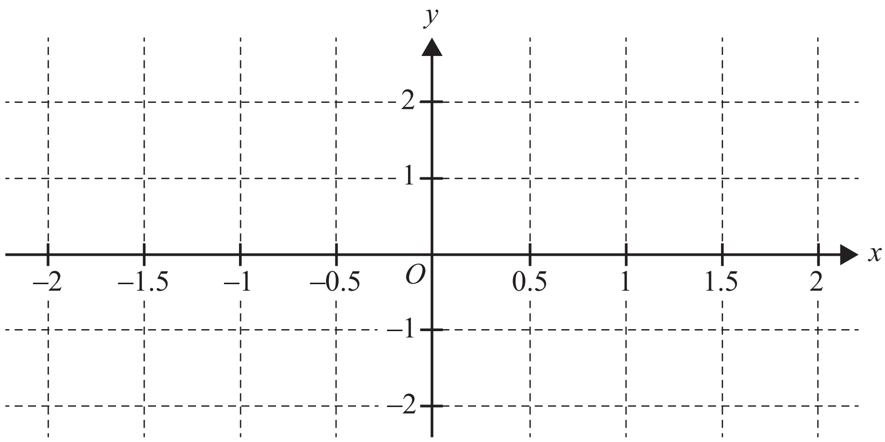
Question 4 (4 marks)

Consider the function $f:[-1, 1] \to R$, $f(x) = \arccos(x) - \dfrac{\pi}{2}$.

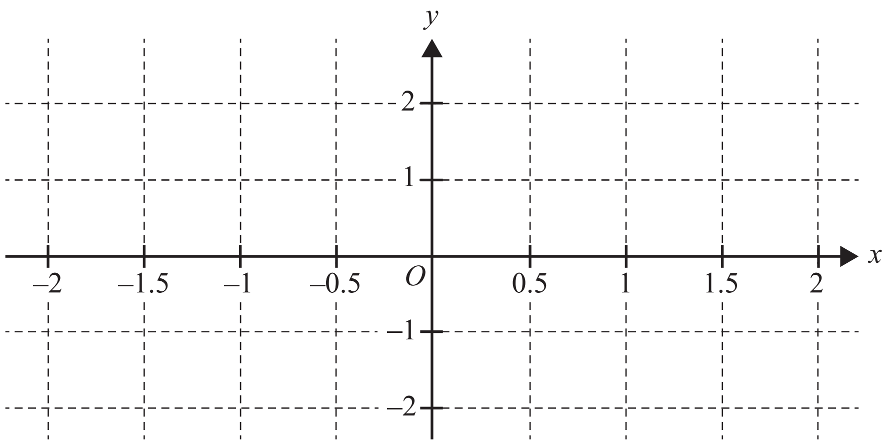
a. Sketch the graph of f on the axes below, labelling the endpoints with their coordinates. 2 marks



b. Sketch the graph of $y = |f(x)|$ on the axes below. 1 mark



c. Sketch the graph of $y = f(|x|)$ on the axes below. 1 mark



Question 5 (4 marks)

a. Solve the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$, given that $y(0) = 1$, to show that $y = \frac{x+1}{1-x}$. 3 marks

b. Find $y\left(\frac{1}{\sqrt{3}}\right)$ in the form $a + \sqrt{b}$, where $a, b \in R$. 1 mark

DO NOT WRITE IN THIS AREA

Question 6 (4 marks)

Find the values of a and b for $x^3 y + ay^2 = b$, given that the tangent to the graph of the relation at (1, 1) has the equation $4x + 5y = 9$.

Question 7 (3 marks)

A particle moves in a straight line so that its acceleration is $(5 + 6x)$ ms^{-2}, where its displacement from a fixed origin is x metres.

If its velocity is 4 ms^{-1} when $x = 0$, find its velocity, where $v > 0$, in metres per second, when $x = 2$.

DO NOT WRITE IN THIS AREA

Question 8 (4 marks)

Find the volume of the solid of revolution formed when the graph of the relation $y = \frac{6}{\sqrt{1-9x^2}}$ from $x = 0$ to $x = \frac{1}{6}$ is rotated about the x-axis.

Question 9 (6 marks)

a. Show that $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$. 1 mark

b. Hence, show that $\frac{d}{dx}\left[\log_e(\sec(x)+\tan(x))\right] = \sec(x)$. 1 mark

Question 9 – continued

c. Find the length of the curve $y = \log_e(\sec(x))$ for the interval $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$. 4 marks

Question 10 (3 marks)

Find a, where $\sqrt{a} \in R$, given that $\sqrt{1+i\sqrt{a}}+\sqrt{1-i\sqrt{a}}=\sqrt{2a}$.

END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2021

SPECIALIST MATHEMATICS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Specialist Mathematics formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc\sin(A)$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	
$\sin(2x) = 2\sin(x)\cos(x)$	$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

Circular functions – continued

Function	$\sin^{-1}$ or arcsin	$\cos^{-1}$ or arccos	$\tan^{-1}$ or arctan
Domain	$[-1, 1]$	$[-1, 1]$	R
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r\left(\cos(\theta) + i\sin(\theta)\right) = r\,\text{cis}(\theta)$	
$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \text{Arg}(z) \le \pi$
$z_1 z_2 = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}\left(\theta_1 - \theta_2\right)$
$z^n = r^n\,\text{cis}(n\theta)$ (de Moivre's theorem)	

Probability and statistics

for random variables X and Y	$\text{E}(aX + b) = a\text{E}(X) + b$ $\text{E}(aX + bY) = a\text{E}(X) + b\text{E}(Y)$ $\text{var}(aX + b) = a^2\text{var}(X)$
for independent random variables X and Y	$\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y)$
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\overline{X}$	mean $\text{E}\left(\overline{X}\right) = \mu$ variance $\text{var}\left(\overline{X}\right) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e \lvert x \rvert + c$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$
	$\int (ax+b)^{-1} dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
Euler's method	If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$
acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2} \sqrt{1 + \left(f'(x)\right)^2}\, dx$ or $\int_{t_1}^{t_2} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$
$\lvert \underset{\sim}{r} \rvert = \sqrt{x^2 + y^2 + z^2} = r$
$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$

Mechanics

momentum	$\underset{\sim}{p} = m\underset{\sim}{v}$
equation of motion	$\underset{\sim}{R} = m\underset{\sim}{a}$

END OF FORMULA SHEET