

STUDENT NUMBER  Letter

# SPECIALIST MATHEMATICS

## Written examination 1

Friday 5 November 2021

Reading time: 9.00 am to 9.15 am (15 minutes)

Writing time: 9.15 am to 10.15 am (1 hour)

### QUESTION AND ANSWER BOOK

#### Structure of book

<i>Number of questions</i>	<i>Number of questions to be answered</i>	<i>Number of marks</i>
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

#### Materials supplied

- Question and answer book of 13 pages
- Formula sheet
- Working space is provided throughout the book.

#### Instructions

- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

#### At the end of the examination

- You may keep the formula sheet.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**

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**Instructions**

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude  $g \text{ ms}^{-2}$ , where  $g = 9.8$

**Question 1** (4 marks)

The net force acting on a body of mass 10 kg is  $\underline{F} = 5\underline{i} + 12\underline{j}$  newtons.

- a. Find the acceleration of the body in  $\text{ms}^{-2}$ . 1 mark

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- b. The initial velocity of the body is  $-3\underline{j} \text{ ms}^{-1}$ .  
Find the velocity of the body, in  $\text{ms}^{-1}$ , at any time  $t$  seconds. 2 marks

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- c. Find the momentum of the body, in  $\text{kg ms}^{-1}$ , when  $t = 2$  seconds. 1 mark

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**Question 2** (3 marks)

Evaluate  $\int_0^1 \frac{2x+1}{x^2+1} dx$ .

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**Question 3** (5 marks)

A company produces a particular type of light globe called Shiny. The company claims that the lifetime of these globes is normally distributed with a mean of 200 weeks and it is known that the standard deviation of the lifetime of Shiny globes is 10 weeks. Customers have complained, saying Shiny globes were lasting less than the claimed 200 weeks. It was decided to investigate the complaints. A random sample of 36 Shiny globes was tested and it was found that the mean lifetime of the sample was 195 weeks.

Use  $\Pr(-1.96 < Z < 1.96) = 0.95$  and  $\Pr(-3 < Z < 3) = 0.9973$  to answer the following questions.

- a. Write down the null and alternative hypotheses for the one-tailed test that was conducted to investigate the complaints. 1 mark

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- b. i. Determine the  $p$  value, correct to three decimal places, for the test. 2 marks

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- ii. What should the company be told if the test was carried out at the 1% level of significance? 1 mark

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- c. The company decided to produce a new type of light globe called Globeplus.

Find an approximate 95% confidence interval for the mean lifetime of the new globes if a random sample of 25 Globeplus globes is tested and the sample mean is found to be 250 weeks. Assume that the standard deviation of the population is 10 weeks. Give your answer correct to two decimal places. 1 mark

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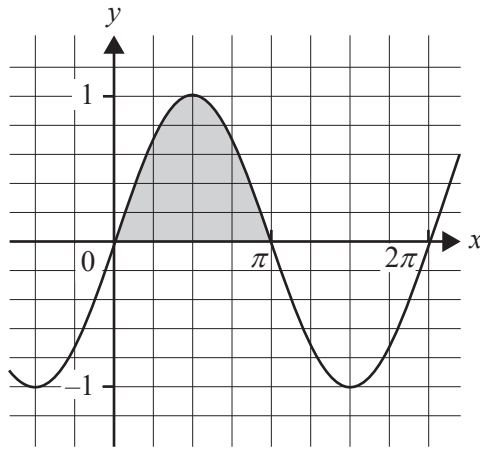
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**Question 4** (4 marks)

- a. The shaded region in the diagram below is bounded by the graph of  $y = \sin(x)$  and the  $x$ -axis between the first two non-negative  $x$ -intercepts of the curve, that is, the interval  $[0, \pi]$ . The shaded region is rotated about the  $x$ -axis to form a solid of revolution.



Find the volume,  $V_s$ , of the solid formed.

3 marks

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- b. Now consider the function  $y = \sin(kx)$ , where  $k$  is a positive real constant. The region bounded by the graph of the function and the  $x$ -axis between the first two non-negative  $x$ -intercepts of the graph is rotated about the  $x$ -axis to form a solid of revolution.

Find the volume of this solid in terms of  $V_s$ .

1 mark

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**Question 5** (3 marks)

Find the gradient of the curve with equation  $e^x e^{2y} + e^{4y^2} = 2e^4$  at the point (2, 1).

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**Question 6** (4 marks)

Consider the three vectors  $\underline{a} = -\underline{i} + 6\underline{j} - 3\underline{k}$ ,  $\underline{b} = 2\underline{i} - 8\underline{j} + 5\underline{k}$  and  $\underline{c} = 3\underline{i} + 2\underline{j} + |1 - p^2|\underline{k}$ , where  $p$  is a real constant.

Find the values of  $p$  for which the three vectors are linearly independent.

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**Question 7** (5 marks)

The velocity of a particle satisfies the differential equation  $\frac{dx}{dt} = x \sin(t)$ , where  $x$  centimetres is its displacement relative to a fixed point  $O$  at time  $t$  seconds.

Initially, the displacement of the particle is 1 cm.

- a. Find an expression for  $x$  in terms of  $t$ .

3 marks

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- b. Find the maximum displacement of the particle and the times at which this occurs.

2 marks

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**Question 8** (4 marks)

a. Solve  $z^2 + 2z + 2 = 0$  for  $z$ , where  $z \in \mathbb{C}$ .

1 mark

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b. Solve  $z^2 + 2\bar{z} + 2 = 0$  for  $z$ , where  $z \in \mathbb{C}$ .

3 marks

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**Question 9** (8 marks)

Let  $\underline{r}(t) = (-1 + 4\cos(t))\underline{i} + \frac{2}{\sqrt{3}}\sin(t)\underline{j}$  and  $\underline{s}(t) = (3\sec(t) - 1)\underline{i} + \tan(t)\underline{j}$  be the position vectors relative to a fixed point  $O$  of particle  $A$  and particle  $B$  respectively for  $0 \leq t \leq c$ , where  $c$  is a positive real constant.

- a. i. Show that the cartesian equation of the path of particle  $A$  is  $\frac{(x+1)^2}{16} + \frac{3y^2}{4} = 1$ . 1 mark

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- ii. Show that the cartesian equation of the path of particle  $A$  in the first quadrant can be written as  $y = \frac{\sqrt{3}}{6}\sqrt{-x^2 - 2x + 15}$ . 1 mark

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- b. i. Show that the particles  $A$  and  $B$  will collide. 1 mark

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- ii. Hence, find the coordinates of the point of collision of the two particles. 1 mark

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- c. i. Show that  $\frac{d}{dx} \left( 8 \arcsin \left( \frac{x+1}{4} \right) + \frac{(x+1)\sqrt{-x^2-2x+15}}{2} \right) = \sqrt{-x^2-2x+15}$ . 2 marks

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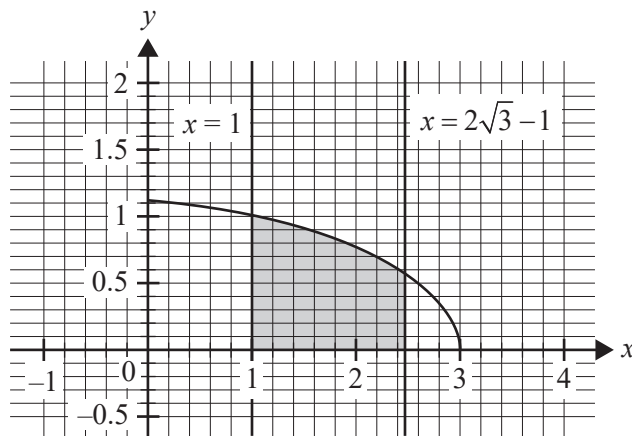


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ii.



Hence, find the area bounded by the graph of  $y = \frac{\sqrt{3}}{6} \sqrt{-x^2-2x+15}$ , the  $x$ -axis and the lines  $x = 1$  and  $x = 2\sqrt{3} - 1$ , as shown in the diagram above. Give your answer in the form  $\frac{a\sqrt{3}\pi}{b}$ , where  $a$  and  $b$  are positive integers. 2 marks

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**Victorian Certificate of Education  
2021**

**SPECIALIST MATHEMATICS**

**Written examination 1**

**FORMULA SHEET**

**Instructions**

This formula sheet is provided for your reference.  
A question and answer book is provided with this formula sheet.

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## Specialist Mathematics formulas

### Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc \sin(A)$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab \cos(C)$

### Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	
$\sin(2x) = 2\sin(x)\cos(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$



**Circular functions – continued**

<b>Function</b>	$\sin^{-1}$ or arcsin	$\cos^{-1}$ or arccos	$\tan^{-1}$ or arctan
<b>Domain</b>	$[-1, 1]$	$[-1, 1]$	$R$
<b>Range</b>	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

**Algebra (complex numbers)**

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r \operatorname{cis}(\theta)$	
$ z  = \sqrt{x^2 + y^2} = r$	$-\pi < \operatorname{Arg}(z) \leq \pi$
$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$
$z^n = r^n \operatorname{cis}(n\theta)$ (de Moivre's theorem)	

**Probability and statistics**

for random variables $X$ and $Y$	$E(aX + b) = aE(X) + b$ $E(aX + bY) = aE(X) + bE(Y)$ $\operatorname{var}(aX + b) = a^2 \operatorname{var}(X)$
for independent random variables $X$ and $Y$	$\operatorname{var}(aX + bY) = a^2 \operatorname{var}(X) + b^2 \operatorname{var}(Y)$
approximate confidence interval for $\mu$	$\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\bar{X}$	mean $E(\bar{X}) = \mu$ variance $\operatorname{var}(\bar{X}) = \frac{\sigma^2}{n}$

**Calculus**

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a} e^{ax} + c$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e x  + c$
$\frac{d}{dx}(\sin(ax)) = a \cos(ax)$	$\int \sin(ax) dx = -\frac{1}{a} \cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a \sin(ax)$	$\int \cos(ax) dx = \frac{1}{a} \sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = a \sec^2(ax)$	$\int \sec^2(ax) dx = \frac{1}{a} \tan(ax) + c$
$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c, a > 0$
$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n dx = \frac{1}{a(n+1)} (ax+b)^{n+1} + c, n \neq -1$
	$\int (ax+b)^{-1} dx = \frac{1}{a} \log_e ax+b  + c$
product rule	$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$
Euler's method	If $\frac{dy}{dx} = f(x)$ , $x_0 = a$ and $y_0 = b$ , then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$
acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2} \sqrt{1+(f'(x))^2} dx$ or $\int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} dt$

**Vectors in two and three dimensions**

$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$
$ \underline{r}  = \sqrt{x^2 + y^2 + z^2} = r$
$\dot{\underline{r}} = \frac{d\underline{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$
$\underline{r}_1 \cdot \underline{r}_2 = r_1 r_2 \cos(\theta) = x_1 x_2 + y_1 y_2 + z_1 z_2$

**Mechanics**

momentum	$\underline{p} = m\underline{v}$
equation of motion	$\underline{R} = m\underline{a}$