The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2021

Trial Written Examination 2 - SOLUTIONS

SECTION A – Multiple-choice questions

ANSWERS

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
C	D	B	A	E	C	E	B	A	D

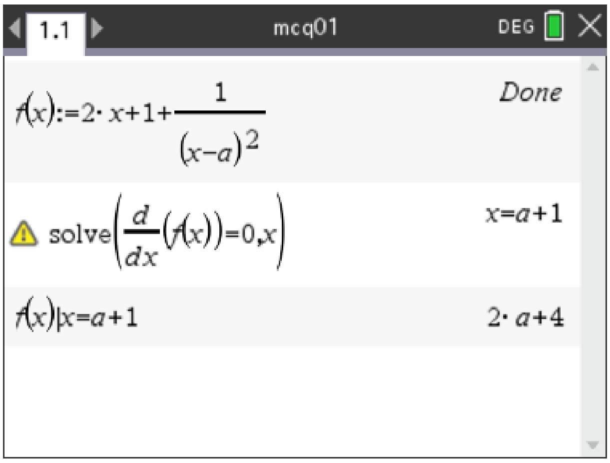
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
C	C	C	A	E	A	A	B	D	B

SOLUTIONS

Question 1 **Answer is C**

$f(x) = 2x + 1 + \frac{1}{(x-a)^2}$ is the sum of a straight line and a truncus.

Using CAS we can find the coordinates of the turning point:



The graph of f has a local minimum at $(a+1, 2a+4)$ and has two asymptotes (the straight line $y = 2x+1$ and the vertical line $x = a$).

Question 2 **Answer is D**

Since $\frac{\pi}{2} < x < \frac{3\pi}{4}$, $\tan(x) < 0$. Note that

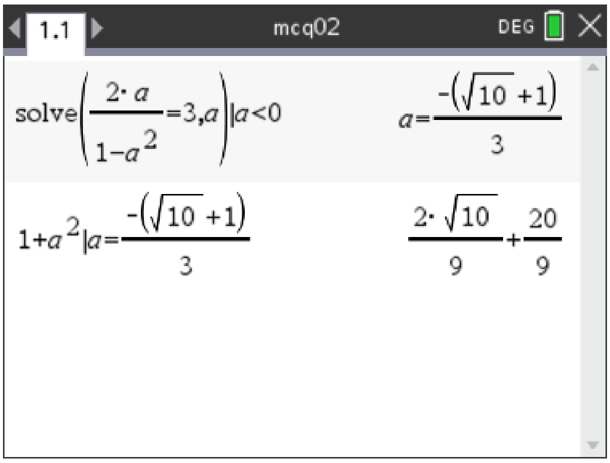
$$\cot(2x) = \frac{1}{3}$$

$$\tan(2x) = 3$$

$$\frac{2\tan(x)}{1-\tan^2(x)} = 3$$

$$\tan(x) = \frac{-1-\sqrt{10}}{3}$$

This may be found using CAS:



Since $\sec^2(x) = 1+\tan^2(x)$ we have $\sec^2(x) = \frac{2}{9}\left(\sqrt{10}+10\right)$.

Question 3 **Answer is B**

The period is π and so $a = 2$. From the graph we see that $\sec\left(2\left(\frac{\pi}{4}-b\right)\right) = -1$ and so

$$\cos\left(2\left(\frac{\pi}{4}-b\right)\right) = -1.$$

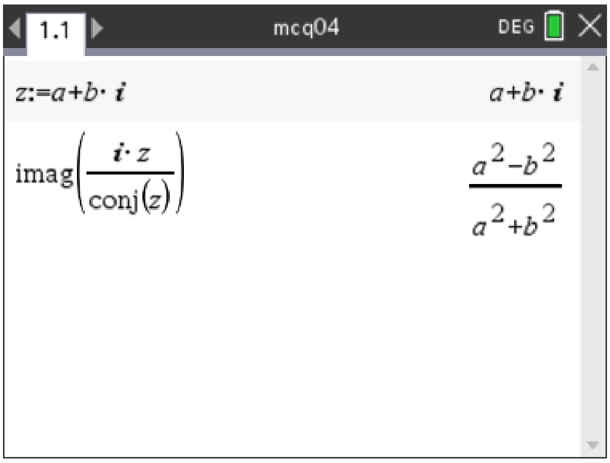
Therefore, it could be the case that $\frac{\pi}{2} - 2b = -\pi$ or $\frac{\pi}{2} - 2b = \pi$ giving $b = \frac{3\pi}{4}$ or $b = -\frac{\pi}{4}$. The second of these options appears as a multiple-choice answer.

Question 4 **Answer is A**

Let $z = a + bi$. Then

$$\begin{aligned}\text{Im}\left(\frac{iz}{\bar{z}}\right) &= \text{Im}\left(\frac{i(a+bi)}{a-bi}\right)\\ &= \text{Im}\left(\frac{i(a+bi)(a+bi)}{a^2+b^2}\right)\\ &= \text{Im}\left(\frac{i\left(a^2-b^2+2abi\right)}{a^2+b^2}\right)\\ &= \frac{a^2-b^2}{a^2+b^2}\end{aligned}$$

Alternatively, CAS can be used to find this result:



Question 5 **Answer is E**

The gradient of the ray is 1 and the ray originates at the point $(-1, 2)$ (not inclusive of this point). Therefore the equation of the line is $y = x + 3$ and the function that describes the ray is $f : (-1, \infty) \to R$, $f(x) = x + 3$.

Question 6 **Answer is C**

Note that $\sqrt{3} + i = 2\,\text{cis}\left(\frac{\pi}{6}\right)$ and so

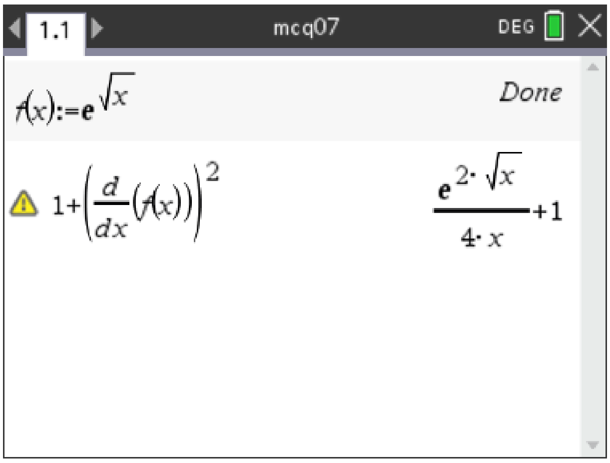
$$\begin{aligned}\left(\sqrt{3}+i\right)^{3n+3} &= \left(\left(\sqrt{3}+i\right)^{n+1}\right)^3\\ &= \left(-a \cdot 2\,\text{cis}\left(\frac{\pi}{6}\right)\right)^3\\ &= -a^3 \cdot 8\,\text{cis}\left(\frac{\pi}{2}\right)\\ &= -8a^3 i\end{aligned}$$

Question 7 **Answer is E**

The length of the curve $f(x)=e^{\sqrt{x}}$ between $x=1$ and $x=4$ is

$$\int_1^4 \sqrt{1+\left(\frac{d}{dx}\left(e^{\sqrt{x}}\right)\right)^2}\,dx=\int_1^4 \sqrt{1+\frac{e^{2\sqrt{x}}}{4x}}\,dx$$

Use CAS to perform the differentiation:



Question 8 **Answer is B**

Note that

$$\int_0^{\frac{\pi}{4}} \cos^3(2x)dx=\int_0^{\frac{\pi}{4}}\left(1-\sin^2(2x)\right)\cos(2x)dx.$$

Let $u=\sin(2x)$ and so $\dfrac{du}{dx}=2\cos(2x)\Rightarrow \dfrac{1}{2}du=\cos(2x)dx$.

Now consider the terminals: When $x=0$, $u=0$ and when $x=\dfrac{\pi}{4}$, $u=1$.

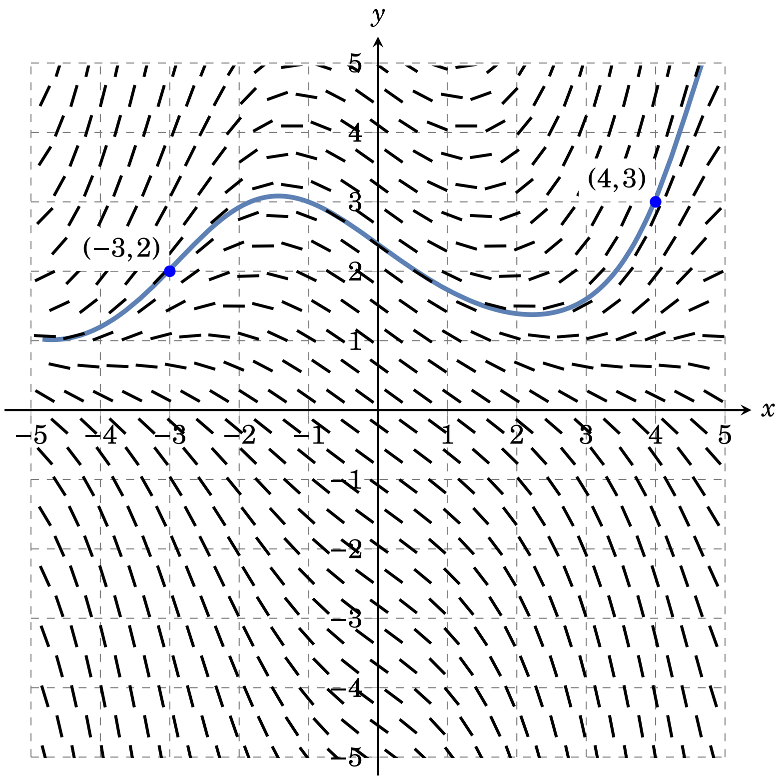
Therefore the integral can be written in terms of u as

$$\frac{1}{2}\int_0^1\left(1-u^2\right)du.$$

Question 9 **Answer is A**

Draw an approximate solution curve that passes through the point $(-3, 2)$.

The curve also passes through the point $(4, 3)$.



Question 10 **Answer is D**

The volume of salt solution in the tank at time $t \geq 0$ is $100+5t$ and so a differential equation for the amount of salt x in the tank at time t is

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$= 0.05 \times 10 - \frac{5x}{100+5t}$$

$$= \frac{1}{2} - \frac{x}{20+t}$$

Question 11 **Answer is C**

Using the scalar product we have

$$\begin{aligned}\cos\theta &= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\\ &= \frac{4-2+2}{\sqrt{9}\sqrt{9}}\\ &= \frac{4}{9}\end{aligned}$$

Using a trigonometric identity we have

$$\begin{aligned}\tan^2(\theta) &= \sec^2(\theta)-1\\ &= \left(\frac{9}{4}\right)^2-1\\ &= \frac{65}{16}\end{aligned}$$

and so $\tan(\theta)=\dfrac{\sqrt{65}}{4}$.

Question 12 **Answer is C**

Suppose that the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are dependent. Therefore $\alpha\underset{\sim}{a}+\beta\underset{\sim}{b}=\underset{\sim}{c}$.

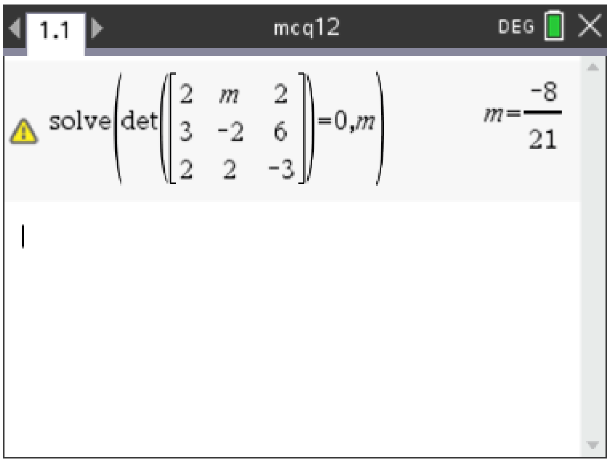
Consider the $\underset{\sim}{i}$ components: $2\alpha+3\beta=2$.
Consider the $\underset{\sim}{j}$ components: $2\alpha+6\beta=-3$.

Solving gives $\alpha=\frac{7}{2}$ and $\beta=-\frac{5}{3}$.

Substituting this into the $\underset{\sim}{j}$ components gives $\frac{7}{2}m-2\left(-\frac{5}{3}\right)=2 \Rightarrow m=-\frac{8}{21}$.

So the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly independent if $m \in R \setminus \left\{-\frac{8}{21}\right\}$.

Alternatively, vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are independent if the determinant of the 3×3 matrix whose rows (or columns) consist of the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, is not zero. This determinant can be evaluated and solved quickly using CAS:

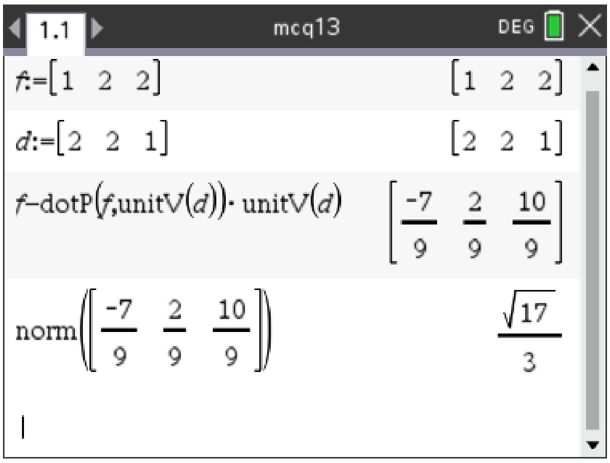


Question 13 **Answer is C**

The component of $\underset{\sim}{F}$ perpendicular to $\underset{\sim}{d}$ is $\underset{\sim}{F}-\left(\underset{\sim}{F}\cdot\hat{\underset{\sim}{d}}\right)\hat{\underset{\sim}{d}}=-\frac{7}{9}\underset{\sim}{i}+\frac{2}{9}\underset{\sim}{j}+\frac{10}{9}\underset{\sim}{k}$.

The magnitude of this vector is $\frac{\sqrt{17}}{3}$.

This can be found using CAS:

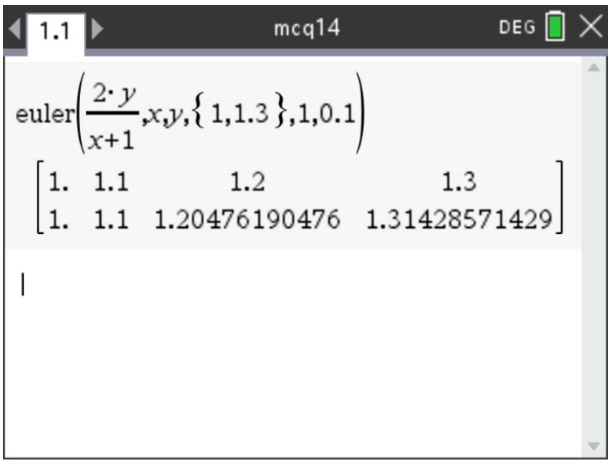


Question 14 **Answer is A**

Using a table to perform the step required for Euler's method is often convenient:

n	x_n	y_n	y_n'
0	1	1	1
1	$\frac{11}{10}$	$1+\frac{1}{10}=\frac{11}{10}$	$\frac{22}{21}$
2	$\frac{12}{10}$	$\frac{11}{10}+\frac{22}{210}=\frac{253}{210}$	$\frac{23}{21}$
3	$\frac{13}{10}$	$\frac{253}{210}+\frac{1}{10}\times\frac{23}{21}=\frac{46}{35}$	

This can also be done on CAS, although a numerical result is obtained which must be compared with the fractional options given.



Question 15 **Answer is E**

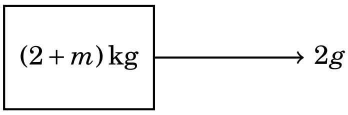
Note that

$$\frac{dy}{dx} = \frac{\sin(x+y)-\sin(x-y)}{2xy}$$
$$= \frac{2\cos(x)\sin(y)}{2xy}$$
$$= \frac{\cos(x)\sin(y)}{xy}$$

Therefore $\int \frac{y}{\sin(y)}dy = \int \frac{\cos(x)}{x}dx$.

Question 16 **Answer is A**

Consider the system as a single mass of $m+2$ kg acted upon by a force of $2g$ Newtons:



This gives $2g = 6(m+2) = 6m+12$ and so

$m = \frac{1}{6}(2g-12) = \frac{g}{3} - 2$ kg.

Question 17 **Answer is A**

Consider the $\underset{\sim}{\mathrm{i}}$ components:

$1+t^2 = 6t-4 \Rightarrow t = 1, 5$

Now consider the $\underset{\sim}{\mathrm{j}}$ components:

$3t+2 = t^2 - 8 \Rightarrow t = 5$.

Therefore, the particles collide when $t = 5$. The position of the point of collision is

$\underset{\sim}{\mathrm{r}}_A(5) = 26\underset{\sim}{\mathrm{i}} + 17\underset{\sim}{\mathrm{j}}$

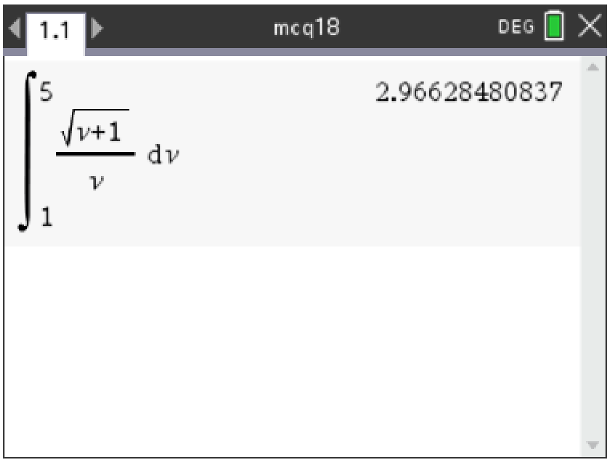
Question 18 **Answer is B**

Since $a = \dfrac{dv}{dt}$ we have

$$\frac{dv}{dt} = \frac{v}{\sqrt{v+1}} \Rightarrow \int \frac{\sqrt{v+1}}{v} dv = \int dt$$

and so the time taken for the particle to increase in velocity from 1 ms^{-1} to 5 ms^{-1} is

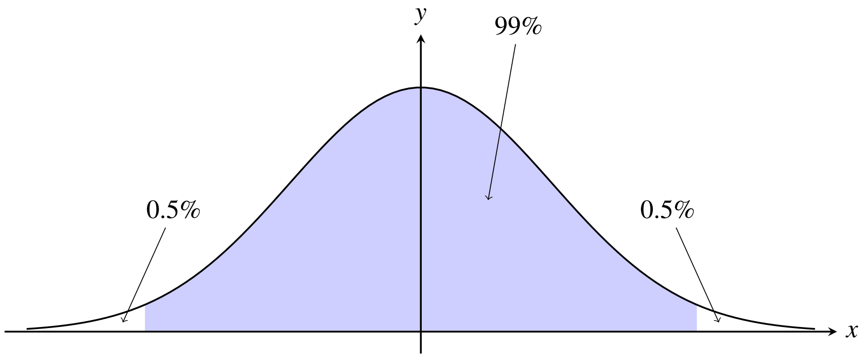
$$\int_1^5 \frac{\sqrt{v+1}}{v} dv = 2.966 \text{ seconds.}$$



Question 19 **Answer is D**

Note that $\bar{x} = \dfrac{382.81 + 387.19}{2} = 385$.

Consider the standard normal distribution:



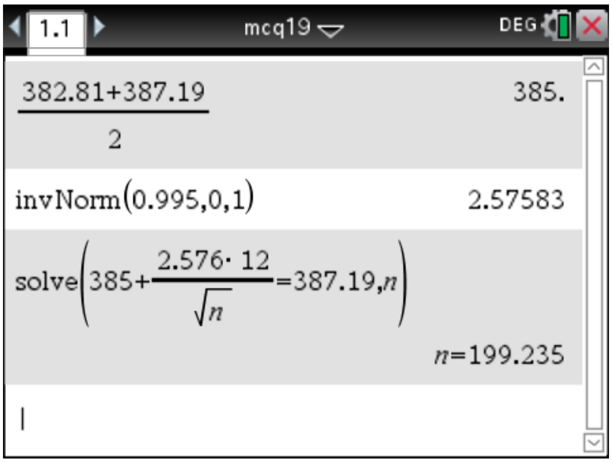
Use CAS to find z if $\Pr(Z > z) = 0.995$. Then $z = 2.576$.

So

$$\bar{x} + z.\frac{s}{\sqrt{n}} = 387.19$$

$$385 + 2.576.\frac{12}{\sqrt{n}} = 387.19$$

$$n \approx 200$$



Question 20 **Answer is B**

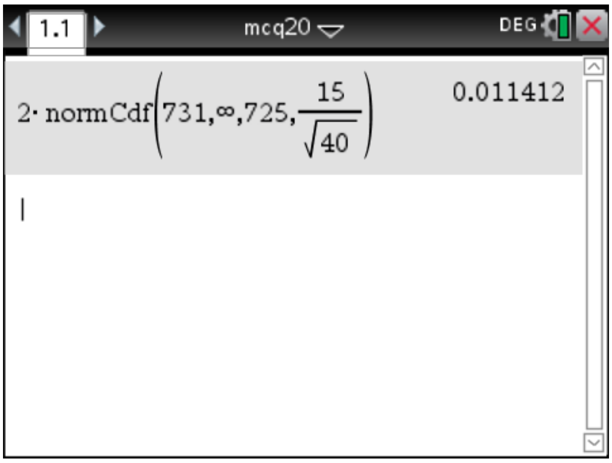
We have

$$E\left(\overline{X}\right)=731$$

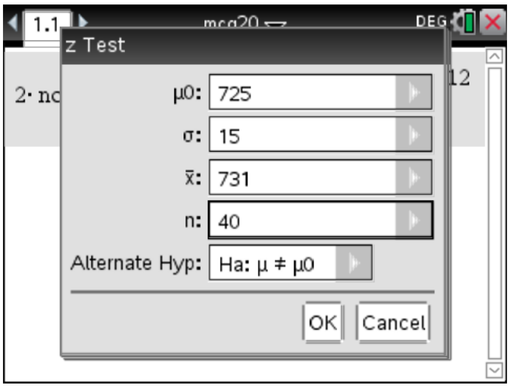
$$\text{sd}\left(\overline{X}\right)=\frac{15}{\sqrt{40}}$$

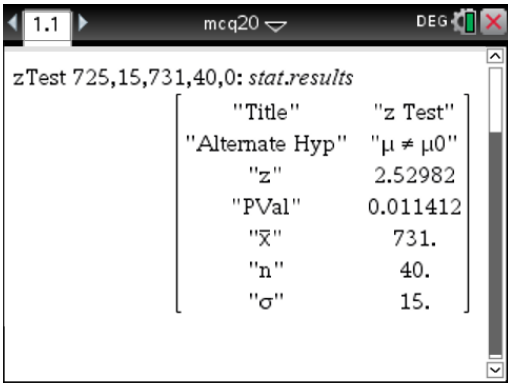
Therefore:

$$\begin{aligned} p-\text{value} &= 2\times \Pr\left(\overline{X}>731 \mid \mu=725\right) \\ &= 0.0114 \end{aligned}$$



Note that this can also be found using the zTest command:



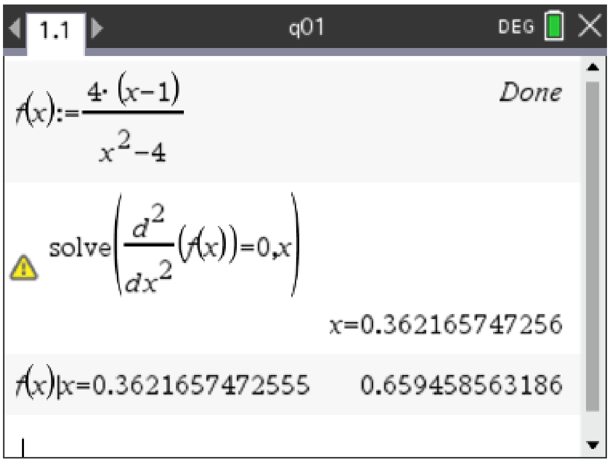


SECTION B

Question 1

a.

Use CAS to find the coordinates of the point of inflection: $(0.362, 0.659)$ **[A2]**



b.

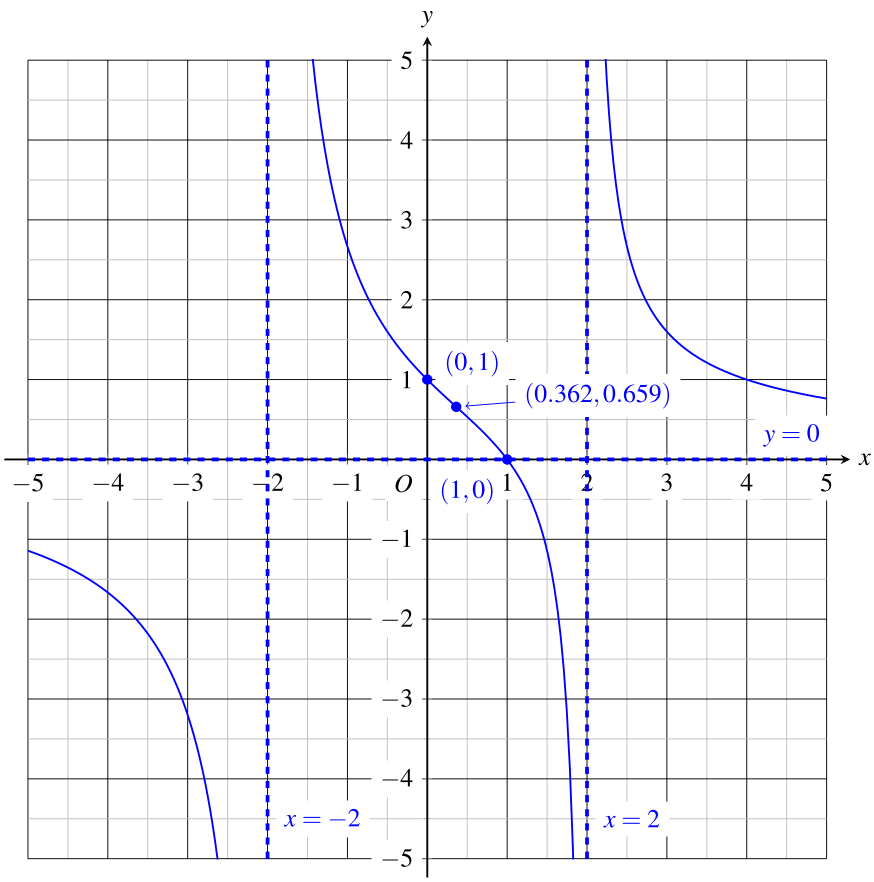
Note that $f(x) = \dfrac{4(x-1)}{x^2-4} = \dfrac{3}{x+2} + \dfrac{1}{x-2}$ and so the asymptotes are $x = 2$, $x = -2$ and $y = 0$.

[A2]

1 mark for vertical asymptotes and 1 for horizontal asymptote

c.

The graph of $y = f(x)$ is plotted below:



The asymptotes, the axis intercepts and the point of inflection are labelled. **[A3]**

1 mark for correct sketch, 1 mark for asymptotes correct and labelled, 1 mark for coordinates

d i.

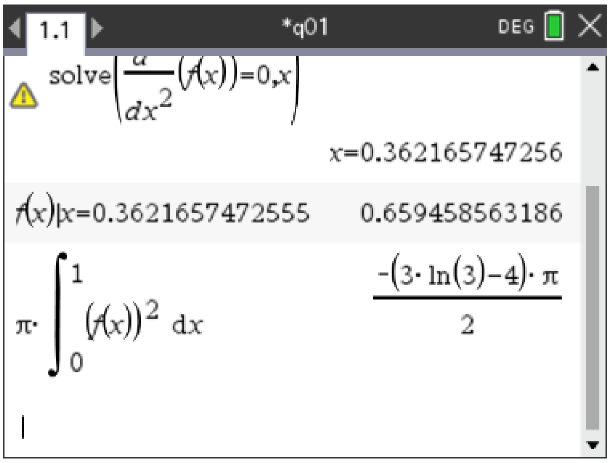
The volume of the solid is

$$V = \pi \int_0^1 \left(\frac{4(x-1)}{x^2-4} \right)^2 dx = \pi \int_0^1 \frac{16(x-1)^2}{\left(x^2-4\right)^2} dx .$$ **[A2]**

1 mark correct terminals and dx, 1 mark integrand and π

ii.

Use CAS to find $V = \frac{\pi}{2}\left(4 - 3\log_e(3)\right)$. **[A1]**



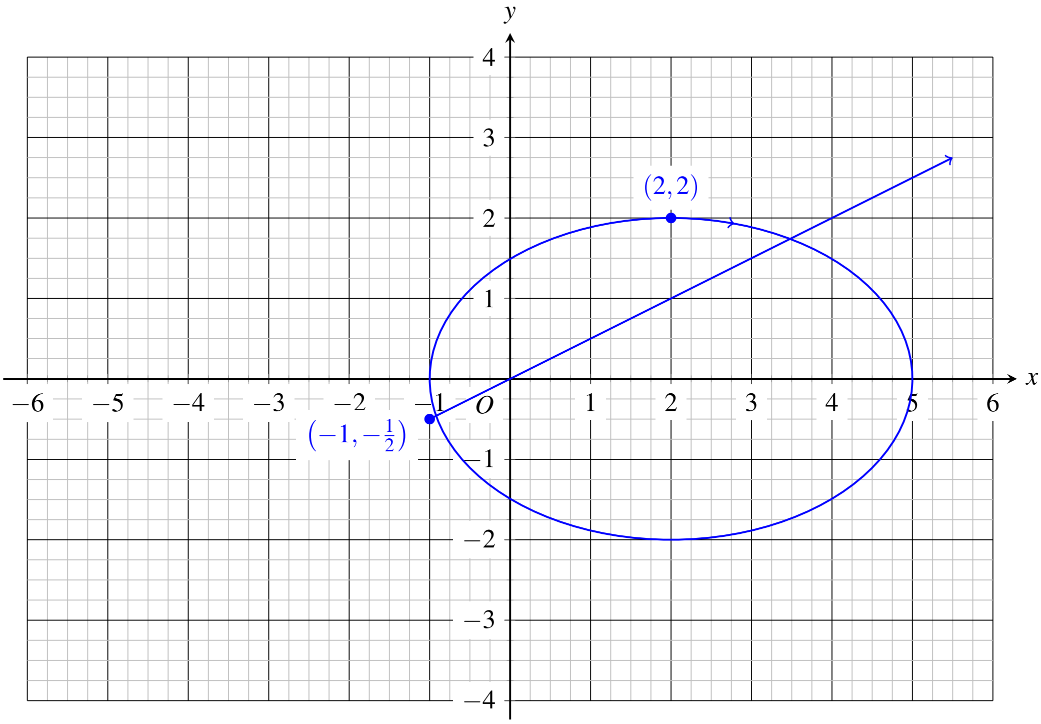
Question 2

a.

The cartesian equation of particle A is $y = \frac{x}{2}$, $x \geq -1$. **[A1]**

The cartesian equation of particle B is $\left(\frac{x-2}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ or $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$. **[A1]**

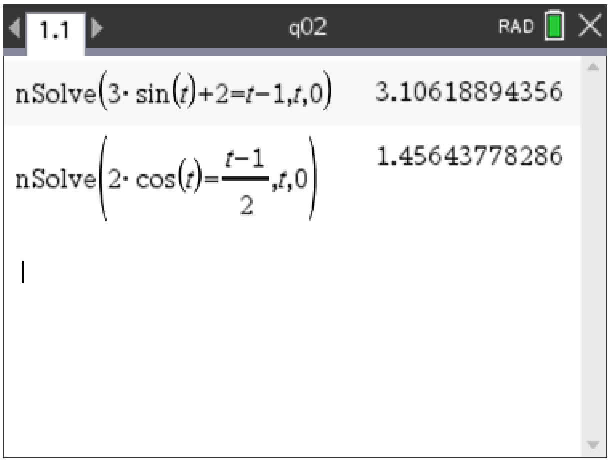
b.
The path of each particle is plotted below:



[A3]
1 mark each particle sketched, 1 mark for directions of both

Note that particle A begins at the point $\left(-1, -\frac{1}{2}\right)$ and particle B begins at the point $(2, 2)$ and moves in a clockwise direction.

c.
The particles are in the same x-position when $t \approx 3.106$ and are in the same y-position when $t \approx 1.456$. **[A1]**
Therefore they do not collide. **[A1]**
With evidence

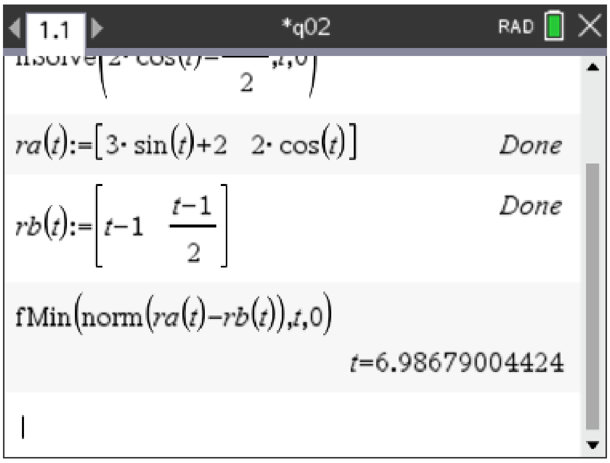


d. i.

The distance between the particles at any time t is

$$\left|\underset{\sim}{r}_A(t)-\underset{\sim}{r}_B(t)\right|=\sqrt{\left(3\sin(t)+2-t+1\right)^2+\left(2\cos(t)-\frac{t-1}{2}\right)^2}$$

Note that this can quickly entered into the calculator using the Norm command:



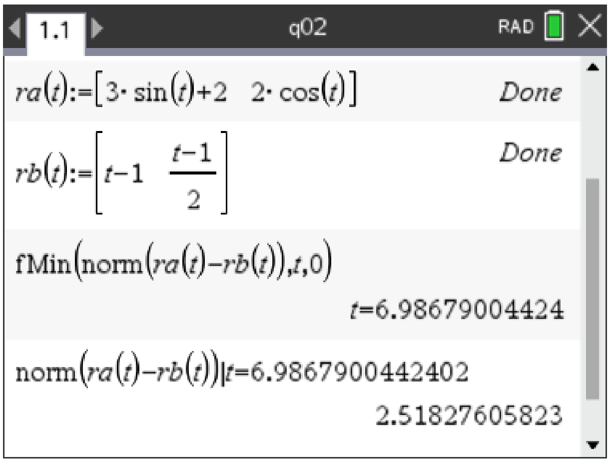
Using CAS we find that the articles are closest to each other when $t = 6.987$ seconds (correct to three decimal places).

[A1]

ii.

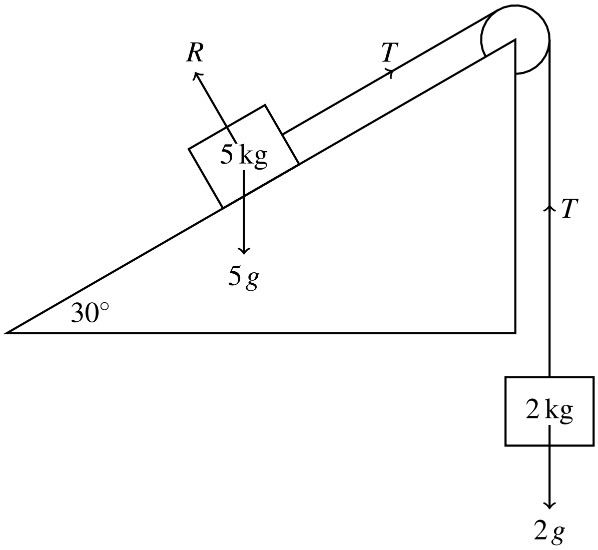
Again, using the Norm command, we find that the closest the particles are to each other is 2.52 metres (correct to two decimal places).

[A1]



Question 3

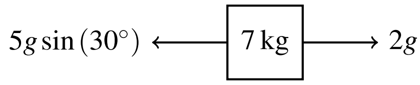
a.
The forces are labelled on the diagram:



[A1]

b.

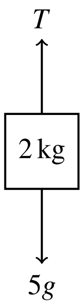
Consider a single 7 kg mass acted upon by forces $2g$ and $5g\sin\left(30^{\circ}\right)=\frac{5}{2}g$:



Then $\frac{5}{2}g-2g=7a \Rightarrow a=\frac{1}{14}g=\frac{7}{10}$ ms^{-2} **[A2]**

c.

Consider the hanging mass:



$$T-2g=2\times\frac{7}{10}$$

$$T=\frac{7}{5}+2g$$ **[A1]**

$$=21$$

d.

Use the constant acceleration formula $v^2 = u^2 + 2as$:

$$v^2 = 2 \times \frac{7}{10} \times 5 = 7$$

Therefore $v = \sqrt{7}$. **[A1]**

e. i.

After the string breaks, the equation of motion for the mass is

$$F = \frac{5g}{2} - 0.2v^2 = 5a$$ **[A1]**

Therefore $a = \frac{g}{2} - \frac{1}{25}v^2$ and so $v\frac{dv}{dx} = \frac{g}{2} - \frac{1}{25}v^2$.

The mass must travel a further 5 metres to reach the bottom of the inclined plane.
An equation which gives the velocity v_1 at the bottom of the plane is

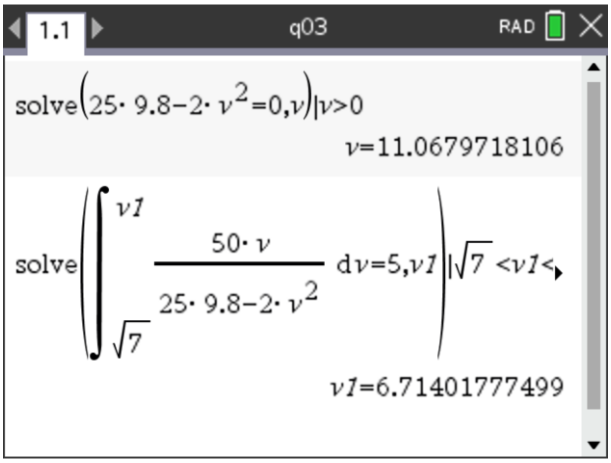
$$\int_{\sqrt{7}}^{v_1} \frac{v}{\frac{g}{2} - \frac{1}{25}v^2} dv = 5 \text{ or } \int_{\sqrt{7}}^{v_1} \frac{50v}{25g - 2v^2} dv = 5$$ **[A2]**

Correct integrand, correct terminals with equation

Note that a different symbol (in this case we have used v_1) should be used.

ii.

Solving using CAS gives $v_1 = 6.71 \text{ ms}^{-1}$ correct to two decimal places as the speed at which the particle reaches the bottom of the plane. **[A1]**



Note that $v < 5\sqrt{\frac{g}{2}} \approx 11.07$ in order for the integral to be defined. This allows bounds to be placed on the solution.

Question 4

a.

Concentration $= \dfrac{x}{V} = \dfrac{x}{10+20t-10t} = \dfrac{x}{10+10t}$.

Answer: $\dfrac{x}{10+10t}$. **[A1]**

b.

$\dfrac{dx}{dt} =$ (inflow of DHA) $-$ (outflow of DHA)

$=$ (rate of inflow of DHA) $\times$ (concentration of DHA in inflow)
$-$ (rate of outflow of DHA) $\times$ (concentration of DHA in outflow).

Substitute concentration of DHA in outflow $= \dfrac{x}{10+10t}$ from **part a.**:

$$\frac{dx}{dt} = (20)e^{-0.2t} - (10)\frac{x}{10+10t} \quad *$$

$$= 20e^{-0.2t} - \frac{x}{1+t}. \quad *$$

$$\Rightarrow \frac{dx}{dt} + \frac{x}{1+t} = 20e^{-0.2t}. \quad *$$

All lines labelled * **[A1]**

c.

Use a CAS to solve the differential equation $\frac{dx}{dt}+\frac{x}{1+t}=20e^{-0.2t}$

subject to the initial condition $x(0)=0$:

$$x=\frac{100e^{-t/5}\left(6e^{t/5}-t-6\right)}{t+1} \text{ or } x=\frac{600-100(t+6)e^{-t/5}}{t+1}.$$ **[A1]**

Use a CAS to solve $x(t)=30$:

$t=3.96$ or $t=16.02$ (correct to two decimal places)

The value of t for which x is **decreasing** is required.

Option 1: Inspect a graph of $x=x(t)$ (draw the graph using a CAS).

Option 2: Choose the value of t such that $\frac{dx}{dt}<0$ when $x=30$.

Substitute $x=30$ into $\frac{dx}{dt}=20e^{-0.2t}-\frac{x}{1+t}$:

$$\frac{dx}{dt}=20e^{-0.2t}-\frac{30}{1+t}.$$

Use a CAS to test the value of $\frac{dx}{dt}$ for $t=3.96$ and $t=16.02$:

$t=3.96$: $\frac{dx}{dt}>0$. $t=16.02$: $\frac{dx}{dt}<0$.

Answer: $t=16.02$. **[A1]**

d.

• Step size: 20 seconds $= \frac{1}{3}$ minute.

Note: The unit of time in the differential equation is minutes therefore the step size must be converted from seconds to minutes.

• From the initial condition $x(0) = 0$: $x_0 = 0$ and $t_0 = 0$.

• $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{1+t}$.

• The number of steps in 3 minutes is 9 therefore the value of x_9 is required.

Use a CAS to run Euler's Method with the above input data.

Answer: 27.81. **[A1]**

e.

Use a CAS to substitute $t = 4$ into the solution to the differential equation found in **part c.**:

$x = 30.134207$. **[A1]**

Note: More accuracy than the final answer requires must be used so as to avoid rounding error.

Substitute $t = 4$ and $x = 30.134207$ into $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{1+t}$:

$\frac{dx}{dt} = 2.960$ grams per minute (correct to three decimal places).

Answer: 2.960 grams per minute. **[A1]**

f.

By inspection of $\frac{dx}{dt} = 20e^{-0.2t} - \frac{x}{1+t}$:

Rate of outflow of DHA $= \frac{x}{1+t}$ **[M1]**

where $x = \frac{600 - 100(t+6)e^{-t/5}}{t+1}$ is the solution to the differential equation found in **part c.**

Therefore the **amount** of DHA that has flowed out of the tank over the first 8 minutes is given by

$$\int_0^8 \frac{x}{1+t}\,dt$$ **[M1]**

where $x = \frac{600 - 100(t+6)e^{-t/5}}{t+1}$

$= 44.5498$ grams (correct to four decimal places).

Answer: 44.5. **[A1]**

Question 5

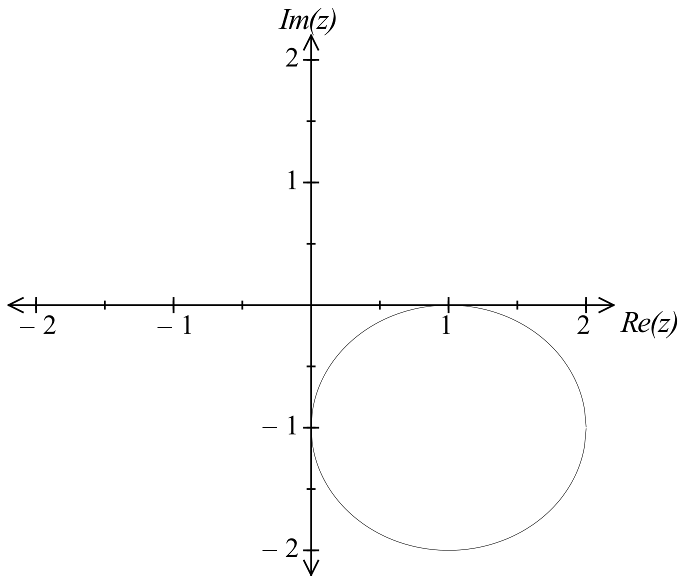
a.

The given relation is a circle. It can be written in standard form as $|z-(1-i)|=1$.

By inspection of the standard form:

Centre at $z=1-i$.

Radius $r=1$.

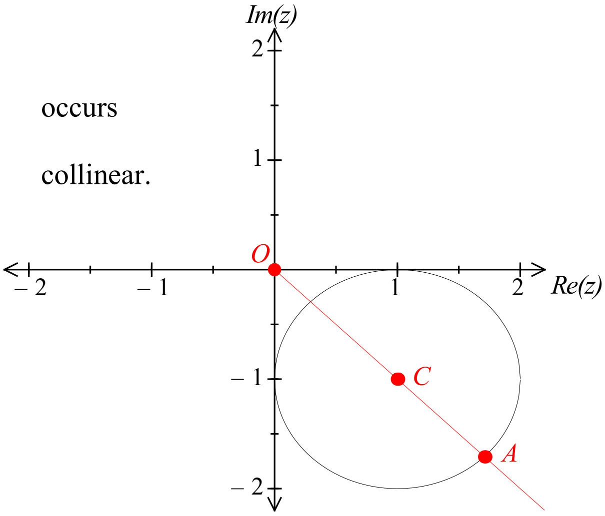


[M1]

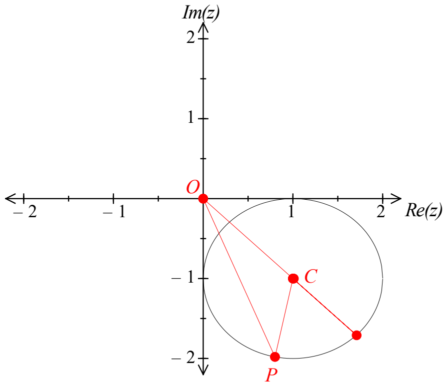
Correct centre and radius are required.

b. i.

By symmetry, the value of z with the largest modulus is represented by the point of intersection A of the circle and the line passing through the origin O and centre $C(1,\ -1)$ of the circle.



Note: Let P be a point on the circle. From triangle OPC: $OP \le OC + CP$ therefore the maximum value of OP occurs at P when the points O, C and P are collinear.



Algebraic Method:

Circle: $(x-1)^2 + (y+1)^2 = 1$. ... (1)

Line: $y = -x$. ... (2)

[M1] Both equations.

Use a CAS to solve equations (1) and (2) simultaneously:

$$x = 1 \pm \frac{1}{\sqrt{2}} \quad = \frac{\sqrt{2} \pm 1}{\sqrt{2}} \quad = \frac{2 \pm \sqrt{2}}{2}.$$

Reject $x = \dfrac{2-\sqrt{2}}{2}$ (corresponds to minimum modulus).

Geometric Method:

$|z| = OC + CA = \sqrt{2} + 1$. $\qquad \text{Arg}(z) = -\dfrac{\pi}{4}$.

Therefore the polar form of z is $z = (\sqrt{2}+1)\text{cis}\left(-\dfrac{\pi}{4}\right)$. **[A1]**

Answer: $z = \left(\dfrac{2+\sqrt{2}}{2}\right) - i\left(\dfrac{2+\sqrt{2}}{2}\right)$. **[A1]**

b. ii.

By inspection of the graph in **part a.** the largest principal argument is 0 (when $z=1$).

Answer: $z=1$. **[A1]**

c. i.

It is required that the distance of the point representing $z=x+iy$ from the origin to the circle is $\sqrt{3}$: $\sqrt{x^2+y^2}=\sqrt{3}$.

Circle: $(x-1)^2+(y+1)^2=1$. ... (1)

$\sqrt{x^2+y^2}=\sqrt{3}$. ... (3)

[M1]
Both equations.

Use a CAS to solve equations (1) and (3) simultaneously:

$$x=\frac{2+\sqrt{2}}{2},\quad y=\frac{-2+\sqrt{2}}{2}.$$

$$x=\frac{2-\sqrt{2}}{2},\quad y=\frac{-2-\sqrt{2}}{2}.$$

Answer: $z=\dfrac{2+\sqrt{2}}{2}+i\dfrac{(-2+\sqrt{2})}{2}$, $\quad z=\dfrac{2-\sqrt{2}}{2}+i\dfrac{(-2-\sqrt{2})}{2}$. **[A1]**

c. ii.

The value of z represented by the point of intersection of the circle and the line passing through the origin with gradient $m=\tan(\theta)$ where $\theta=\tan^{-1}(-2)$ is required.

Circle: $(x-1)^2+(y+1)^2=1$. ... (1)

Line: $y=-2x$. ... (4)

[M1]
Both equations.

Use a CAS to solve equations (1) and (4) simultaneously:

$$x=1,\quad y=-2.$$

$$x=\frac{1}{5},\quad y=-\frac{2}{5}.$$

Answer: $z=1-2i$, $\quad z=\dfrac{1}{5}-i\dfrac{2}{5}$. **[A1]**

d.

Compare $|z-1+i|=1$ with $\sqrt{2}\left|z-(1+\sqrt{2})+ai\right|=|2z-b+2i|$:

- $|z-1+i|=1$

$\Rightarrow |z-1+i|^2=1$

$\Rightarrow (z-1+i)(\overline{z-1+i})=1$

$\Rightarrow (z-1+i)(\bar{z}-1-i)=1$.

Expand using a CAS: $z\bar{z}+(-1-i)z+(-1+i)\bar{z}+1=0$. …. (1)

- $\sqrt{2}\left|z-(1+\sqrt{2})+ai\right|=|2z-b+2i|$

$\Rightarrow 2\left|z-(1+\sqrt{2})+ai\right|^2=|2z-b+2i|^2$

$\Rightarrow 2\left(z-(1+\sqrt{2})+ai\right)\left(\overline{z-(1+\sqrt{2})+ai}\right)=\left(2z-b+2i\right)(\overline{2z-b+2i})$

$\Rightarrow 2\left(z-(1+\sqrt{2})+ai\right)\left(\bar{z}-(1+\sqrt{2})-ai\right)=\left(2z-b+2i\right)(2\bar{z}-b-2i)$ since $a,b\in R$.

Expand both sides using a CAS:

$$2z\bar{z}+2\left(-(1+\sqrt{2})-ai\right)z+2\left(-(1+\sqrt{2})+ai\right)\bar{z}+6+4\sqrt{2}+2a^2$$
$$=4z\bar{z}+(-2b-4i)z+(-2b+4i)\bar{z}+b^2+4$$

$$\Rightarrow 2z\bar{z}+\left(-2b-4i+2(1+\sqrt{2})+2ai\right)z+\left(-2b+4i+2(1+\sqrt{2})-2ai\right)\bar{z}$$
$$+b^2-2-4\sqrt{2}-2a^2=0. \quad …. (2)$$

Compare equations (1) and (2).

Consider the coefficients of either z or $\bar{z}$:

$2(-1-i)=-2b-4i+2(1+\sqrt{2})+2ai$

$\Rightarrow -1-i=-b+(1+\sqrt{2})+(a-2)i$. …. (3) **[A1]**

Equate real and imaginary parts of equation (3).

Real parts: $-1 = -b + (1+\sqrt{2}) \quad \Rightarrow b = 2+\sqrt{2}$.

Imaginary parts: $-1 = a - 2 \quad \Rightarrow a = 1$.

Answer: $a = 1$, $b = 2+\sqrt{2}$. **[A1]**

Note: These answers can be checked by comparing the constant terms of equations (1) and (2).

$2 = b^2 - 2 - 4\sqrt{2} - 2a^2$

$\Rightarrow 2 = \left(2+\sqrt{2}\right)^2 - 2 - 4\sqrt{2} - 2 = 0$ ✓.

Question 6

a.

- Let X be the random variable "*Mass (grams) of a Wakandan apple*".

- $X \sim \text{Normal}\left(\mu_X = 125,\ \ \sigma_X = 20\right)$.

- Let the number of apples in a paper bag be n.

- Let W be the random variable "*Sum of mass (grams) of n apples*".

$W = X_1 + X_2 + \cdots + X_n$

where $X_1,\ X_2,\ \ldots\ X_n$ are independent copies of X.

Note: Using the random variable nX is incorrect: $X_1 + X_2 + \cdots + X_n \neq nX$.

- The largest value of n such that $\Pr(W < 2000) > 0.9$ is required.

Note: Must convert 2 kg into 2000 grams since the unit of X is grams.

- W follows a normal distribution since $X_1,\ X_2,\ \ldots\ X_n$ are independent normal random variables.

- $\text{E}(W) = \mu_W = \mu_{X_1} + \mu_{X_2} + \cdots \mu_{X_n} = n\mu_X = 125n$.

- $\text{Var}(W) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots \text{Var}(X_n) \qquad = n\text{Var}(X) = n(20)^2$

$\Rightarrow \text{sd}(W) = \sigma_W = 20\sqrt{n}$.

- Therefore $W \sim \text{Normal}\left(\mu_W = 125n,\ \ \sigma_W = 20\sqrt{n}\right)$. **[M1]**

- The largest value of n such that $\Pr(W < 2000) > 0.9$ is required.

Answer: 15. **[A1]**

Method 1:

• Define the function

$$f(x) = \text{normCdf}\left(\underset{\substack{\uparrow\\ \text{Lower}\\ \text{value}}}{-\infty},\ \underset{\substack{\uparrow\\ \text{Upper}\\ \text{value}}}{2000},\ \underset{\substack{\uparrow\\ \mu_W}}{125x},\ \underset{\substack{\uparrow\\ \sigma_W}}{20\sqrt{x}}\right).$$

The smallest value of $x \in Z^+$ such that $f(x) > 0.9$ is required.

• Solve using a CAS from either a table of values, solving $f(x) = 0.9$ or trial-and-error: $x = 15$.

Method 2:

• Find the value of z such that $\Pr(Z < z) = 0.9$.

Use the inverse normal command on a CAS: $z = 1.282$.

Note: Sufficient accuracy is required to ensure that the final answer is correct to the nearest integer.

• $Z = \dfrac{W - \mu_W}{\sigma_W} \quad \Rightarrow 1.282 = \dfrac{2000 - 125n}{20\sqrt{n}}$.

Solve using a CAS: $n = 15.2$.

b. i.

Answer: $H_0: \ \mu_X = 125$.

$H_1: \ \mu_X \neq 125$.

[A1]

Both statements are required.

b. ii.

The probability of rejecting H_0 when it is true is the level of significance of the statistical test.

2% level of significance $\Leftrightarrow \alpha = 0.02$.

Answer: 0.02. **[A1]**

b. iii.

$\left(C_1^*, \ C_2^*\right)$ is the interval such that H_0 is accepted at the 2% level of significance when the sample mean $\bar{x} \in \left(C_1^*, \ C_2^*\right)$.

Note: $\left(C_1^*, \ C_2^*\right)$ is **not** a 98% confidence interval. A 98% confidence interval is the interval such that H_0 is accepted at the 2% level of significance when it contains μ_X (the population mean under H_0).

- H_0 is accepted at the 2% level of siginificance if $\bar{x} \in \left(C_1^*, \ C_2^*\right)$

therefore H_0 is rejected at the 2% level of significance if $\bar{x} < C_1^*$ or $\bar{x} > C_2^*$.

- Sample of size 30 therefore $\bar{X} \sim \text{Normal}\left(\mu_{\bar{X}} = \mu_X = 125, \ \ \sigma_{\bar{X}} = \dfrac{20}{\sqrt{30}}\right)$. **[A1]**

- 2% level of significance $\Leftrightarrow \alpha = 0.02$.

- $\Pr\left(\bar{X} < C_1^*\right) = \dfrac{0.02}{2} = 0.01$. $\qquad \Pr\left(\bar{X} > C_2^*\right) = \dfrac{0.02}{2} = 0.01$. **[M1]**

Use the inverse normal command on a CAS:

Answer: $C_1^* = 116.51$. $\qquad C_2^* = 133.50$. **[A1]**

Both values are required.

b. iv.

Use a CAS.

Answer: $(115.21,\ 132.20)$. **[A1]**

b. v.

Answer: H_0 should not be rejected at the 2% level of significance.

Accept either of the following justifications:

- $\bar{x} \in \left(C_1^*,\ C_2^*\right)$ where $\bar{x}$ is the observed sample mean: $123.51 \in (116.51,\ 133.49)$.

- 2% level of significance $\Leftrightarrow$ 98% confidence interval.

$\mu_X = 125$ lies inside the 98% confidence interval $(115.21,\ 132.20)$.

[H1]

Consequential on answers to **part iii.** or **part iv.**

Note: Calculating the *p*-value ($p = 0.72 > \alpha$ therefore H_0 is not rejected) is a valid but ridiculous justification given the intervals calculated in **part iii.** and **part iv.**

b. vi.

- $\bar{X} \sim \text{Normal}\left(\mu_{\bar{X}} = \mu_X,\ \sigma_{\bar{X}} = \dfrac{20}{\sqrt{30}}\right)$.

- H_0 is accepted if $\bar{x} \in \left(C_1^*,\ C_2^*\right)$ where $C_1^* = 116.51$ and $C_2^* = 133.49$ (from **part iii.**)

- Therefore the required probability is given by

$$\Pr\left(C_1^* < \bar{X} < C_2^* \mid H_1 \text{ true}\right) = \Pr\left(C_1^* < \bar{X} < C_2^* \mid \mu_X = 114\right)$$

$$= \Pr\left(116.51 < \bar{X} < 133.49 \mid \mu_X = 114\right).$$

[H1]

Consequential on answers to **part iii.**

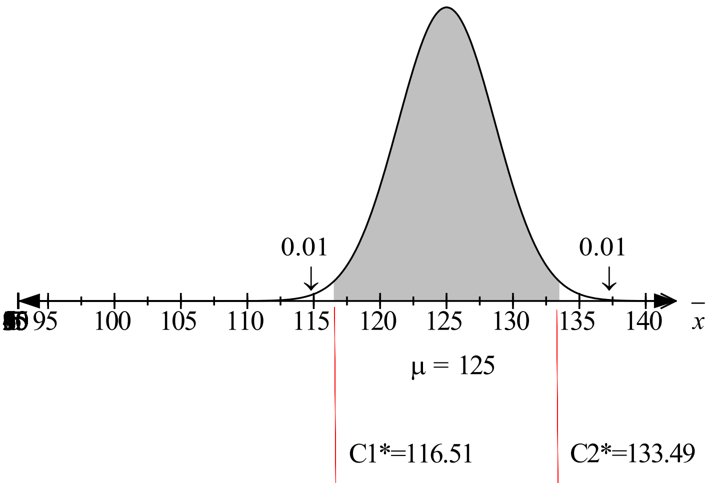
- Use the normal distribution command on a CAS:

$$\Pr\left(116.51 < \bar{X} < 133.49\right) = 0.2459.$$

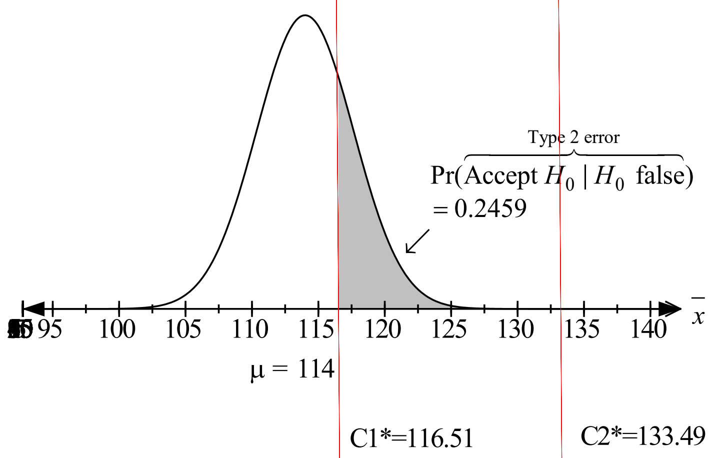
Answer: 0.2459. **[A1]**

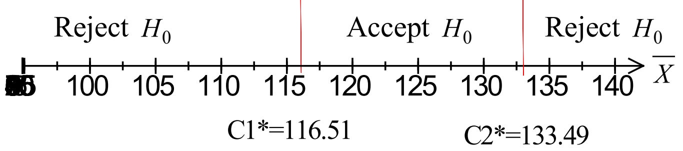
Remark: To accept H_0 when H_1 is true is to commit a type 2 error.

Distribution of $\overline{X}$ when H_0 is true:



Distribution of $\overline{X}$ when H_0 is false:





END OF SOLUTIONS