VCAA 2020 SPECIALIST MATHEMATICS EXAMINATION 2 SOLUTIONS

By TWM Publications

SECTION A

Question 1 (D)

$$f'(x) = \frac{x^2-4x+5a-6}{(x-2)^2}. \quad \text{Thus,}$$

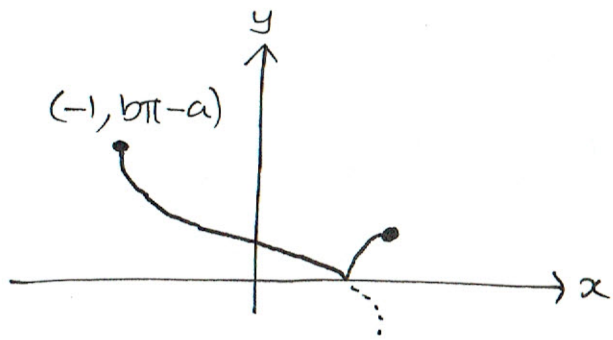
$$f'(0) = 0 \Rightarrow a = \frac{6}{5}.$$

Question 2 (C)

Using the figure, we that

$$\operatorname{ran}(f) = [0, b\pi - a]$$

(noting that $0 < a < \frac{b\pi}{2}$, where $b > 0$)



Question 3 (A)

For $0 \le t \le 30$, $v(t) = \frac{10t}{30} = \frac{t}{3}$.

For $30 \le t \le 230$, $v(t) \equiv 10$.

For $230 \le t \le 260$, $v(t) = \frac{10-0}{230-260}(t-230)+10 = \frac{1}{3}(260-t)$.

Question 4 (E)

$$f(g(x)) = \frac{\sqrt{\csc^2(x)-1}}{\csc^2(x)} = \cot(x)\sin^2(x) = \sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

(note: $0 < x < \frac{\pi}{2}$)

$$\left(0, \tfrac{\pi}{2}\right) \xrightarrow{g} (1, \infty) \xrightarrow{f} \left(0, \tfrac{1}{2}\right].$$

Question 5 (A)

If $z = a+bi$, then

$$\frac{4z\bar{z}}{(z+\bar{z})^2} = \frac{a^2+b^2}{a^2} = 1+\left(\frac{b}{a}\right)^2 = 1+\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)^2$$

Question 6 (C)

Since $P(z)$ has real coefficents, by the conjugate root theorem, $P(3i)=0 \Rightarrow P(-3i)=0$.

Thus, $P(z) = (z+2)(z-3i)(z+3i)$

$$= z^3+2z^2+9z+18$$

$\Rightarrow a=2,\ b=9,\ c=18$

Question 7 (D)

$$\frac{1}{ax(x^2+b)} = \frac{1}{a}\left(\frac{A}{x}+\frac{B}{x-\sqrt{|b|}}+\frac{C}{x+\sqrt{|b|}}\right)$$

$$= \frac{A'}{x}+\frac{B'}{x-\sqrt{|b|}}+\frac{C'}{x+\sqrt{|b|}},$$

where $A' = \frac{A}{a}$, $B' = \frac{B}{a}$ and $C' = \frac{C}{a}$.

Question 8 (A)

$$(y-xi)^{14} = (-i(x+yi))^{14}$$

$$= -(x+yi)^{14}$$

$$= -a-bi$$

~~Question 9~~

~~At (a,b), the tangent is $y = \left.\frac{dy}{dx}\right|_{(a,b)}(x-a)+b$. But, this tangent~~

Question 9 (B)

We have $\frac{dy}{dx} = \frac{0-y}{y-x} = \frac{y}{x-y}$ since the tangent passes through (x,y) and $(y,0)$. When $x=0$, $\left.\frac{dy}{dx}\right|_{x=0} = -1$ for $y \neq 0$, leaving only option B.

Question 10 (D)

$$\frac{dm_{in}}{dt} = 2 \times 15 = 30 \text{ g/min}.$$ The volume of the mixture is

$v(t) = -3t + 50$, so

$$\frac{dm_{OUT}}{dt} = 5 \frac{m}{-3t+50} = \frac{5m}{50-3t}.$$

$$\Rightarrow \frac{dm}{dt} = 30 - \frac{5m}{50-3t}.$$

Question 11 (C)

$$I = \int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\sec^2(x) - 3\tan(x) + 1}\,dx = \int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan^2(x) - 3\tan(x) + 2}\,dx$$

Let $u = \tan(x) \Rightarrow \frac{du}{dx} = \sec^2(x)$, $u(\frac{\pi}{4}) = 1$, $u(\frac{\pi}{3}) = \sqrt{3}$.

$$\Rightarrow I = \int_1^{\sqrt{3}} \frac{1}{u^2 - 3u + 2}\,du$$

$$= \int_1^{\sqrt{3}} \left(\frac{1}{u-2} - \frac{1}{u-1} \right) du$$

Question 12 (C)

$$y_1 = e + 0.1e \Rightarrow y_2 = e + 0.1e + 0.1e^{\cos(0.1)}$$
$$= e + 0.1(e + e^{\cos(0.1)})$$

$$\Rightarrow y_3 = e + 0.1(e + e^{\cos(0.1)}) + 0.1e^{\cos(0.2)}$$
$$= e + 0.1(e + e^{\cos(0.1)} + e^{\cos(0.2)})$$

Question 13 (E)

Since $\underset{\sim}{a} \nparallel \underset{\sim}{c}$, we set $\underset{\sim}{b} = m\underset{\sim}{a} + n\underset{\sim}{c}$.

$$\Rightarrow \begin{cases} \lambda = m+n \\ 3 = 2m+0 \\ 2 = -m+n \end{cases} \qquad \Rightarrow m = \frac{3}{2}, \; n = \frac{7}{2}$$

$\Rightarrow \lambda = 5$ for linear dependence

Question 14 (B)

$$\left| \frac{\underset{\sim}{F} \cdot \underset{\sim}{d}}{|\underset{\sim}{d}|} \right| = \frac{92}{7}$$

Question 15 (E)

$$\underset{\sim}{r}''(t) = \frac{1}{3}(\underset{\sim}{F}_A + \underset{\sim}{F}_B) = 2\underset{\sim}{i} + \underset{\sim}{j} \text{ m s}^{-2}.$$

$$\Rightarrow \underset{\sim}{r}'(t) = \int_0^t (2\underset{\sim}{i} + \underset{\sim}{j})\, d\tau = 2t\underset{\sim}{i} + t\underset{\sim}{j}$$

$$\Rightarrow \underset{\sim}{r}(t) = \int_0^t (2\tau\underset{\sim}{i} + \tau\underset{\sim}{j})\, d\tau + \underset{\sim}{i} + \underset{\sim}{j}$$

$$= \left(t^2 + 1\right)\underset{\sim}{i} + \left(\frac{t^2}{2} + 1\right)\underset{\sim}{j}$$

$$\Rightarrow x = t^2 + 1 \text{ and } y = \frac{t^2}{2} + 1$$

$$\Rightarrow y = \frac{x-1}{2} + 1 = \frac{x}{2} + \frac{1}{2}$$

Question 16 (D)

$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{1}{9} \text{ with } 0 < \theta < \frac{\pi}{2}$$

$$\Rightarrow \sin(\theta) = \sqrt{1 - \left(\frac{1}{9}\right)^2} = \frac{4\sqrt{5}}{9}$$

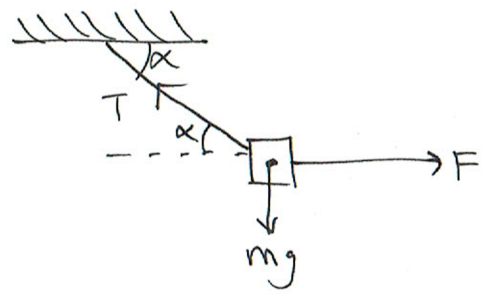
$$\Rightarrow \sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \cdot \frac{4\sqrt{5}}{9} \cdot \frac{1}{9} = \frac{8\sqrt{5}}{81}$$

Question 17 (B)

$$v = \frac{1}{x} \Rightarrow a = v\frac{dv}{dx} = \frac{1}{x}\cdot\frac{-1}{x^2} = \frac{-1}{x^3}$$

$$\Rightarrow a = \frac{-1}{8}\ \text{m s}^{-2}.$$

Question 18 (E)



Resolving forces vertically,

$$mg = T\sin(\alpha).$$

Resolving forces horizontally,

$$F = T\cos(\alpha)$$

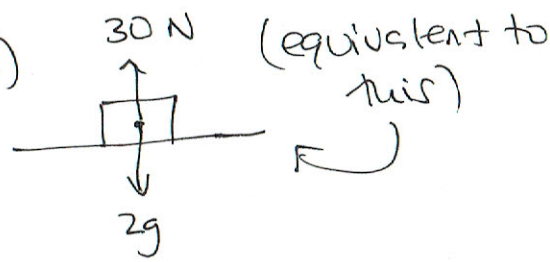
$$\Rightarrow \tan(\alpha) = \frac{mg}{F}.$$

Question 19 (B)

$$\Delta\underset{\sim}{p} = 0.02\left(2\underset{\sim}{i} - 7\underset{\sim}{j} - (2\underset{\sim}{i} - 10\underset{\sim}{j})\right) = 0.06\underset{\sim}{j}$$

$$\Rightarrow |\Delta\underset{\sim}{p}| = 0.06\ \text{kg m s}^{-1}.$$

Question 20 (A)



We have (where positive is down)

$$2a = 2g - 30$$

$$\Rightarrow a = -5.2\ \text{m s}^{-2}\ \text{down}$$

$$= 5.2\ \text{m s}^{-2}\ \text{up}.$$

SECTION B

Question 1a (2 marks)

$$\underset{\sim}{r}(t) = 2\sin(2t)\underset{\sim}{i} + 3\cos(t)\underset{\sim}{j}$$

$$\Rightarrow \left|\underset{\sim}{r}\left(\tfrac{\pi}{6}\right)\right| = \left|\sqrt{3}\underset{\sim}{i} + \frac{3\sqrt{3}}{2}\underset{\sim}{j}\right| = \frac{\sqrt{39}}{2}\ \text{m}$$

Question 1b.i (3 marks)

$\frac{dx}{dt} = 4\cos(2t)$ and $\frac{dy}{dt} = -3\sin(t)$, so by the chain rule,

$$\frac{dy}{dx} = \frac{-3\sin(t)}{4\cos(2t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\pi} = 0$$

The tangent to the path is $y=-3$.

Question 1b.ii (2 marks)

$$\underset{\sim}{v}(t) = 4\cos(2t)\underset{\sim}{i} - 3\sin(t)\underset{\sim}{j}$$

$$\Rightarrow \underset{\sim}{v}(\pi) = 4\underset{\sim}{i}$$

Question 1b.iii (2 marks)

$$\underset{\sim}{a}(t) = -8\sin(2t)\underset{\sim}{i} - 3\cos(t)\underset{\sim}{j}$$

$$\Rightarrow |\underset{\sim}{a}(\pi)| = |3\underset{\sim}{j}| = 3 \text{ m s}^{-2}.$$

Question 1c (1 mark)

$\underset{\sim}{r}(t) = \underset{\sim}{0} \Rightarrow t = \frac{\pi}{2}(2k-1)$, $k \in \mathbb{Z}^+$ by CAS.

Thus the first time is $t = \frac{\pi}{2}$ seconds.

Question 1d (2 marks)

$$d = \int_0^{\pi/6} |\underset{\sim}{v}(t)|\,dt = \int_0^{\pi/6} \sqrt{16\cos^2(2t) + 9\sin^2(t)}\,dt$$

$$= 1.804 \text{ m} \quad (3\text{DP})$$

$\boxed{\text{* Note: Question 2 is at the end!}}$

Question 3a (2 marks)

$f'(x) = (2x - x^2)\, e^{-x}$. So

$f'(x) = 0 \Rightarrow x = 0,\ 2.$

$f(0) = 0$ and $f(2) = 4e^{-2}$

$\Rightarrow$ Stationary points are $(0, 0)$ and $(2, 4e^{-2})$

Question 3b (1 mark)

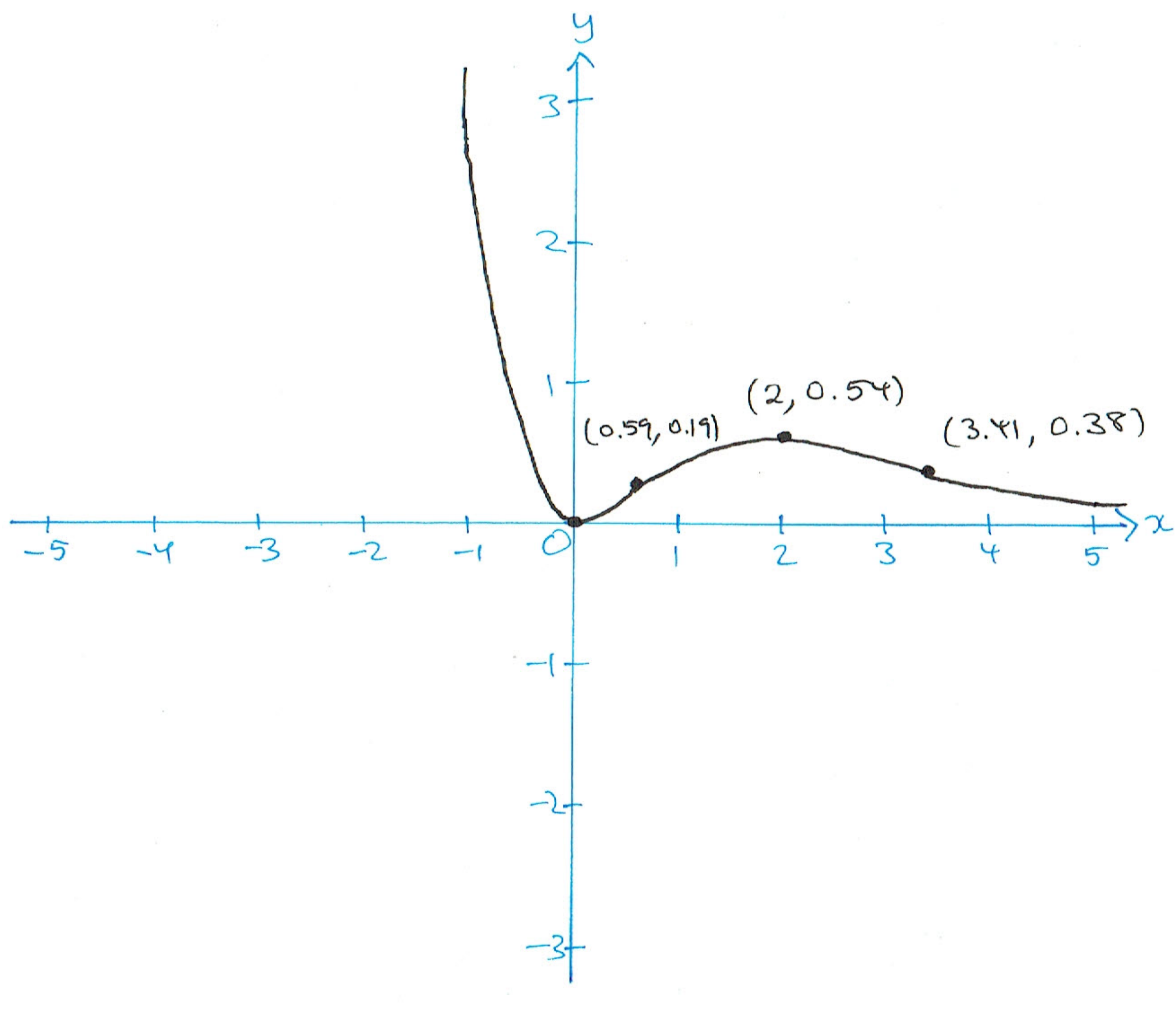
As $x \to \infty$, $f(x) \to 0$ so $y = 0$ is an asymptote of f.

Question 3c (3 marks)

$f''(x) = (x^2 - 4x + 2)e^{-x} = 0 \Rightarrow x = 2 \pm \sqrt{2}$.

The points of inflection are

$(3.41, 0.38)$ and $(0.59, 0.19)$ (2DP)



Question 3d (1 mark)

$$g''(x) = \left(x^n - 2nx^{n-1} + n(n-1)x^{n-2}\right)e^{-x}$$

Question 3e.i (1 mark)

$g''(x) = 0, \; x \neq 0 \implies x = n \pm \sqrt{n}$ provided $n \neq 0$.

Question 3e.ii (2 marks)

We could write $g''(x) = x^{n-2}\left(x^2 - 2nx + n(n-1)\right)e^{-x}$.

Thus, g has a point of inflection at $x = 0$ provided $n - 2 \geq 1$ and $n-2$ is odd so that the sign of g'' changes. g has a point of inflection at $x = n \pm \sqrt{n}$ provided $n \geq 0$ and $n - \sqrt{n} \neq n + \sqrt{n}$ (otherwise, the sign of g'' does not change). Thus:

Number of points of inflection	Value(s) of $n \in \mathbb{Z}$
0	$\mathbb{Z}^- \cup \{0\}$
1	$\{1\}$
2	$2\mathbb{Z}^+$
3	$\{2k+1 \mid k \in \mathbb{Z}^+\}$

Question 4a (3 marks)

$$\underset{\sim}{r}_A'(t) = -25\pi\cos\left(\frac{\pi t}{6}\right)\underset{\sim}{i} + \frac{100}{3}\pi\sin\left(\frac{\pi t}{6}\right)\underset{\sim}{j}$$

$$\implies |\underset{\sim}{r}_A'(t)| = \frac{25\pi}{3}\sqrt{7\sin^2\left(\frac{\pi t}{6}\right) + 9}$$

$$\implies |\underset{\sim}{r}_A'(t)|_{MAX} = \frac{25\pi}{3}\sqrt{7 \times 1 + 9} = \frac{100\pi}{3} \text{ m s}^{-1}$$

(ie. when $\sin^2\left(\frac{\pi t}{6}\right) = 1$).

Question 4b.i (2 marks)

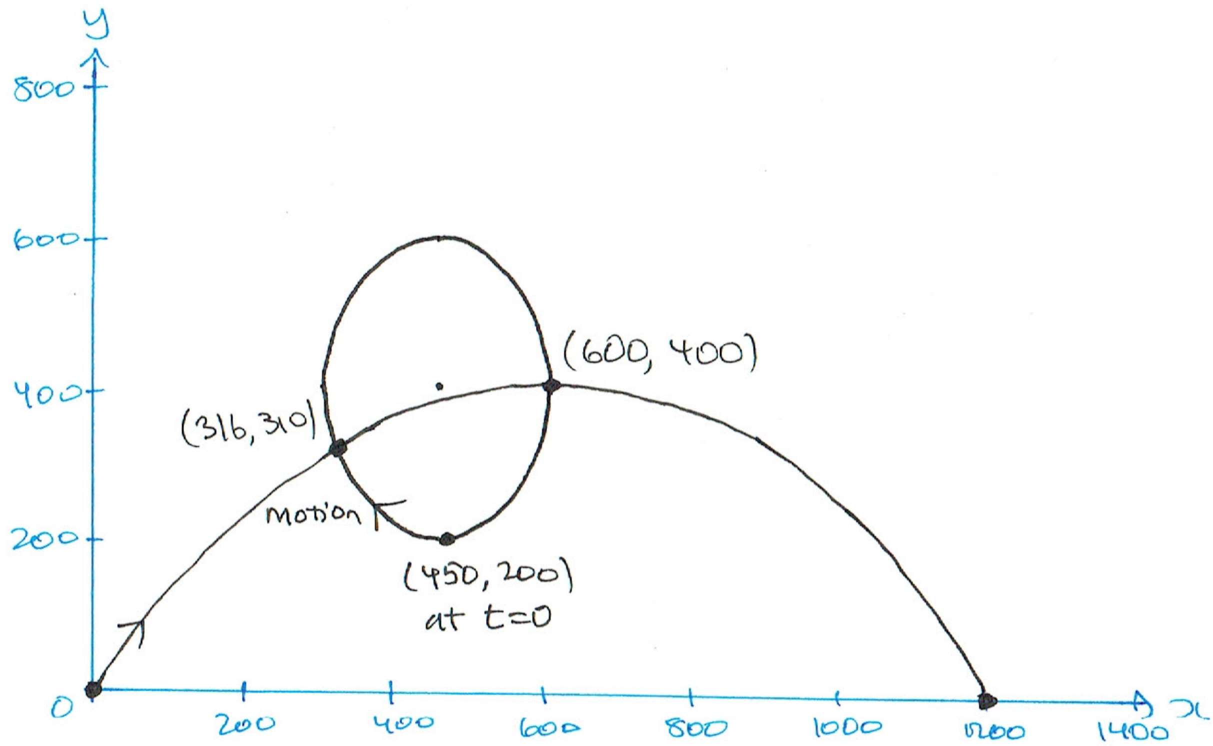
$$x = 450 - 150\sin\left(\frac{\pi t}{6}\right), \quad y = 400 - 200\cos\left(\frac{\pi t}{6}\right)$$

$$\Rightarrow \sin^2\left(\frac{\pi t}{6}\right) = \frac{(x-450)^2}{150^2}, \quad \cos^2\left(\frac{\pi t}{6}\right) = \frac{(y-400)^2}{200^2}$$

Thus, $1 = \sin^2\left(\frac{\pi t}{6}\right) + \cos^2\left(\frac{\pi t}{6}\right)$ gives

$$\frac{(x-450)^2}{22500} + \frac{(y-400)^2}{40000} = 1. \quad ②$$

Question 4b.ii (3 marks)



Question 4c (3 marks)

$$\underset{\sim}{r}_D(t) = 30t\underset{\sim}{i} + (-t^2 + 40t)\underset{\sim}{j}$$

$$\Rightarrow x = 30t \text{ and } y = -t^2 + 40t$$

$$\Rightarrow y = -\left(\frac{x}{30}\right)^2 + \frac{40x}{30} = \frac{-x^2}{900} + \frac{4x}{3} \quad ①$$

(sketch shown above)

Solving ① and ② simultaneously gives

(316, 310) and (600, 400) (ODP)

Question 4d (3 marks)

The drone will be at (316, 310) when

$$30t = 315.921... \Rightarrow t = 10.5307...\ s.$$

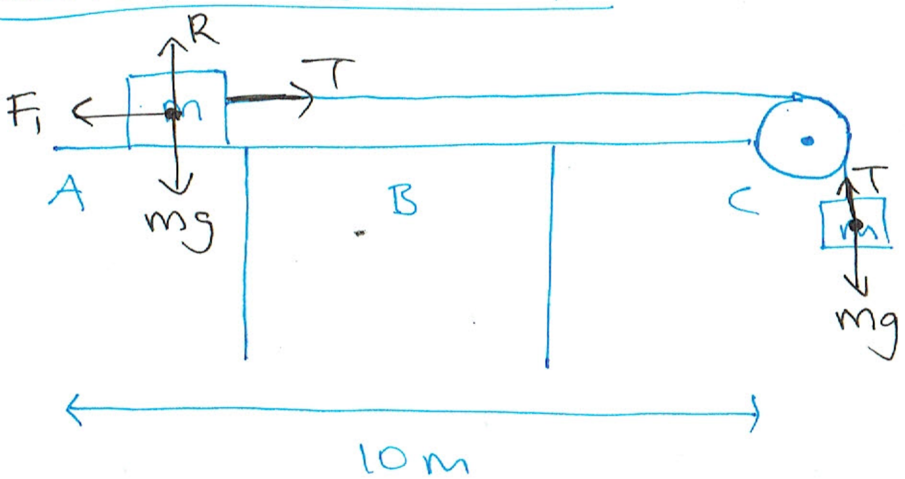
But, $\underset{\sim}{r}_A(10.5307) = 554.347...\underset{\sim}{i} + 256.324...\underset{\sim}{j}$,

So the airplane will not be located at (316, 310) at the same time as the drone. Similarly, the drone will be at (600, 400) when

$$30t = 600 \Rightarrow t = 20\ s.$$

But $\underset{\sim}{r}_A(20) = 579.909...\underset{\sim}{i} + 500\underset{\sim}{j}$, and so will not be located at (600, 400) at the same time as the drone. Thus, the drone and airplane will never make contact. as these are the only points where their paths cross.

Question 5a (2 marks)



Question 5bi (2 marks)

$ma = T - F_1 = T - kmg$ by the block on the platform. ~~But, T = mg by the suspended~~

But, $ma = mg - T$ by the suspended block.

$$\Rightarrow 2ma = mg - kmg = mg(1-k)$$

$$\Rightarrow a = \frac{g(1-k)}{2}\ ms^{-2}.$$

Question 5b.ii (1 mark)

We require that $a>0 \Rightarrow 1-k>0$

$\Rightarrow k \in (0,1)$.

Question 5c (2 marks)

$a = \frac{g(1-k)}{2}, \quad s=5, \quad u=0$

$\Rightarrow 5 = \frac{1}{2}\cdot\frac{g(1-k)}{2}t^2 \quad (t>0)$

$\Rightarrow t = \frac{10}{7}\sqrt{\frac{1}{1-k}}$

Question 5d (2 marks)

$V_B = u+at = \frac{g(1-k)}{2}\cdot\frac{10}{7}\sqrt{\frac{1}{1-k}}$

$= 7\sqrt{1-k}$

(or $V_B^2 = u^2+2as = 2\,\frac{g(1-k)}{2}\cdot 5$

$\Rightarrow V_B = \sqrt{49(1-k)} = 7\sqrt{1-k}$)

Question 5e (4 marks)

We have $ma = mv\frac{dv}{dx} = -0.075mg - 0.4mv^2$
with $v=V_B=2.5$ where $x=0$. The body comes to rest at a distance of $5-x(0)$ from C.

$\cancel{\Rightarrow x(0) = \int_{2.5}^{0} (-0.075g - 0.4v^2)\,d}$

$\Rightarrow 5-x(0) = 5+\int_{2.5}^{0}\frac{v}{0.075g+0.4v^2}\,dv$

$= 3.15$ m.

Question 2a (3 marks)

Let $z = x+yi$. Then,

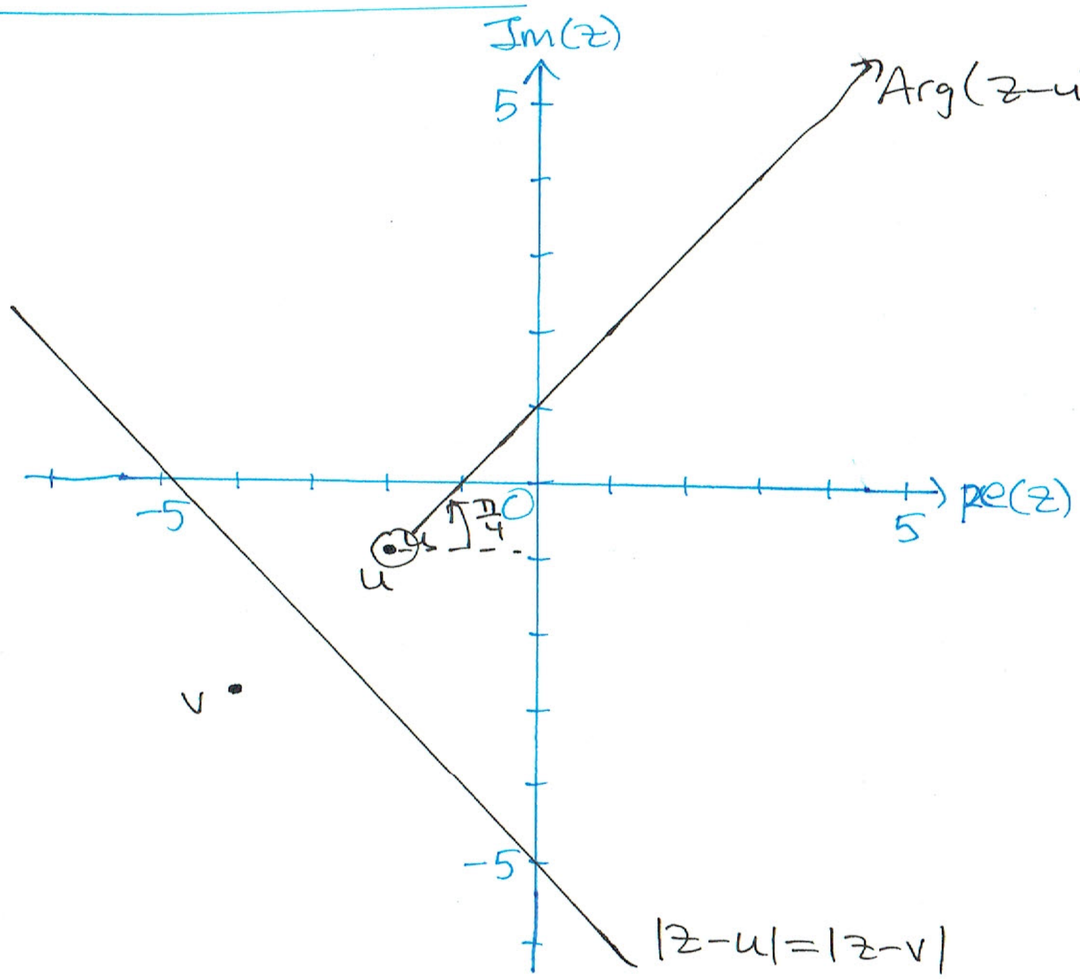
$$|z-u| = |z-v| \Rightarrow (x+2)^2 + (y+1)^2 = (x+4)^2 + (y+3)^2$$

$$\Rightarrow \cancel{x^2} + 4x + 4 + \cancel{y^2} + 2y + 1 = \cancel{x^2} + 8x + 16 + \cancel{y^2} + 6y + 9$$

$$\Rightarrow 4y = -4x - 20$$

$$\Rightarrow y = -x - 5.$$

Question 2b (2marks)



Question 2c (1mark)

The line $|z-u| = |z-v|$ is the perpendicular bisector of the line segment joining u and v.

Question 2d.i (1 mark)

(see above)

Question 2d.ii (1 mark)

$\operatorname{Arg}(z-u) = \frac{\pi}{4} \longleftrightarrow y = x+1, \quad x > -2$

$\Rightarrow f: (-2, \infty) \to \mathbb{R}, \; f(x) = x+1.$

Question 2e. (3 marks)

Let $z_c = \alpha + \beta i$, where $\alpha, \beta \in \mathbb{R}$. Then

$$|z - z_c| = r \longleftrightarrow (x-\alpha)^2 + (y-\beta)^2 = r^2$$

Thus, we have

$$\begin{cases} (-2-\alpha)^2 + (-1-\beta)^2 = r^2 \\ (-4-\alpha)^2 + (-3-\beta)^2 = r^2 \\ (0-\alpha)^2 + (-5-\beta)^2 = r^2 \end{cases}$$

Solving these simultaneously yields

$$a = \frac{-5}{3}, \quad b = \frac{-10}{3}, \quad r = \frac{5\sqrt{2}}{3}.$$

Hence, $z_c = \frac{-5}{3} - \frac{10}{3} i$ and $r = \frac{5\sqrt{2}}{3}$.