The Mathematical Association of Victoria

Trial Examination 2020

SPECIALIST MATHEMATICS

Trial Written Examination 1 - SOLUTIONS

Question 1

Substitute $x = 0$ into $y^3 - xy + e^{-x} = -7$:

$y^3 + 1 = -7 \Rightarrow y^3 = -8 \Rightarrow y = -2$.

Use implicit differentiation.

Differentiate both sides of $y^3 - xy + e^{-x} = -7$ with respect to *x*:

$$\frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(xy) - e^{-x} = 0$$

$$\Rightarrow \underbrace{3y^2\frac{dy}{dx}}_{\text{From the chain rule}} - \underbrace{\left(y + x\frac{dy}{dx}\right)}_{\text{From the product rule}} - e^{-x} = 0.$$ **[M1]**

Substitute $x = 0$ and $y = -2$:

$$3(-2)^2\frac{dy}{dx} - \big((-2) + 0\big) - 1 = 0$$

$$\Rightarrow 12\frac{dy}{dx} + 2 - 1 = 0 \qquad \Rightarrow \frac{dy}{dx} = -\frac{1}{12}.$$

Note: A less efficient approach is to first solve for $\frac{dy}{dx}$ in terms of *y* and *x* using algebra and then to substitute $x = 0$ and $y = -2$.

Answer: $-\frac{1}{12}$. **[A1]**

Question 2

a.

D is the set of all real numbers excluding values of x such that $|x-1|-2=0$.

$|x-1|-2=0$

$\Rightarrow |x-1|=2$. **[M1]**

Case 1: $x-1=2 \Rightarrow x=3$.

Case 2: $x-1=-2 \Rightarrow x=-1$.

Answer: $R\setminus\{-1,\ 3\}$. **[A1]**

b.

$x-2-\dfrac{2}{|x-1|-2}=0$.

Substitute $|x-1|=\begin{cases} x-1 & \text{for } x \ge 1 \\ -(x-1) & \text{for } x<1 \end{cases}$.

Case 1: $x \ge 1$.

$x-2-\dfrac{2}{(x-1)-2}=0$ and $x \ge 1$

$\Rightarrow x-2-\dfrac{2}{x-3}=0$ **[M1]**

$\Rightarrow (x-3)(x-2)-2=0$ $\quad\Rightarrow x^2-5x+4=0$ $\quad\Rightarrow (x-4)(x-1)=0$

$\Rightarrow x=4$ or $x=1$.

Case 2: $x<1$.

$x-2-\dfrac{2}{-(x-1)-2}=0$ and $x<1$

$\Rightarrow x-2+\dfrac{2}{x+1}=0$ **[M1]**

$\Rightarrow (x+1)(x-2)+2=0$ $\quad\Rightarrow x^2-x=0$ $\quad\Rightarrow x(x-1)=0$

$\Rightarrow x=0$.

Note: $x=1$ is rejected because of the restriction $x<1$.

Answer: $x=0,\ 1,\ 4.$ **[A1]**

c.

Use the results from **part b.**

Answer: $g(x)=\begin{cases} x-2-\dfrac{2}{x-3} & \text{for } x \ge 1 \\ x-2+\dfrac{2}{x+1} & \text{for } x<1 \end{cases}$. **[A1]**

Question 3

a.

$$\frac{4}{4-x^2}=\frac{4}{(2-x)(2+x)}.$$

Two non-repeated linear factors in the denominator therefore use the partial fraction form $\frac{A}{2-x}+\frac{B}{2+x}$.

$$\frac{4}{4-x^2}=\frac{4}{(2-x)(2+x)}=\frac{A}{2-x}+\frac{B}{2+x}=\frac{A(2+x)+B(2-x)}{(2-x)(2+x)}.$$

Therefore

$$4=A(2+x)+B(2-x)$$

is required to be true for all values of x.

Method 1:
Substitute convenient values of x and solve for A and B:

$x=-2$: $4=4B\Rightarrow B=1$.

$x=2$: $4=4A\Rightarrow A=1$.

Method 2:
Expand, group like terms, equate coefficients and solve the resulting simultaneous equations.

$4=A(2+x)+B(2-x) \quad \Rightarrow 4=(A-B)x+2A+2B$.

Coefficient of x: $0=A-B$. .... (1)

Constant term: $4=2A+2B\Rightarrow 2=A+B$. .... (2)

Solve equations (1) and (2) simultaneously: $A=1$ and $B=1$.

Answer: $\frac{1}{2-x}+\frac{1}{2+x}$ **[A1]**

b.

- $\dfrac{dy}{dx}=\dfrac{4y\cos(x)}{3+\cos^2(x)}$ is a separable differential equation:

$$\int \frac{1}{y}\,dy=\int \frac{4\cos(x)}{3+\cos^2(x)}\,dx\,.$$ **[M1]**

Perform the integrations:

- $\displaystyle\int \frac{1}{y}\,dy=\log_e|y|+c_1$ where c_1 is an arbitrary real constant of integration.

- Let $I=\displaystyle\int \frac{4\cos(x)}{3+\cos^2(x)}\,dx$.

The numerator of the integrand suggests substituting $\cos^2(x)=1-\sin^2(x)$ in the denominator so that the integrand has the form $\dfrac{f'(x)}{f(x)}$ and therefore the integral can calculated by making an appropriate substitution:

$$I=\int \frac{4\cos(x)}{4-\sin^2(x)}\,dx\,.$$ **[M1]**

Substitute $u=\sin(x)$:

$$I=\int \frac{4\cos(x)}{4-u^2}\left(\frac{du}{\cos(x)}\right)=\int \frac{4}{4-u^2}\,du\,.$$

Use the answer from **part a.**:

$$I=\int \frac{1}{2-u}+\frac{1}{2+u}\,du$$

$$=-\log_e|2-u|+\log_e|2+u|+c_2 \qquad =\log_e\left|\frac{2+u}{2-u}\right|+c_2$$

where c_2 is an arbitrary real constant of integration.

Back-substitute $u=\sin(x)$:

$$I=\log_e\left|\frac{2+\sin(x)}{2-\sin(x)}\right|+c_2\,.$$ **[H1]**

Consequential on answer to **part a.**

- Therefore:

$$\log_e|y|+c_1=\log_e\left|\frac{2+\sin(x)}{2-\sin(x)}\right|+c_2$$

$\Rightarrow \log_e|y|=\log_e\left|\dfrac{2+\sin(x)}{2-\sin(x)}\right|+c_3$ where c_3 is an arbitrary real constant.

- Substitute $y(\pi)=1$: $\log_e|1|=\log_e\left|\dfrac{2+\sin(\pi)}{2-\sin(\pi)}\right|+c_3$ $\Rightarrow 0=\log_e\left|\dfrac{2+0}{2+0}\right|+c_3=c_3$.

- Therefore:

$$\log_e|y|=\log_e\left|\frac{2+\sin(x)}{2-\sin(x)}\right|$$

$$\Rightarrow y=\frac{2+\sin(x)}{2-\sin(x)}$$

since $y(\pi)=1\Rightarrow y\ge 0$ and $\dfrac{2+\sin(x)}{2-\sin(x)}>0$ (because $-1\le\sin(x)\le 1$).

- Re-arrange into the required form $y=\dfrac{a}{b-\sin(x)}+c$:

Method 1: Perform a long division.

$$\begin{array}{r|l} & -1 \\ -\sin(x)+2 & \sin(x)+2 \\ & -\left(\sin(x)-2\right) \\ \hline & \qquad 4 \end{array}$$

Method 2:

$$\frac{2+\sin(x)}{2-\sin(x)}=\frac{a}{b-\sin(x)}+c=\frac{a+c\left(b-\sin(x)\right)}{b-\sin(x)}=\frac{a+cb-c\sin(x)}{b-\sin(x)}.$$

By inspection:

$b=2$ and $c=-1$.

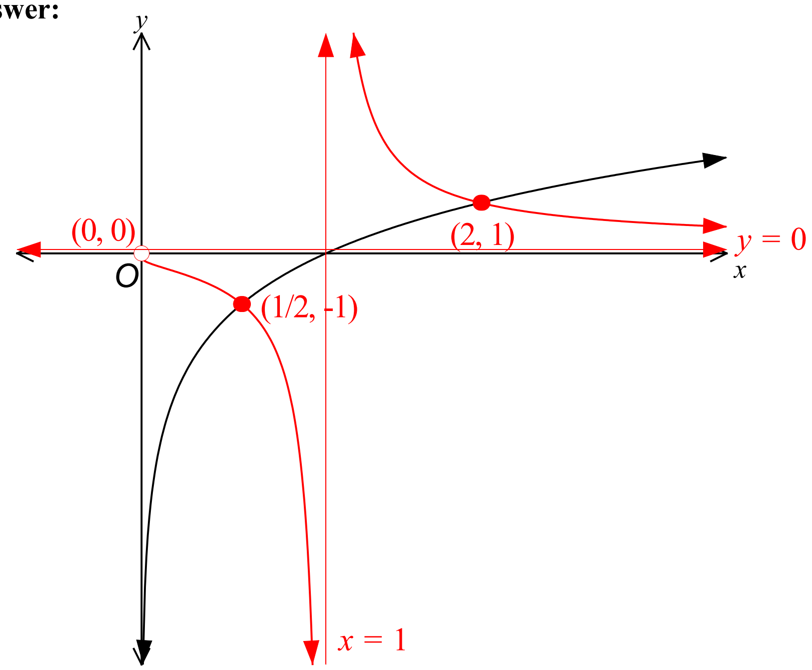
$a+cb=2\Rightarrow a=4$.

Therefore $y=\dfrac{4}{2-\sin(x)}-1$.

Answer: $y=\dfrac{4}{2-\sin(x)}-1$. **[A1]**

Question 4

Answer:



Marking scheme:

- Vertical asymptote: $x = 1$. $\left[\text{A}\frac{1}{2}\right]$

- Horizontal asymptote: $y = 0$. $\left[\text{A}\frac{1}{2}\right]$

- '**Hole**' at the endpoint $(0,\ 0)$ (must be labelled with an open circle). $\left[\text{A}\frac{1}{2}\right]$

Note: $x = 0$ is **not** a vertical asymptote. Students showing an asymptote should be penalised.

- Intersection point at $(2,\ 1)$. $\left[\text{A}\frac{1}{2}\right]$

- Intersection point at $\left(\frac{1}{2},\ -1\right)$. $\left[\text{A}\frac{1}{2}\right]$

- Shape to the right of the vertical asymptote. $\left[\text{A}\frac{1}{2}\right]$

- Shape to the left of the vertical asymptote. $\left[\text{A}\frac{1}{2}\right]$

Note: Shape must show a point of inflection somewhere in the interval $\left(0,\ \frac{1}{2}\right)$. However, its location does not have to be exactly consistent with the scale.

Add up all $\frac{1}{2}$ marks and **round down** to the nearest integer

Calculation of required features:

- Vertical asymptote: $\log_2(x) = 0 \Rightarrow x = 1$.

- $\lim\limits_{x \to 0^+} \dfrac{1}{\log_2(x)} = 0$ therefore (0, 0) is an endpoint and a 'hole'.

Note: If $x = 0$ was a vertical asymptote then $\lim\limits_{x \to 0^+} \dfrac{1}{\log_2(x)} = \pm\infty$. This is **not** the case therefore $x = 0$ is **not** a vertical asymptote.

- Intersection points:

Method 1: Intersection points occur where $y = \pm 1$ (since the reciprocal of $y = \pm 1$ is also $y = \pm 1$).

$\log_2(x) = \pm 1 \Rightarrow x = 2, \dfrac{1}{2}$.

Method 2: $\log_2(x) = \dfrac{1}{\log_2(x)} \Rightarrow \left(\log_2(x)\right)^2 = 1 \Rightarrow \log_2(x) = \pm 1 \Rightarrow x = \dfrac{1}{2}, 2$.

Calculation of the coordinates of the point of inflection (NOT required in this question)

***x*-coordinate:** Solve $\dfrac{d^2y}{dx^2} = 0$.

$$y = \frac{1}{\log_2(x)}, \quad x \neq 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\frac{1}{kx}}{\left(\log_2(x)\right)^2} \quad \text{where } k = \log_e(2) \text{ (for convenience)}$$

$$= \frac{-k}{x\left(k\log_2(x)\right)^2} \qquad = \frac{-k}{x\left(\log_e(x)\right)^2} \quad \text{using the change of base rule}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{k\frac{d}{dx}\left[x\left(\log_e(x)\right)^2\right]}{x^2\left(\log_e(x)\right)^4} \qquad = \frac{k\left(\left(\log_e(x)\right)^2 + 2\log_e(x)\right)}{x^2\left(\log_e(x)\right)^4}.$$

$$\frac{d^2y}{dx^2} = 0 \qquad \Rightarrow \left(\log_e(x)\right)^2 + 2\log_e(x) = 0 \qquad \Rightarrow \log_e(x)\left(\log_e(x) + 2\right) = 0.$$

Case 1: $\log_e(x) = 0 \Rightarrow x = 1$ is rejected because of the restriction $x \neq 1$.

Case 2: $\log_e(x) + 2 = 0 \Rightarrow x = \dfrac{1}{e^2}$.

Therefore $y = \dfrac{1}{\log_2(x)}$ has a **potential** point of inflection at $x = \dfrac{1}{e^2}$.

Since it is not known whether a point of inflection actually exists, this solution must be checked:

The solution to $\frac{d^2y}{dx^2} = 0$ will correspond to a point of inflection of $y = \frac{1}{\log_2(x)}$ only if it corresponds to a turning point of $\frac{dy}{dx} = -\frac{k}{x(\log_e(x))^2}$.

Let $g(x) = \frac{dy}{dx} = \frac{-k}{x(\log_e(x))^2}$.

Then $g(x)$ has a stationary point at $x = \frac{1}{e^2}$.

The nature of this stationary point can be investigated using the sign test:

x	$\frac{1}{e^3}$	$\frac{1}{e^2}$	$\frac{1}{e}$
$g'(x) = \frac{k\left((\log_e(x))^2 + 2\log_e(x)\right)}{x^2(\log_e(x))^4}$	$\frac{k\left((-3)^2 + 2(-3)\right)}{e^{-6}(-3)^4}$ $= \frac{3k}{81e^{-6}} > 0$	0	$\frac{k\left((-1)^2 + 2(-1)\right)}{e^{-2}(-1)^4}$ $= \frac{-k}{e^{-6}} < 0$

Note: $k = \log_e(2) > 0$.

Therefore $g(x)$ has a (maximum) turning point at $x = \frac{1}{e^2}$.

Therefore $y = \frac{1}{\log_2(x)}$ has a point of inflection at $x = \frac{1}{e^2}$.

***y*-coordinate:**

Substitute $x = \frac{1}{e^2}$ into $y = \frac{1}{\log_2(x)}$: $y = -\frac{1}{2\log_2(e)} = -\frac{1}{2}\log_e(2)$ (using the change of base rule).

Therefore $y = \frac{1}{\log_2(x)}$ has a point of inflection at $\left(\frac{1}{e^2},\ -\frac{1}{2}\log_e(2)\right)$.

Since $\frac{dy}{dx} = \frac{-k}{x(\log_e(x))^2} \neq 0$ for $x = \frac{1}{e^2}$ (or any value of x) the point of inflection is non-stationary.

Question 5

The direction of motion is in the direction of the velocity:

$\underset{\sim A}{r}(t) = \sin(at)\underset{\sim}{i} + \cos(t)\underset{\sim}{j} \qquad \Rightarrow \underset{\sim A}{v} = \dfrac{d\underset{\sim A}{r}}{dt} = a\cos(at)\underset{\sim}{i} - \sin(t)\underset{\sim}{j}$.

$\underset{\sim B}{r}(t) = \sin(t)\underset{\sim}{i} - \cos(at)\underset{\sim}{j} \qquad \Rightarrow \underset{\sim B}{v} = \dfrac{d\underset{\sim B}{r}}{dt} = \cos(t)\underset{\sim}{i} + a\sin(at)\underset{\sim}{j}$.

Particles moving perpendicular to each other:

$\underset{\sim A}{v} \cdot \underset{\sim B}{v} = 0$

$\Rightarrow \left(a\cos(at)\underset{\sim}{i} - \sin(t)\underset{\sim}{j}\right) \cdot \left(\cos(t)\underset{\sim}{i} + a\sin(at)\underset{\sim}{j}\right) = 0$ **[M1]**

$\Rightarrow a\cos(at)\cos(t) - a\sin(at)\sin(t) = 0$

$\Rightarrow \cos(at + t) = 0$ **[M1]**

using the compound angle formula (from the VCAA formula sheet)

$\Rightarrow (a+1)t = (2n+1)\dfrac{\pi}{2}, \quad n \in Z$.

Substitute $t = 2$:

$2(a+1) = (2n+1)\dfrac{\pi}{2}$

$\Rightarrow a = (2n+1)\dfrac{\pi}{4} - 1$.

The two smallest positive values of a are required:

$n = 0: \quad a = \dfrac{\pi}{4} - 1 < 0$ therefore reject.

$n = 1: \quad a = \dfrac{3\pi}{4} - 1 > 0$.

$n = 2: \quad a = \dfrac{5\pi}{4} - 1$.

Answer: $a = \dfrac{3\pi}{4} - 1$ and $a = \dfrac{5\pi}{4} - 1$. **[A1]**

Question 6

a.

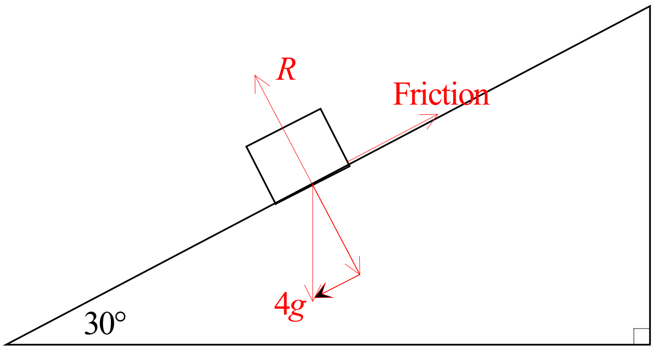
Forces acting on the object:

- Weight force of size $4g$ in the downwards vertical direction.

  Component of weight force parallel to plane: $4g\sin(30^0) = 2g$.

  Component of weight force parallel to plane: $4g\cos(30^0) = 2\sqrt{3}g$.

- Friction force up the plane (opposite to the direction of motion).
- Normal reaction force of size R perpendicular to the plane.



Component of the weight force parallel to the plane:

$4g\sin(30^0) = 2g$. **[A1]**

Forces on the object acting parallel to the slope (take down the slope [direction of motion] as the positive direction):

$F_{net} = ma = 4(3) = 12$. .... (1)

$F_{net} = \underbrace{2g}_{\text{Component of the weight force parallel to the plane}} - \text{Friction}$. .... (2)

Equate equations (1) and (2):

$12 = 2g - \text{Friction}$

$\Rightarrow \text{Friction} = 2g - 12$.

Answer: $2g - 12$ newtons. **[A1]**

b.

Method 1:

- Substitute $a = 3$ into $a = v\dfrac{dv}{dx}$:

$$v\frac{dv}{dx} = 3.$$

This is a separable differential equation:

$$\int v \, dv = \int 3 \, dx$$

$$\Rightarrow \frac{1}{2}v^2 = 3x + c \qquad \ldots. (1)$$ **[M1]**

where c is an arbitrary real constant of integration.

- A boundary condition on v and x is required so that the value of c can be determined.

Note: It is **not** correct to take $v = 0$ when $x = 0$. This is because the object was "… given an initial push …". and so is given an initial velocity greater than zero.

It is known that "After four seconds the velocity of the object is $14\,\text{ms}^{-1}$." This information can be used to calculate the initial velocity of the object:

Substitute $a = 3$ into $a = \dfrac{dv}{dt}$:

$$\frac{dv}{dt} = 3 \qquad \Rightarrow v = 3t + k$$

where k is an arbitrary real constant of integration.

Substitute $v = 14$ when $t = 4$ and solve for k: $k = 2$.

Therefore $v = 3t + 2$. $\qquad \ldots. (2)$

Substitute $t = 0$ into equation (2) to get the initial velocity:

$v = 2$. **[A1]**

- Substitute $v = 2$ when $x = 0$ and solve for c: $c = 2$.

Substitute $c = -2$ into equation (1):

$$\frac{1}{2}v^2 = 3x + 2.$$

Substitute $x = 4$ and solve for v:

$v = \sqrt{28}\ \text{ms}^{-1}$ (taking down the plane as the positive direction).

Answer: $2\sqrt{7}\ \text{ms}^{-1}$. Accept $\sqrt{28}\ \text{ms}^{-1}$. **[A1]**

Method 2:

- Substitute $a = 3$ into $a = \frac{dv}{dt}$:

$\frac{dv}{dt} = 3 \qquad \Rightarrow v = 3t + c$

where c is an arbitrary real constant of integration.

Substitute $v = 14$ when $t = 4$ and solve for c: $c = 2$.

Therefore:

$v = 3t + 2 \qquad \text{.... (1)}$

$\Rightarrow \frac{dx}{dt} = 3t + 2$

$\Rightarrow x = \frac{3}{2}t^2 + 2t + k$

where k is an arbitrary real constant of integration.

Substitute $x = 0$ when $t = 0$ and solve for k: $k = 0$.

Therefore:

$x = \frac{3}{2}t^2 + 2t$. .... (2) **[M1]**

- Substitute $x = 4$ into equation (2) and solve for t:

$4 = \frac{3}{2}t^2 + 2t, \quad t > 0$

$\Rightarrow 3t^2 + 4t - 8 = 0 \qquad \Rightarrow t = \frac{-4 \pm \sqrt{112}}{6} = \frac{-4 \pm 4\sqrt{7}}{6} = \frac{-2 \pm 2\sqrt{7}}{3}$.

$t = \frac{-2 + 2\sqrt{7}}{3}$ **[A1]**

since $t > 0$.

- Substitute $t = \frac{-2 + 2\sqrt{7}}{3}$ into equation (1):

$v = 3\left(\frac{-2 + 2\sqrt{7}}{3}\right) + 2 = \sqrt{28}$.

Answer: $\sqrt{28}$ ms^{-1}. **[A1]**

Question 7

a.

Let $\theta = \arcsin\left(\sqrt{x}\right)$:

$$\sin\left(2\arcsin\left(\sqrt{x}\right)\right) = \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

using the double angle formula (from the VCAA formula sheet).

$\theta = \arcsin\left(\sqrt{x}\right) \Rightarrow \sin(\theta) = \sqrt{x}$.

$\sin(\theta) = \sqrt{x} \Rightarrow \cos(\theta) = \pm\sqrt{1-x}$.

But $0 < x < 1$

therefore $\theta = \arcsin\left(\sqrt{x}\right) \Rightarrow 0 < \theta < \dfrac{\pi}{2}$

therefore $\cos(\theta) = +\sqrt{1-x}$.

Substitute $\sin(\theta) = \sqrt{x}$ and $\cos(\theta) = \sqrt{1-x}$ into $\sin\left(2\arcsin\left(\sqrt{x}\right)\right) = \sin(2\theta) = 2\sin(\theta)\cos(\theta)$:

$\sin\left(2\arcsin\left(\sqrt{x}\right)\right) = 2\sqrt{x}\sqrt{1-x}$. **[M1]**

Justification for choosing the positive square root $\sqrt{x-1}$ is required

Therefore:

$$f(x) = \frac{x}{\sin\left(2\arcsin\left(\sqrt{x}\right)\right)} = \frac{x}{2\sqrt{x}\sqrt{1-x}} = \frac{\sqrt{x}}{2\sqrt{1-x}}.$$

Note: The simplification is valid because $x \neq 0$.

Answer: $f(x) = \dfrac{\sqrt{x}}{2\sqrt{1-x}}$. **[A1]**

b.

$$V = \pi \int_{\frac{1}{4}}^{\frac{1}{2}} y^2 \, dx.$$

Substitute $y = \dfrac{\sqrt{x}}{2\sqrt{1-x}}$:

$$V = \pi \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{4(1-x)} \, dx$$ **[H1]**

Consequential on answer to **part a.**

$$= \frac{\pi}{4} \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{1-x} \, dx$$

$$= \frac{\pi}{4} \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{1-x} - 1 \, dx$$ **[H1]**

$$= -\frac{\pi}{4} \left[\log_e(1-x) + x \right]_{\frac{1}{4}}^{\frac{1}{2}}$$

Note: Modulus is not necessary because $(1-x) > 0$ for $x \in \left[\frac{1}{4}, \frac{1}{2}\right]$.

$$= -\frac{\pi}{4}\left(\left(\log_e\left(\frac{1}{2}\right) + \frac{1}{2} \right) - \left(\log_e\left(\frac{3}{4}\right) + \frac{1}{4} \right) \right)$$

$$= -\frac{\pi}{4}\left(\log_e\left(\frac{1}{2}\right) - \log_e\left(\frac{3}{4}\right) + \frac{1}{4} \right) \qquad = -\frac{\pi}{4}\left(\log_e\left(\frac{2}{3}\right) + \frac{1}{4} \right)$$

$$= \frac{\pi}{4}\left(\log_e\left(\frac{3}{2}\right) - \frac{1}{4} \right).$$

Answer: $\frac{\pi}{4}\left(\log_e\left(\frac{3}{2}\right) - \frac{1}{4} \right)$ cubic units. **[H1]**

Accept all equivalent answers

Question 8

a.

$|z| = r$ since $z = r\text{cis}(\theta)$.

Therefore the values of r are required.

Substitute $z = r\text{cis}(\theta) \Rightarrow \bar{z} = r\text{cis}(-\theta)$:

$z^n - 3i\bar{z} = 0 \quad \Rightarrow z^n = 3i\bar{z}$

$\Rightarrow r^n\text{cis}(n\theta) = 3ir\text{cis}(-\theta)$

$\Rightarrow r^n\text{cis}(n\theta) = 3r\text{cis}\left(\frac{\pi}{2}\right)\text{cis}(-\theta)$

$\Rightarrow r^n\text{cis}(n\theta) = 3r\text{cis}\left(\frac{\pi}{2} - \theta\right)$. …. (1) **[M1]**

Note: Part b. also requires equation (1).

Equate moduli in equation (1):

$r^n = 3r \quad \Rightarrow r^n - 3r = 0 \quad \Rightarrow r\left(r^{n-1} - 3\right) = 0$.

Case 1: $r = 0$.

Case 2: $r^{n-1} - 3 = 0 \Rightarrow r = 3^{\frac{1}{n-1}}$.

Answer: $r = 0$ or $r = 3^{\frac{1}{n-1}}$. **[A1]**

b.

Case 1: $r = 0 \quad \Rightarrow z = 0$. Therefore $\theta = \text{Arg}(z)$ is undefined since $\text{Arg}(0)$ is undefined. **[A1]**

Case 2: $r = 3^{\frac{1}{n-1}}$.

From **part a.**: $r^n\text{cis}(n\theta) = 3r\text{cis}\left(\frac{\pi}{2}\right)\text{cis}(-\theta)$. …. (1) **[M1]**

Equate arguments in equation (1):

$n\theta = \frac{\pi}{2} - \theta + 2k\pi$, $k \in Z$ **[M1]**

$\Rightarrow (n+1)\theta = \frac{\pi}{2} + 2k\pi \quad = \frac{\pi + 4k\pi}{2} \quad = \frac{\pi(4k+1)}{2}$.

c.

Case 1: $z = 0$ is a solution.

Case 2: Substitute $n = 5$ into $r = 3^{\frac{1}{n-1}}$ and $\theta = \dfrac{(1+4k)}{2(n+1)}\pi$.

$r = 3^{\frac{1}{4}}$ and $\theta = \dfrac{(1+4k)}{12}\pi$.

$\underline{k=0}$: $\theta = \dfrac{\pi}{12}$. $\underline{k=1}$: $\theta = \dfrac{5\pi}{12}$. $\underline{k=2}$: $\theta = \dfrac{3\pi}{4}$.

$\underline{k=-1}$: $\theta = -\dfrac{\pi}{4}$. $\underline{k=-2}$: $\theta = -\dfrac{7\pi}{12}$. $\underline{k=-3}$: $\theta = -\dfrac{11\pi}{12}$.

Other values of k give arguments equivalent to one of those listed above.

Eg $k = 3$ gives $\theta = \dfrac{13\pi}{12} \equiv -\dfrac{11\pi}{12}$ ($k = -3$).

Therefore case 2 has 6 solutions.

Answer: **Number of solutions:** 7.

Justification: $z = 0$ plus 6 other solutions corresponding to $r = 3^{\frac{1}{4}}$ and the 6 unique arguments found from $\theta = \dfrac{(1+4k)}{12}\pi$.

[A1]

Note: $z^5 - 3i\bar{z}$ is not a polynomial in the variable z because one of the terms is $3i\bar{z}$ **not** $3iz$. In particular, $z^5 - 3i\bar{z}$ is **not** a polynomial of degree 5 and the Fundamental Theorem of Algebra is not valid for this equation. Therefore 5 roots should not be expected.

Question 9

a.

- Let $y = x^2 \arccos\left(\frac{1}{\sqrt{x}}\right)$.

Use the product rule:

$u = x^2$.

$v = \arccos\left(\frac{1}{\sqrt{x}}\right)$.

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

- Use the chain rule to calculate $\frac{dv}{dx}$:

Let $w = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow \frac{dw}{dx} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2}\frac{1}{x\sqrt{x}}$.

$v = \arccos(w) \Rightarrow \frac{dv}{dw} = \frac{-1}{\sqrt{1-w^2}}$ (from the VCAA formula sheet).

$$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx} \qquad = -\frac{1}{2}\frac{1}{x\sqrt{x}} \times \frac{-1}{\sqrt{1-w^2}} \qquad = \frac{1}{2x\sqrt{x}\sqrt{1-w^2}}.$$

Back-substitute $w = \frac{1}{\sqrt{x}}$:

$$\frac{dv}{dx} = \frac{1}{2x\sqrt{x}\sqrt{1-\frac{1}{x}}} \qquad = \frac{1}{2x\sqrt{x}\sqrt{\frac{x-1}{x}}} \qquad = \frac{\sqrt{x}}{2x\sqrt{x}\sqrt{x-1}} \qquad = \frac{1}{2x\sqrt{x-1}}.$$

$\frac{dv}{dx} = \frac{1}{2x\sqrt{x-1}}$. **[M1]**

- Therefore $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \qquad = \frac{x^2}{2x\sqrt{x-1}} + 2x\arccos\left(\frac{1}{\sqrt{x}}\right) \qquad = \frac{x}{2\sqrt{x-1}} + 2x\arccos\left(\frac{1}{\sqrt{x}}\right)$.

Answer: $\frac{x}{2\sqrt{x-1}} + 2x\arccos\left(\frac{1}{\sqrt{x}}\right)$. **[A1]**

b.

Use integration by recognition.

From **part a.**:

- $\dfrac{dy}{dx} = \dfrac{x}{2\sqrt{x-1}} + 2x\arccos\left(\dfrac{1}{\sqrt{x}}\right)$ where $y = x^2 \arccos\left(\dfrac{1}{\sqrt{x}}\right)$.

- Integrate both sides with respect to x:

$$y = \int \frac{x}{2\sqrt{x-1}} + 2x\arccos\left(\frac{1}{\sqrt{x}}\right) dx \qquad = \int \frac{x}{2\sqrt{x-1}}\, dx + 2\int x\arccos\left(\frac{1}{\sqrt{x}}\right) dx .$$

- Substitute $y = x^2 \arccos\left(\dfrac{1}{\sqrt{x}}\right)$:

$$x^2 \arccos\left(\frac{1}{\sqrt{x}}\right) = \int \frac{x}{2\sqrt{x-1}}\, dx + 2\int x\arccos\left(\frac{1}{\sqrt{x}}\right) dx .$$ **[M1]**

- Re-arrange to make $\displaystyle\int x\arccos\left(\frac{1}{\sqrt{x}}\right) dx$ the subject:

$$\int x\arccos\left(\frac{1}{\sqrt{x}}\right) dx = \frac{1}{2}x^2 \arccos\left(\frac{1}{\sqrt{x}}\right) - \int \frac{x}{4\sqrt{x-1}}\, dx .$$

- Calculate $I = \displaystyle\int \frac{x}{4\sqrt{x-1}}\, dx$.

Substitute $t = x-1$: $I = \displaystyle\int \frac{t+1}{4\sqrt{t}}\, dt$ **[M1]**

$$= \int \frac{1}{4}t^{1/2} + \frac{1}{4}t^{-1/2}\, dt \qquad = \frac{1}{6}t^{3/2} + \frac{1}{2}t^{1/2} \qquad = \frac{1}{6}(x-1)^{3/2} + \frac{1}{2}(x-1)^{1/2} .$$

- Therefore $\int x \arccos\left(\frac{1}{\sqrt{x}}\right) dx = \frac{1}{2}x^2 \arccos\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{6}(x-1)^{3/2} - \frac{1}{2}(x-1)^{1/2}$.

Answer: $\frac{1}{2}x^2 \arccos\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{6}(x-1)^{3/2} - \frac{1}{2}(x-1)^{1/2}$. **[A1]**

Accept all equivalent forms (such as $\frac{1}{2}x^2 \arccos\left(\frac{1}{\sqrt{x}}\right) - \frac{1}{6}(x+2)\sqrt{x-1}$)

Answers that include a constant term are acceptable: The arbitrary constant of anti-differentiation can be taken as zero since only *an* anti-derivative (rather than *the* anti-derivative) is required.

END OF SOLUTIONS