SPECIALIST MATHEMATICS

Units 3 & 4 – Written examination 2

TSSM™
Creating VCE Success

2019 Trial Examination

SOLUTIONS

SECTION A

Question 1

C

Explanation:

$\cot\theta = \frac{3}{2}$

$\tan\theta = \frac{2}{3}$

$\cos\theta = \pm\frac{3}{\sqrt{13}}$ (First and third quadrants.)

$\sec\theta = \pm\frac{\sqrt{13}}{3}$

Question 2

A

Explanation:

$$\frac{\cos 2\alpha}{cos^4\alpha - sin^4\alpha}$$

$$\frac{cos^2\alpha - sin^2\alpha}{(cos^2\alpha - sin^2\alpha)(cos^2\alpha + sin^2\alpha)}$$

$$\frac{1}{(cos^2\alpha + sin^2\alpha)}$$

= 1

Question 3

D

Explanation:

$f(x) = \tan(20x - \pi)\,, x \in (0, 5\pi]$ has a period of $\frac{\pi}{20}$

So, there will be $\frac{5\pi}{\frac{\pi}{20}}$ = 100

Remove one intercept (at x = 0, due to domain restriction), so 99.

The phase shift of π units will not affect the number of intercepts.

Question 4

A

Explanation:

$z^5 = i = cis\left(\frac{\pi}{2}\right)$

$z_1 = cis\left(\frac{\pi}{10}\right)$

$Solutions\ are\ z = cis\left(\frac{\pi}{10} \pm \frac{2n\pi}{5}\right), where\ n\ is\ an\ integer$

$cis\left(\frac{\pi \pm 4n\pi}{10}\right), where\ n\ is\ an\ integer$

Question 5

B

Explanation:

$12\hat{\boldsymbol{a}}$ has a length of $12 \times 1 = 12$ and is by definition in the direction of **a**.

Question 6

C

Explanation:

$\overrightarrow{AC} = 8\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{AM} = 4\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{MB} = -2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{MA}.\overrightarrow{MB} = 9$

$\cos\theta = \frac{9}{\sqrt{18} \times \sqrt{17}}$

$\theta \approx 59°$

Question 7

E

Explanation:

P(z) **could** have two real and one non-real solution is true. For example, solutions of $z = 0, z = 1$ and $z = i$ implies $P(z) = z^3 - (1 + i)z^2 + iz$ as required.

Question 8

B

Explanation:

$m \times n = (4a - b) + (a + 4b)i$
$4a - b = 11, \qquad a + 4b = 10$

$a = -2, \qquad b = 3$

$m + \bar{n} = (4 + i) + (-2 - 3i) = 2 - 2i$

Question 9

A

Explanation:

$|z - 2i| = 2$ defines the circle $x^2 + (y - 2)^2 = 4$

$Re(z) - Im(z + 4i) = 2\ defines\ the\ straight\ line\ y = x - 6$

The shortest distance will be the intersection of $y = -x + 2\ and\ x^2 + (y - 2)^2 = 4$

This will be the points $(4, -2)\ and\ (\sqrt{2}, 2 - \sqrt{2})$

Distance = $\sqrt{\left(4 - \sqrt{2}\right)^2 + (4 - \sqrt{2})^2} = \sqrt{2}\left(4 - \sqrt{2}\right) = 4\sqrt{2} - 2$

Question 10

C

Explanation:

$z^3 + az^2 - iz^2 + bz + iz + c = (z-1)(z+i)(z-2i)$

$z^3 + az^2 - iz^2 + bz + iz + c = z^3 - z^2 - iz^2 + 2z + iz - 2$

$a = -1, b = 2, c = -2$

Question 11

B

Explanation:

$f(x) = x\tan^{-1} 2x$

$f'(x) = \tan^{-1} 2x + \left(\frac{2x}{1+4x^2}\right)$

$f'\left(\frac{1}{2a}\right) = \tan^{-1}\left(\frac{2}{2a}\right) + \left(\frac{\frac{1}{a}}{1+\frac{4}{4a^2}}\right)$

$= \tan^{-1}\left(\frac{1}{a}\right) + \left(\frac{\frac{1}{a}}{1+\frac{1}{a^2}}\right)$

$= \tan^{-1}\left(\frac{1}{a}\right) + \left(\frac{a}{a^2+1}\right)$

Question 12

D

Explanation:
$Let\ u = 2x + 1$

$x = 0, u = 1\ and\ x = 1, u = 3$

$\frac{du}{dx} = 2, so\ dx = \frac{du}{2}$

$\int_1^3 \frac{u-1}{\sqrt{u}} \frac{du}{2}$

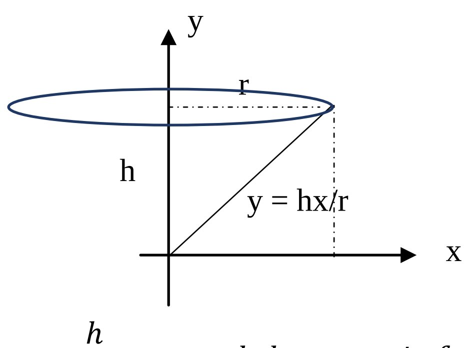
$\frac{1}{2}\int_1^3 (\sqrt{u} - \frac{1}{\sqrt{u}})du$

$\frac{1}{2}\int_1^3 (u^{½} - u^{-½})du$

Question 13

D

Explanation:



$y = \frac{h}{r}x\ \ around\ the\ y - axis\ from\ y = 0\ \ to\ \ y = h$

Question 14

E

Explanation:

Find the three points of intersection.

$solve(x^3 - 2x^2 = x^2 - 1, x)$

$x = -0.5321, x = 0.6527, x = 2.8794$

$\int_{-0.5321}^{0.6527}(x^3 - 3x^2 + 1)dx + \int_{0.6527}^{2.8794}(-x^3 + 3x^2 - 1)dx$

≈ 5.01

Question 15

D

Explanation:

$$\frac{d}{dx}(x^2 - 6x + y^2 + 8y) = 0$$

$$2x - 6 + (2y).\frac{dy}{dx} + (8).\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x + 3}{y + 4}$$

Question 16

B

Explanation:

$f(x) = x^2e^x$

$f'(x) = 2xe^x + x^2e^x$

$f''(x) = 2e^x + 4xe^x + x^2e^x$

Extreme gradients occur at points of inflection.

$f''(x) = 2e^x + 4xe^x + x^2e^x \ = 0$

$e^x(x^2 + 4x + 2) \ = 0$

$x = \ -2 \pm \sqrt{2}$

Look at the graph of $f(x) = x^2e^x$ or otherwise, it is clear that $x = \ -2 + \sqrt{2}$ is the negative gradient.

Question 17

A

Explanation:

$a(x) = f(x)(g(x))^2$

$a'(x) = f'(x)(g(x))^2 \ + \ \ 2f(x)g'(x)g(x)$

$a'(2) = f'(2)(g(2))^2 \ + \ \ 2f(2)g'(2)g(2) \ = 0$

$All\ terms\ in\ a''(2)\ have\ g(2)\ terms\ except\ 2f(2)g'(2)g'(2)$

$2f(2)g'(2)g'(2) = 2 \times 2 \times 3 \times 3 = 36$

Question 18

D

Explanation:
$Let\ \bar{x} = \mu + 1$

$$sd(\bar{x}) = \frac{1.5}{\sqrt{n}}$$

$$z = \frac{\mu + 1 - \mu}{\frac{1.5}{\sqrt{n}}} = 2.579$$

$n \approx 14.96$, so at least 15 pickets

Question 19

B

Explanation:

Friction $= 10g\sin 20°$

Up plane: $F = 10 \times 2 + 10g\sin 20° + 10g\sin 20°$

$F \approx 87\ newtons$

Question 20

A

Explanation:

$\boldsymbol{Resolving\ horizontally}$

$T\cos 30° = F\cos 20°$

$\boldsymbol{Resolving\ vertically}$

$8g = T\sin 30° + F\sin 20°$

$T \approx 96\ newtons, F \approx 89\ newtons$

SECTION B

Question 1

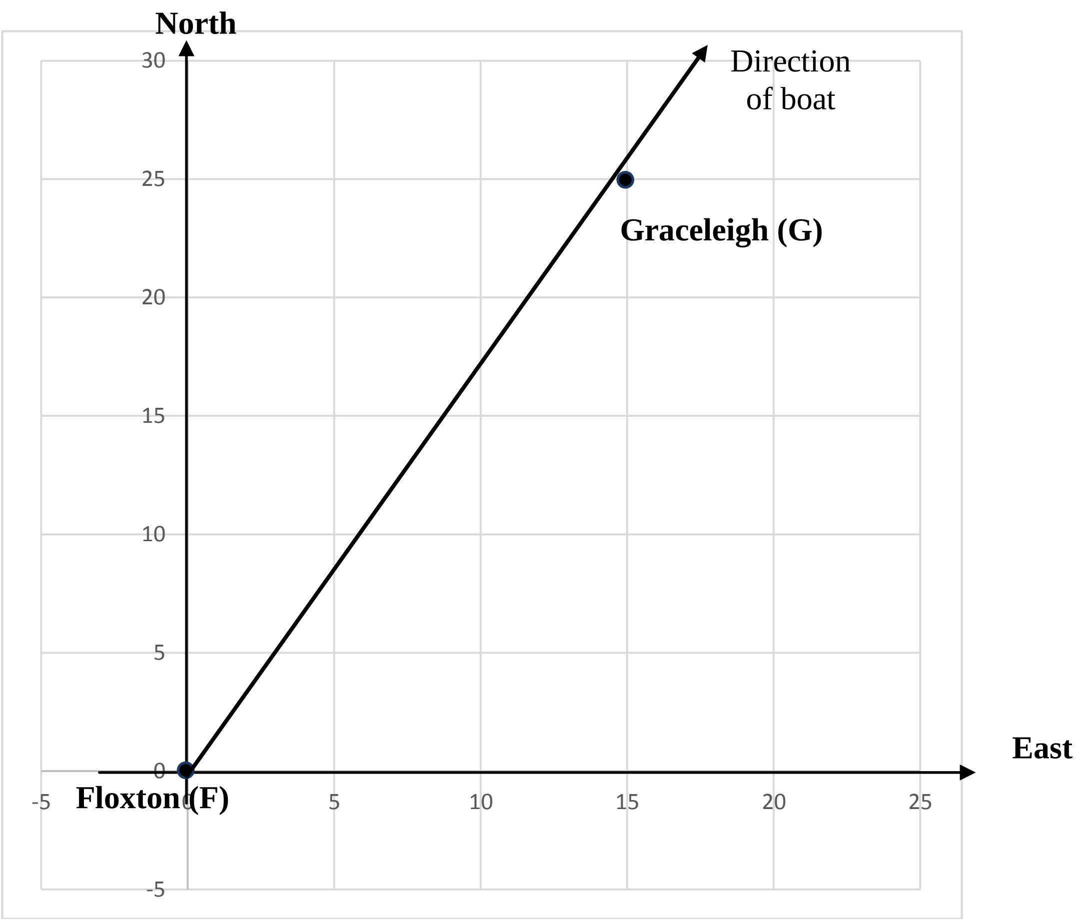
a. (1 mark)

Answer:

$\overrightarrow{OG} = 15\underset{\sim}{i} + 25\underset{\sim}{j}$

b. (1 mark)

Answer:



c. (2 marks)

Answer:

When the boat is 15 km east, (x = 15) it will be $15\tan 60°$ north.

$15\tan 60° \approx 25.98\ km$

So the boat passes about 0.98 km north of Graceleigh.

d. (2 marks)

Answer:

$\hat{p} = \frac{1}{2}\underset{\sim}{i} + \frac{\sqrt{3}}{2}\underset{\sim}{j}$

$\overrightarrow{OP} = \underset{\sim}{p} = 10t(\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j})$

k = 10

e. (1 mark)

Answer:

$10t = 15$
$t = 1.5\ hours$

f. (2 marks)

Answer:

$\overrightarrow{PG} = \overrightarrow{PO} + \overrightarrow{OG}$

$= -10t(\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}) + 15\underset{\sim}{i} + 25\underset{\sim}{j}$

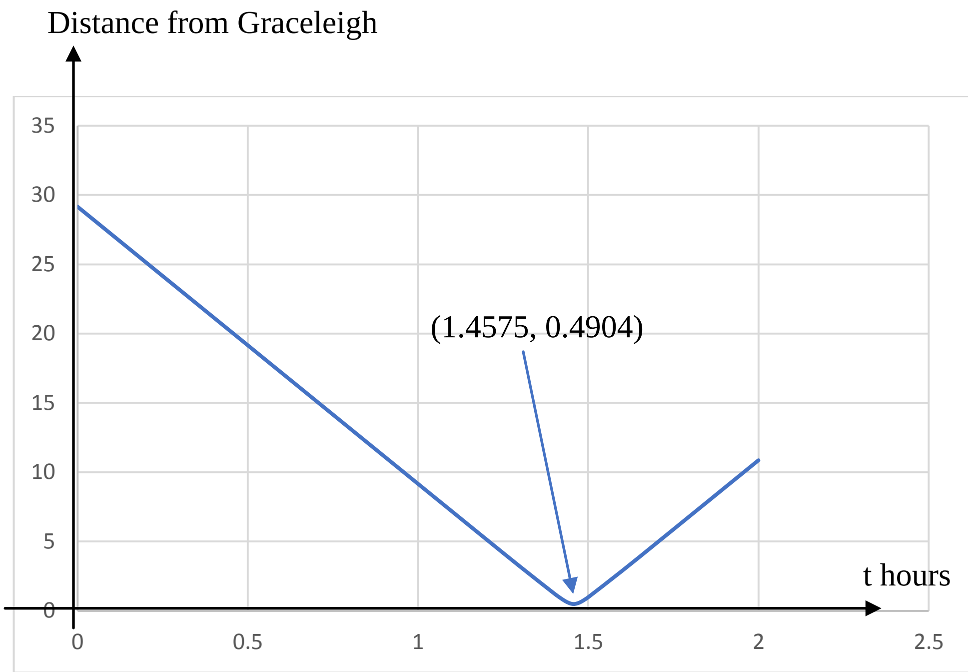
$= (15 - 10t)\underset{\sim}{i} + \left(25 - 10\sqrt{3}t\right)\underset{\sim}{j}$

g. (3 marks)

Answer:

$|\overrightarrow{PG}| = \sqrt{(15 - 10t)^2 + (25 - 10\sqrt{3}t)^2}$

Sketch $d = \sqrt{(15 - 10t)^2 + (25 - 10\sqrt{3}t)^2}$



The boat comes within 490 m of Graceleigh after 1 hour 27 minutes.

h. (1 mark)

Answer:

$\underset{\sim}{b}(0) = (15\cos 0° + 15)\,\underset{\sim}{i} + (10\sin 0°)\,\underset{\sim}{j}$

$\underset{\sim}{b}(0) = 30\,\underset{\sim}{i} + 0\,\underset{\sim}{j}$

So, the boats are 30 km apart.

i. (1 mark)

Answer:

$x = 15\cos 2t + 15, y = 10\sin 2t$

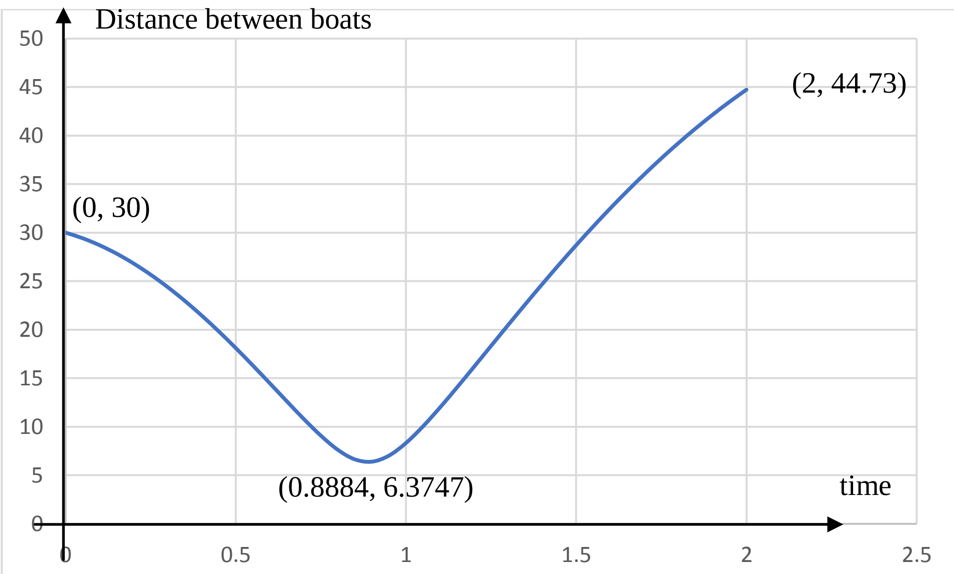
$\cos^2 2t + \sin^2 2t = 1$

$$\frac{(x-15)^2}{225} + \frac{(y)^2}{100} = 1$$

j. (3 marks)

Answer:

$$|\overrightarrow{PB}| = \sqrt{(15\cos 2t + 15 - 10t)^2 + (10\sin 2t - 10\sqrt{3}t)^2}$$



After 53 minutes, the two boats are at their closest distance of 6.375 km.

Question 2

a. (1 mark)

Answer:

$$f'(x) = \frac{-x^2 - 2x + 1}{(x^2+1)^2}$$

b. (3 marks)

Answer:

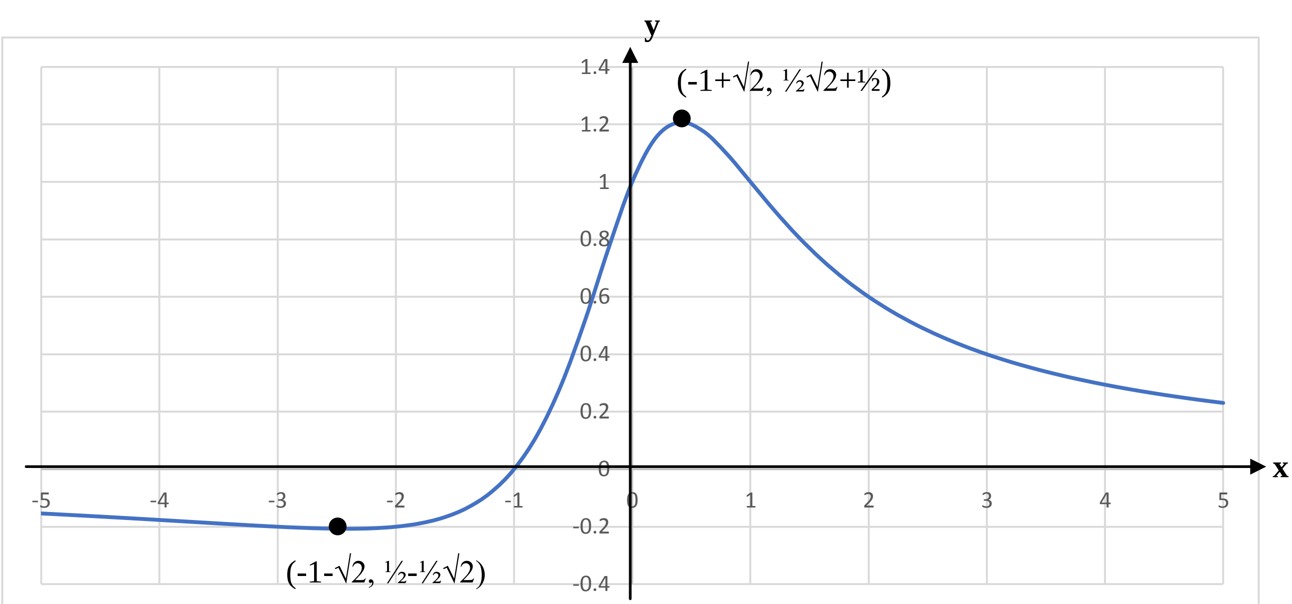
$solve(-x^2 - 2x + 1 = 0, x)$

$x = -1 \pm \sqrt{2}$

Substitute into original function.

When $x = -1 - \sqrt{2}, y = \frac{1-\sqrt{2}}{2}$

When $x = -1 + \sqrt{2}, y = \frac{1+\sqrt{2}}{2}$



c. (3 marks)

Answer:

$$f'(x) = \frac{-x^2 - 2x + 1}{(x^2+1)^2}$$

The $x - values$ of stationary points on the original graph will be $x - intercepts$ on the derivative graph.

Use CAS to find the stationary points on the derivative graph.

$$f''(x) = \frac{2(x^3 + 3x^2 - 3x - 1)}{(x^2+1)^3}$$

$solve(x^3 + 3x^2 - 3x - 1 = 0, x)$

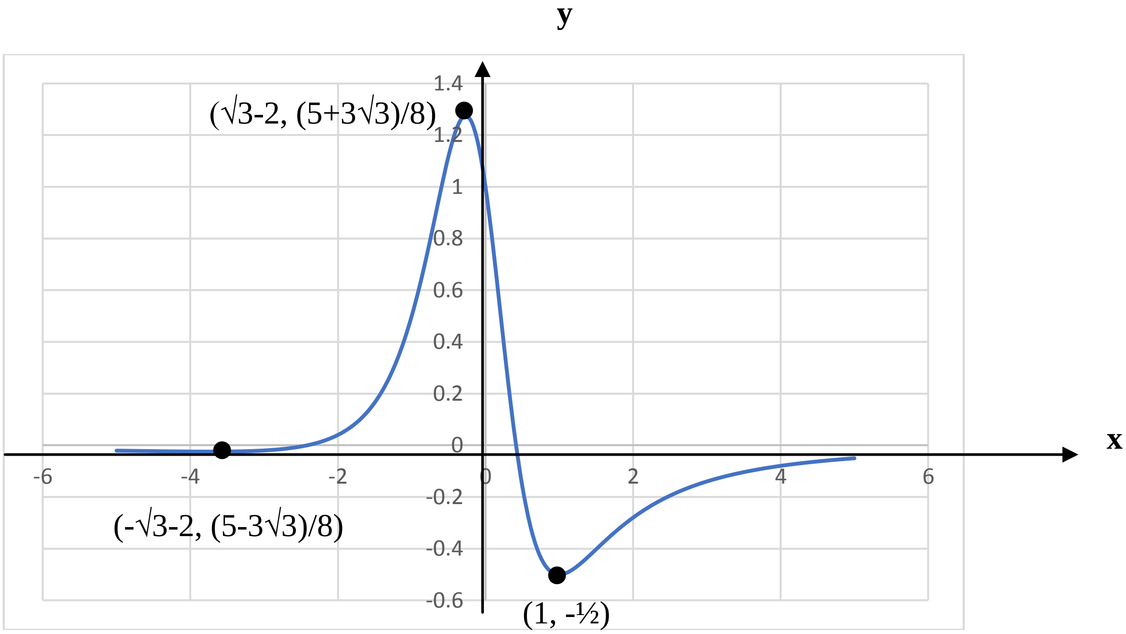
$x = 1 \; or \; x = \pm\sqrt{3} - 2$

Substitute into derivative function.

When $x = 1,\ f'(1) = -\frac{1}{2}$

When $x = \sqrt{3} - 2,\ f'\left(\sqrt{3} - 2\right) = \frac{5+3\sqrt{3}}{8}$

When $x = -\sqrt{3} - 2,\ f'\left(-\sqrt{3} - 2\right) = \frac{5-3\sqrt{3}}{8}$



d. (1 mark)

Answer:

Minimum when $x = 1,\ g(1) = -\frac{1}{2}$

Maximum when $x = \sqrt{3} - 2,\ g\left(\sqrt{3} - 2\right) = \frac{5+3\sqrt{3}}{8}$

Range: $\left[-\frac{1}{2}, \frac{5+3\sqrt{3}}{8}\right]$

e. (1 mark)

Answer:

Gradients on the original function vary from a minimum of $-\frac{1}{2}$ to a maximum of $\frac{5+3\sqrt{3}}{8}$

f. (3 marks)

Answer:

$n = 0$

Area = $\int_0^1 \left(\frac{x+1}{x^2+1} - f'(x) \right) dx$

$= \int_0^1 \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} - f'(x) \right) dx$

$= \left[\tfrac{1}{2} log_e(x^2+1) + \tan^{-1} x - f(x) \right]_0^1$

$= \tfrac{1}{2} log_e(2) + \tan^{-1} 1 - f(1) - (\tfrac{1}{2} log_e(1) + \tan^{-1} 0 - f(0))$

$= \tfrac{1}{2} log_e(2) + \frac{\pi}{4} - 1 - 0 - 0 + 1$

$= \tfrac{1}{2} log_e(2) + \frac{\pi}{4}$

g. (2 marks)

Answer:

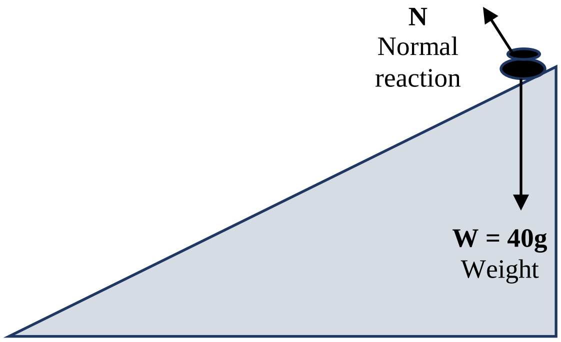
Volume = $\pi \int_{-1}^0 \left(\frac{x+1}{x^2+1} \right)^2 dx$

$= \frac{\pi(\pi-2)}{4} \, cubic \; units$

Question 3

a. (1 mark)

Answer:



b. (1 mark)

Answer:

$u = 0$, x = 5

$F = ma$

$40gsin30^{\circ} = 40a$

$a = 4.9\, ms^{-2}$
$x = ½at^2 + ut$

$5 = ½ \times 4.9 \times t^2 + 0$

$t \approx 1.43\, s$

c. (1 mark)

Answer:

$v^2 = u^2 + 2ax$

$v^2 = 0^2 + 2 \times 4.9 \times 5$

$v = 7\, ms^{-1}$

d. (2 marks)

Answer:

$v^2 = u^2 + 2ax$

$5^2 = 0^2 + 2 \times a \times 5$

$a = 2.5\ ms^{-2}$

$40gsin30^{\circ} - F_R = 40 \times 2.5$

$F_R = 20g - 100$

$F_R = 96\ netwons$

e. (2 marks)

Answer:

$P + 40gsin30^{\circ} - 96 = 40a$

$960t + 20g - 96 = 40a$

$a = 24t + 0.5g - 2.4$

$\frac{dv}{dt} = 24t + 2.5$

$v = 12t^2 + 2.5t$
$v(0.5) = 12 \times 0.5^2 + 2.5 \times 0.5$

$v = 4.25\ ms^{-1}$

f. (1 mark)

Answer:

$\frac{dx}{dt} = 12t^2 + 2.5t$

$x = 4t^3 + 1.25t^2 + 0$

$x(0.5) = 0.813\ m$

g. (2 marks)

Answer:

$v^2 = u^2 + 2ax$

$v^2 = 4.25^2 + 2 \times 2.5 \times (5 - 0.813)$

$v = 6.245\ ms^{-1}$

h. (2 marks)

Answer:

$F = ma$

$40a = -100x$

$a = -2.5x$

$\frac{d}{dx}(½v^2) = -2.5x$

$½v^2 = -1.25x^2 + c$

$v^2 = -2.5x^2 + c$

$x = 0, v = 6.245, c = 39$

$v^2 = -2.5x^2 + 39$

$v = 0$

$0^2 = -2.5x^2 + 39$

$2.5x^2 = 39$

$x \approx 3.95\ m$

Question 4

a. (1 mark)

Answer:

$(4i)^3 = 64i^3 = 64i \times i^2 = -64i$

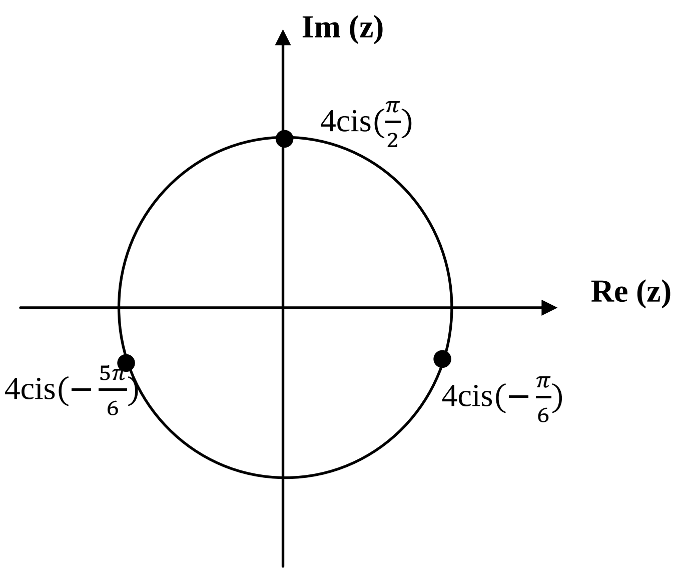
b. (1 mark)

Answer:

$z_1 = 4cis(\frac{\pi}{2})$

c. (2 marks)

Answer:



d. (2 marks)

Answer:

$z_1 = 4cis\left(\frac{\pi}{2}\right) = 4i$

$z_2 = 4cis\left(\frac{7\pi}{6}\right) = -2\sqrt{3} - 2i$

$z_3 = 4cis\left(-\frac{\pi}{6}\right) = 2\sqrt{3} - 2i$

e. (2 marks)

Answer:

$z^2 = -64i$

$z^2 = 64cis\left(-\frac{\pi}{2}\right)$

$z_1 = 8cis\left(-\frac{\pi}{4}\right) = 4\sqrt{2} - 4\sqrt{2}\,i$

$z_2 = 8cis\left(\frac{3\pi}{4}\right) = -4\sqrt{2} + 4\sqrt{2}\,i$

f. (2 marks)

Answer:

From $z^3 = -64i$

$z_a = 4cis\left(\frac{\pi}{2}\right)$

From $z^2 = -64i$

$z_b = 8cis\left(-\frac{\pi}{4}\right)$

$\frac{\boldsymbol{z_b}}{\boldsymbol{z_a}} = \frac{\boldsymbol{z}^{1/2}}{\boldsymbol{z}^{1/3}} = \boldsymbol{z}^{1/6} = \frac{8cis\left(-\frac{\pi}{4}\right)}{4cis\left(\frac{\pi}{2}\right)} = 2cis\left(-\frac{3\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$

g. (3 marks)

Answer:

$\int_0^{20} (a\,t(t-10))\,dt = 1$

$a = -\frac{3}{4000}$

$$E(T) = -\frac{3}{4000}\int_0^{20} (\, t^2(t-10)\,)\, dt = 10$$

$$E(T^2) = -\frac{3}{4000}\int_0^{20} (\, t^3(t-10)\,)\, dx = 120$$

$$Var(T) = E(T^2) - (E(T))^2$$

$$Var(T) = 120 - 10^2 = 20$$

$$\sigma(T) = 2\sqrt{5}$$

h. (2 marks)

Answer:

$$2.326 = \frac{\bar{t} - 10}{\frac{\sqrt{20}}{\sqrt{23}}}$$

$$\bar{t} \approx 12.17\ minutes$$

i. (2 marks)

Answer:

Let y be the average time of the entire class of 25 students.

$$2.326 = \frac{\bar{y} - 10}{\frac{\sqrt{20}}{\sqrt{25}}}$$

$$\bar{y} \approx 12.080$$

Let a be the average time of the two students who missed the original test.

$$12.080 \times 25 = 23 \times 12.169 + 2a$$

$$a \approx 11.1\ minutes$$