Fortify Sample Exam A

SPECIALIST MATHEMATICS

Written examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 75

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages.
- Formula sheet.

Instructions
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

The corresponding cartesian equation for parametric equations $x = 2\tan(2t)$ and $y = 3\sec(2t)$ is

A. $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

B. $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$

C. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

D. $(y-3)^2 - (x-2)^2 = 1$

E. $\dfrac{y^2}{3} - \dfrac{x^2}{2} = 1$

Question 2

$\dfrac{2}{x^2-1}$ can be expressed as

A. $\dfrac{1}{x+1} - \dfrac{1}{x-1}$

B. $\dfrac{1}{x-1} + \dfrac{1}{x+1}$

C. $\dfrac{1}{x-1} - \dfrac{1}{x+1}$

D. $\dfrac{2}{x-1} - \dfrac{1}{x+1}$

E. $\dfrac{1}{x-1} - \dfrac{2}{x+1}$

Question 3

Using a suitable substitution, $\int_3^4 x(x-3)^{17}dx$ can be expressed as

A. $\int_3^4 (u+3)u^{17}du$

B. $\int_0^1 (u+3)u^{17}du$

C. $\int_0^1 u^{18}du$

D. $\int_3^4 (u-3)u^{17}du$

E. $\int_0^1 3u^{17}du$

Question 4

A basketball is shot in the air from the ground at an angle of 45° to the horizontal with a velocity of 16 ms$^-1$. Ignoring air resistance, what is the maximum height, in metres, the ball reaches.

A. $\frac{64}{g}$

B. $8\sqrt{2}g$

C. $\frac{8}{g}$

D. $\frac{16}{g}$

E. $\frac{8\sqrt{2}}{g}$

Question 5

Given that $\frac{dy}{dx} = \mathrm{e}^x y^2$, the value of $\frac{d^2y}{dx^2}$ at the point $(0,\ 2)$ is

A. e

B. $2\mathrm{e}^2 + \mathrm{e}$

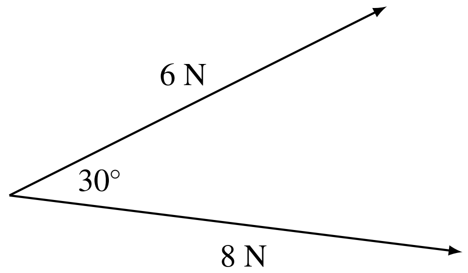
C. 0

D. 20

E. 4

Question 6

Forces of 8 N and 6 N act on a body as shown below:



The magnitude of the resultant force correct to one decimal place is

A. 13.5 N

B. 4.1 N

C. 10 N

D. 12.2 N

E. 7.2 N

Question 7

Given that $x = \cos(t) + \tan(t)$ and $y = tan(t)$, then $\frac{dy}{dx}$ in terms of t is

A. $-\sin(t)$

B. $\frac{1}{1 - \sin(t)\cos^2(t)}$

C. $1 - \sin(t)\cos^2(t)$

D. $\frac{1 - \sin(t)\cos^2(t)}{\cos^4(t)}$

E. $\sec^2(t)$

Question 8

Let $\underset{\sim}{a} = 2\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + \alpha\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$, where $\alpha \in R$.

If the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{2\sqrt{14}}{7}$, then α equals

A. $3\sqrt{3}$

B. 3

C. $\sqrt{3}$

D. 2

E. -1

Question 9

Consider the function $f(x) = \dfrac{1}{\sqrt{\cos^{-1}(x-3)}}$. The maximal domain of f is

A. $x \in R$

B. $x \in [2,\ 4]$

C. $x \in R \setminus \{4\}$

D. $x \in [2,\ 4)$

E. $x \in R^+$

Question 10

If $\text{Arg}(-2 + ai) = \dfrac{3\pi}{4}$, then a is

A. $\sqrt{3}$

B. $\sqrt{2}$

C. 2

D. -1

E. -2

Question 11

When simplified, $(\sin(\theta) + i\cos(\theta))(\sin(\phi) + i\cos(\phi))$ is equal to

A. $\text{cis}\left(\dfrac{\pi}{2} - \theta - \phi\right)$

B. $\text{cis}\,(\theta + \phi)$

C. $\text{cis}\,(\pi - \theta - \phi)$

D. $\text{cis}\,(\pi - \theta + \phi)$

E. $\text{cis}\left(\dfrac{\pi}{2} + \theta + \phi\right)$

Question 12

A body of mass 2 kg is subject to a force of $5\underset{\sim}{i}$ and $12\underset{\sim}{j}$ N.

If no other forces act on the particle, what is the magnitude of the particle's acceleration, in ms^{-2}?

A. 13

B. $5\underset{\sim}{i} + 12\underset{\sim}{j}$

C. 6.5

D. $2.5\underset{\sim}{i} + 6\underset{\sim}{j}$

E. 34

Question 13

A curve is given by its parametric equations $x = t^3 + 3t^2$ and $y = t^3 - 3t^2$ for $0 \le t \le 3$. The length of the curve is

A. $13\sqrt{26} - 8\sqrt{2}$

B. $13 - 8\sqrt{2}$

C. $\sqrt{13} - 8$

D. $13\sqrt{26} + 8$

E. 54

Question 14

On an Argand diagram, a point that does not lie on the path defined by $|z + 2i| = 2|z - i|$ is

A. $(2,\ 2)$

B. $(0,\ 2)$

C. $(-2,\ 2)$

D. $(0,\ 4)$

E. $(0,\ 0)$

Question 15

Consider $\dfrac{dy}{dx} = \mathrm{e}^x y$, where $y(0) = y_0 = 1$. Using Euler's method with step size of 0.1, $y(0.2) = y_2$ is approximately

A. 1.37

B. 1

C. 0.24

D. 1.1

E. 1.22

Question 16

A function f, its derivative f' and its second derivative f'' are defined for $x \in R$ with the following properties.

$f(a) = 1$, $f(b) = -1$

$f'(a) = 0$, $f'(b) = 0$

$f''(x) = c - x$, where $a > c > b$

The values at which the function are concaving up are

A. $x \in (c,\ \infty)$

B. $x \in (-\infty,\ c)$

C. $x \in (a,\ b)$

D. $x \in (b,\ a)$

E. $x \in R$

Question 17

A constant force of F newtons accelerates a particle of mass 5 kg in a straight line from a speed of
5 ms^{-1} to 13 ms^{-1} over a distance of 16 m. The magnitude of F is

A. 2.5

B. 4.5

C. 22.5

D. 36.4

E. 40

Question 18

U and V are independent normally distributed random variables. U has a mean of 10 and a standard deviation of 1. V has a mean of 6 and a standard deviation of 2. The random variable W is defined by $W = 2U - 3V$. In terms of the standard normal variable Z, $\Pr(W > 4)$ is equal to

A. $\Pr\left(Z > \dfrac{\sqrt{22}}{11}\right)$

B. $\Pr\left(Z < \dfrac{-1}{\sqrt{10}}\right)$

C. $\Pr\left(Z > \dfrac{-1}{\sqrt{10}}\right)$

D. $\Pr\left(Z < \dfrac{1}{\sqrt{10}}\right)$

E. $\Pr\left(Z > \dfrac{-\sqrt{22}}{11}\right)$

Question 19

The mean time (in hours per week) spend exercising is found to be 2.5 with a standard deviation of 0.5 for a sample of 49 random year 12 students. Assuming that the standard deviation obtained from the sample sufficiently and accurately estimates the population standard deviation, an approximate 95% confidence interval for the mean time spent exercising by all year 12 students is

A. $(2.48,\ 2.52)$

B. $(2.43,\ 2.57)$

C. $(2,\ 3)$

D. $(2.36,\ 2.64)$

E. $(1.52,\ 3.48)$

Question 20

The scores achieved on a certain maths test is normally distributed with a mean of 85 and a variance of 25. The probability that the average test score of 4 students is greater than 83 is closest to

A. 0.2119

B. 0.7881

C. 0.2858

D. 0.7142

E. 0.6534

END OF SECTION A

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SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question..
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (11 marks)

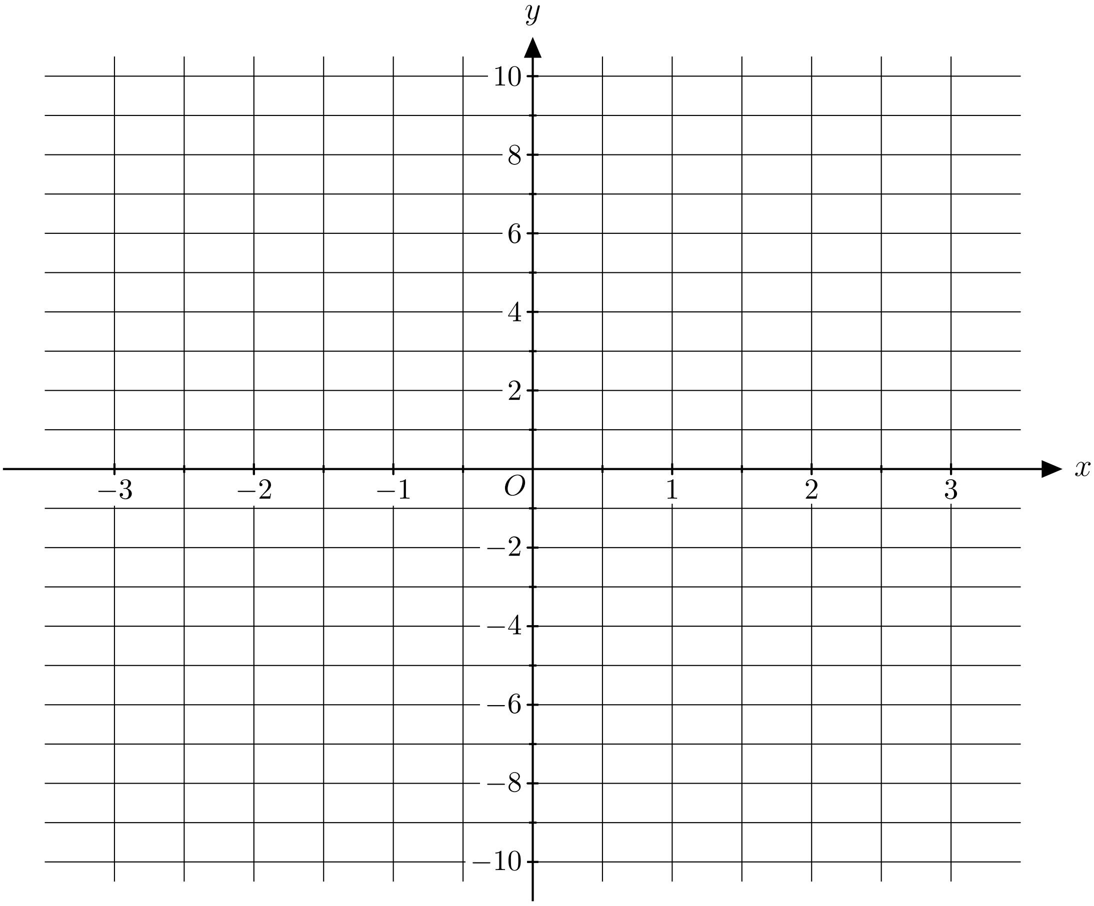
Let $f(x) = \dfrac{x^2 + x + 7}{\sqrt{2x+1}}$.

a. Find the maximal domain of f. 1 mark

b. Find $f'(x)$, and hence, find all the coordinates of the stationary point(s) of f and state the nature of the stationary point(s). 3 marks

c. Find the equations of the asymptote(s). 1 mark

d. Sketch the graph of $f(x)$, labelling all asymptotes, intercepts and stationary points. 3 marks



A vase is to be made by rotating the curve of the part of the graph where $x \in [0,\ 3]$ around the x-axis to form a solid of revolution.

e. Write a definite integral, in terms of x, which gives the length of the curve to be rotated. 1 mark

f. Find the volume of this vase, correct to 2 decimal places. 2 marks

Question 2 (12 marks)

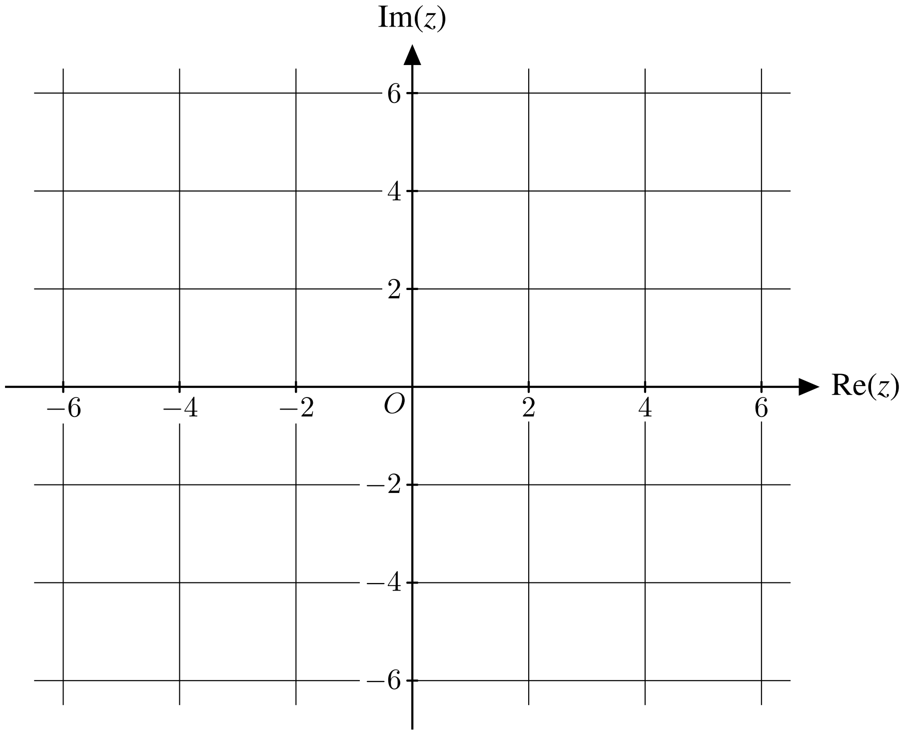
A line in the complex plane is given by $|z-2| = |z+2-2\sqrt{2}i|$, $x \in C$.

a. Find the cartesian equation of this line in the form $y = mx + c$, where $m, c \in R$. 2 marks

b. Find the points of intersection, correct to two decimal places, of the line $|z-2| = |z+2-2\sqrt{2}i|$ with the circle $|z-2| = 4$. 2 marks

c. Sketch the two graphs on the axes below. 2 marks

Show the coordinates of the points of intersection.



d. Show that the roots to the equation $z^2 - 2z + 3 = 0$ are $z = 1 + \sqrt{2}i$ and $z = 1 - \sqrt{2}i$. 2 marks

e. Find the equation of the line which joins the two roots of $z^2 - 2z + 3 = 0$. 1 mark

f. Find the area of the triangle enclosed by the x-axis, the equation of the line in **part e.** and the line $|z-2| = |z+2-2\sqrt{2}i|$. 2 marks

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Question 3 (9 marks)

A cricket ball is thrown directly upwards from the ground at a speed of 49 ms^{-1}. Assume that air resistance is negligible and that the ball is only subject to gravitational acceleration.

a. Find the time, in seconds, it takes for the ball to reach the top of its path. 2 marks

b. Find the maximum distance, in metres, that the ball reaches from the ground. 2 marks

A second cricket ball is thrown from the same spot at the same time in a horizontal direction. The velocity of this particle is given by $v = \frac{-1}{4}t^2 + 16$, $0 \leq v \leq 8$.

c. Find the acceleration of the particle at $t = 2$. 1 mark

d. Find the distance travelled, in metres, of the second cricket ball after 5 seconds, correct to two decimal places. 2 marks

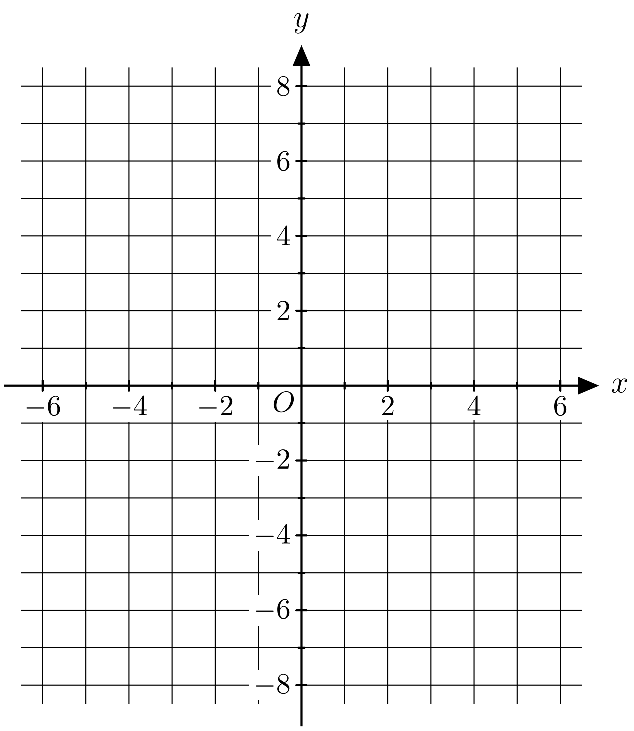
e. After 5 seconds, calculate the distance between the two balls, correct to two decimal places. 2 marks

Question 4 (12 marks)

Two athletes, Adam and Beth, are running at a park. Their paths are represented by the equations $r_a = (3 - 2\cos(t))\underset{\sim}{i} + (4 + 3\sin(t))\underset{\sim}{j}$ and $r_b = (t^2 - 1)\underset{\sim}{i} + (t^2 + 1)\underset{\sim}{j}$, for $t \geq 0$. The distances are measured in kilometres and time is measured in minutes.

a. Find the Cartesian equations for Adam and Beth's running paths. 3 marks

b. Sketch the graphs of their paths on the axes below, labelling the initial positions and their direction of movement. 3 marks



c. Find the coordinates of the points at which their paths cross, correct to two decimal places. 2 marks

A third runner, Charlie, also begins running at the same time, with his path defined by the equation $r_c = (t+1)\underset{\sim}{i} + (t^2 - 4)\underset{\sim}{j}$ for $t \geq 0$.

d. Find the time at which Charlie and Beth are running at the same speed. 2 marks

e. **i.** Write down an expression for the distance between Beth and Charlie at any time t. 1 mark

ii. Find the minimum distance between Beth and Charlie, in kilometres, correct to two decimal places. 1 mark

Question 5 (8 marks)

A cylindrical tank containing a liquid has a vent at the top and an outlet down the bottom where the liquid drains out. The height of the liquid in the tank decreases proportional to the square root of the height. The differential equation used to model the height of the liquid at t minutes is $\dfrac{dh}{dt} = \dfrac{-\sqrt{h}}{A}$, where A m^2 is the surface area of the liquid. The tank has a height of 8 m and a radius of 2 m.

a. Assuming that the tank begins full, solve the differential equation to find h in terms of t. 3 marks

b. Find the time at which all the water has drained out. 1 mark

A second tank is conical in shape and has the same dimensions as the first tank (8 m height and 2 m radius). Water drains out of the tank at $4h^3$ m^3 per minute.

c. Show that the differential equation is $\dfrac{dh}{dt} = \dfrac{-64h}{\pi}$. 2 marks

d. Given that the tank is initially full, find the height of the water after 0.1 minutes, correct to two decimal places. 2 marks

Question 6 (8 marks)

The time taken for a customer to be served at a fast-food chain has a mean of 5 minutes and a standard deviation of 1.25 minutes. Customers have been complaining of slow service, and so the service time was recorded for a random sample of 100 customers. The average service time was found to be 5.200 minutes.

a. Write down the mean and standard deviation of $\bar{X}$. 2 marks

b. State suitable hypotheses H_0 and H_1 for the statistical test. 1 mark

c. Write down an expression for the p value and evaluate it correct to four decimal places. 2 marks

d. State with a reason whether H_0 should be rejected at a 5% significance level. 1 mark

e. Which type of error may have been committed based on the decision made in **part d.**? 1 mark

f. For this test, what is the smallest possible value of the sample mean which would provide significant evidence that the mean service time has increased at the 5% significance level? Give your answer correct to three decimal places. 1 mark

END OF QUESTION AND ANSWER BOOK

SPECIALIST MATHEMATICS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Formula Sheet

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc\sin(A)$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	
$\sin(2x) = 2\sin(x)\cos(x)$	$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

Circular functions – continued

Function	$\sin^{-1}$ or arcsin	$\cos^{-1}$ or arccos	$\tan^{-1}$ or arctan
Domain	$[-1,\ 1]$	$[-1,\ 1]$	R
Range	$\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$	$[0,\ \pi]$	$\left(-\frac{\pi}{2},\ \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r\left(\cos(\theta) + i\sin(\theta)\right) = r\text{cis}(\theta)$	
$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \text{Arg}(z) \leq \pi$
$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$
$z^n = r^n \text{cis}(n\theta)$ (de Moivre's theorem)	

Probability and statistics

for random variables X and Y	$\text{E}(aX + b) = a\text{E}(X) + b$ $\text{E}(aX + bY) = a\text{E}(X) + b\text{E}(Y)$ $\text{var}(aX + b) = a^2\text{var}(X)$
for independent random variables X and Y	$\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y)$
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\bar{X}$	mean $\text{E}(\bar{X}) = \mu$ variance $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, \; n \neq -1$
$\frac{d}{dx}(\mathrm{e}^{ax}) = a\mathrm{e}^{ax}$	$\int \mathrm{e}^{ax} \, dx = \frac{1}{a}\mathrm{e}^{ax} + c$
$\frac{d}{dx}(\log_\mathrm{e}(x)) = \frac{1}{x}$	$\int \frac{1}{x} \, dx = \log_\mathrm{e}\|x\| + c$
$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$	$\int \sin(ax) \, dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$	$\int \cos(ax) \, dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = a\sec^2(ax)$	$\int \sec^2(ax) \, dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c, \; a > 0$
$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}} \, dx = \cos^{-1}\left(\frac{x}{a}\right) + c, \; a > 0$
$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2} \, dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, \; n \neq -1$
	$\int (ax+b)^{-1} \, dx = \frac{1}{a}\log_\mathrm{e}\|ax+b\| + c$
produce rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
Euler's method	If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$
acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2} \sqrt{1 + (f'(x))^2} \, dx$ or $\int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + \underset{\sim}{j} + z\underset{\sim}{k}$
$\|\underset{\sim}{r}\| = \sqrt{x^2 + y^2 + z^2} = r$
$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2 \cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$

Mechanics

momentum	$\underset{\sim}{p} = m\underset{\sim}{v}$
equation of motion	$\underset{\sim}{R} = m\underset{\sim}{a}$

END OF FORMULA SHEET