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SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2019

Question 1 (3 marks)

$$3y^2 + 2xy = 7$$

$$6y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$$ **(1 mark)**

At $(2,1)$, we have $6\frac{dy}{dx} + 4\frac{dy}{dx} + 2 = 0$

$$10\frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = -\frac{1}{5}$$

(1 mark)

Equation of tangent is

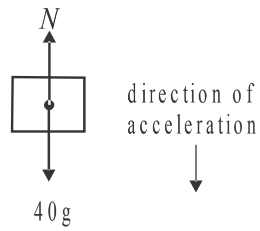
$$y - 1 = -\frac{1}{5}(x - 2)$$

$$y = -\frac{1}{5}x + \frac{7}{5}$$

(1 mark)

Question 2 (4 marks)

a. Draw a force diagram.
Let N be the reaction force of the lift floor on the trolley.

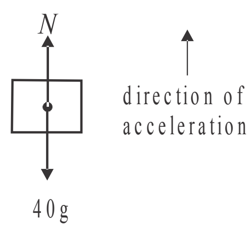


$40g - N = 40a$ (equation of motion) **(1 mark)**

$$N = 40g - 40 \times 1.8$$
$$= 320 \text{ newtons}$$

(1 mark)

b. Draw a force diagram.



$N - 40g = 40a$ (equation of motion) **(1 mark)**

$$a = \frac{448 - 40g}{40} \quad \text{since } N = 448$$

$$a = 1.4\,\text{ms}^{-2}$$

(1 mark)

Question 3 (4 marks)

a.

$$z = 4\text{cis}\left(\frac{\pi}{3}\right)$$
$$= 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$
$$= 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$
$$= 2 + 2\sqrt{3}\,i$$

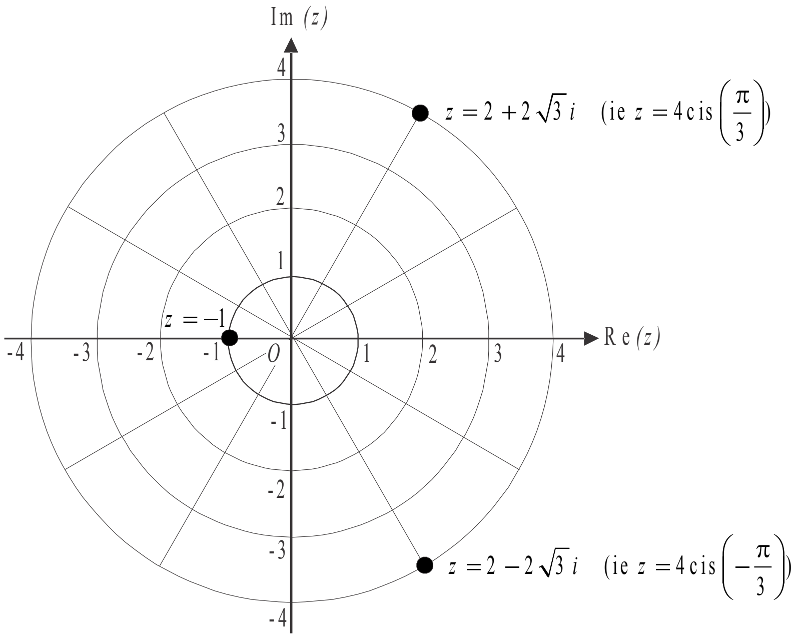
(1 mark)

Another root is $2 - 2\sqrt{3}\,i$ (conjugate root theorem). **(1 mark)**

Because the third root must be real we can use the factor theorem.
If $P(z) = z^3 - 3z^2 + 12z + 16$,
$P(1) = 1 - 3 + 12 + 16 = 26$
$P(-1) = -1 - 3 - 12 + 16 = 0$
so the third root is -1.

(1 mark)

b.



(1 mark)

Question 4 (3 marks)

Let X represent the mass, in grams, of mussels farmed in the bay.

$X \sim \text{N}\,(\text{unknown},\ 3^2)$

For the sample,

$$\text{E}(\bar{X}) = \frac{2400}{100} \quad \text{and sd}\,(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

$$= 24 \qquad\qquad = \frac{3}{10}$$

(1 mark) $\quad = 0.3$ **(1 mark)**

$$95\%\ \text{confidence interval} = \left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right) \qquad \text{(formula sheet)}$$

$$= (24 - 2\times 0.3,\ \ 24 + 2\times 0.3)$$

$$= (23.4,\ \ 24.6)$$

(1 mark)

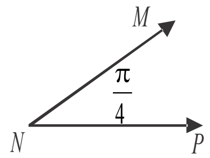
Note that we use the integer multiple of 2 for z, remembering that $\Pr(-1.96 < Z < 1.96) = 0.95$

Question 5 (4 marks)

$\underset{\sim}{m} = 2\underset{\sim}{i} + a\underset{\sim}{j} \qquad (= \overrightarrow{OM})$

$\underset{\sim}{n} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} \qquad (= \overrightarrow{ON})$

$\underset{\sim}{p} = \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k} \qquad (= \overrightarrow{OP})$



$$\overrightarrow{NM} = \overrightarrow{NO} + \overrightarrow{OM} \quad \text{where } O \text{ is the fixed origin}$$

$$= -\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} + 2\underset{\sim}{i} + a\underset{\sim}{j}$$

$$= \underset{\sim}{i} + (a-1)\underset{\sim}{j} + \underset{\sim}{k}$$

$$\overrightarrow{NP} = \overrightarrow{NO} + \overrightarrow{OP}$$

$$= -\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$$

$$= -2\underset{\sim}{j} - \underset{\sim}{k}$$

$$\overrightarrow{NM}\ \overrightarrow{NP} = -2(a-1) - 1$$

$$= 1 - 2a$$

(1 mark)

$$\text{Also, } \overrightarrow{NM}\ \overrightarrow{NP} = |\overrightarrow{NM}|\,|\overrightarrow{NP}|\cos\left(\frac{\pi}{4}\right)$$

$$= \sqrt{1^2 + (a-1)^2 + 1^2}\ \sqrt{(-2)^2 + (-1)^2} \times \frac{1}{\sqrt{2}}$$

$$= \sqrt{a^2 - 2a + 3}\ \sqrt{\frac{5}{2}}$$

$$= \sqrt{\frac{5}{2}(a^2 - 2a + 3)}$$

$$\text{So } 1 - 2a = \sqrt{\frac{5}{2}(a^2 - 2a + 3)} \qquad \underline{\qquad\qquad\qquad}(*)$$

(1 mark)

Squaring both sides,

$$(1-2a)^2 = \frac{5}{2}(a^2-2a+3)$$

$$2(1-4a+4a^2) = 5(a^2-2a+3)$$

$$2-8a+8a^2 = 5a^2-10a+15$$

$$3a^2+2a-13=0$$

$$a = \frac{-2 \pm \sqrt{4-4\times 3\times -13}}{6}$$

$$= \frac{-2 \pm \sqrt{160}}{6}$$

$$= \frac{-2 \pm 4\sqrt{10}}{6}$$

$$a = \frac{-1 \pm 2\sqrt{10}}{3}$$

(1 mark)

Note that from (*), $1-2a \geq 0$ because $\sqrt{\frac{5}{2}(a^2-2a+3)} \geq 0$ and $a \in R$

so $2a \leq 1$

$$a \leq \frac{1}{2}$$

Reject $a = \frac{-1+2\sqrt{10}}{3}$ since $a \leq \frac{1}{2}$.

So $a = \frac{-1-2\sqrt{10}}{3}$

(1 mark)

Question 6 (4 marks)

$$\int_0^{\sqrt{3}} \frac{3+x}{x^2+3}\,dx$$

$$= \int_0^{\sqrt{3}} \frac{3}{x^2+3}\,dx + \int_0^{\sqrt{3}} \frac{x}{x^2+3}\,dx$$

$$= \sqrt{3}\int_0^{\sqrt{3}} \frac{\sqrt{3}}{x^2+3}\,dx + \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{x^2+3}\,dx$$

$$= \sqrt{3}\left[\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^{\sqrt{3}} + \frac{1}{2}\left[\log_e(x^2+3)\right]_0^{\sqrt{3}}$$

(1 mark)

(1 mark) **(1 mark)**

$$= \sqrt{3}\left(\tan^{-1}(1) - \tan^{-1}(0)\right) + \frac{1}{2}\left(\log_e(6) - \log_e(3)\right)$$

$$= \sqrt{3}\left(\frac{\pi}{4} - 0\right) + \frac{1}{2}\log_e(2)$$

$$= \frac{\sqrt{3}\pi}{4} + \frac{1}{2}\log_e(2)$$

(1 mark)

Question 7 (4 marks)

$$y=\frac{1-x}{x^2-2x}$$
$$=\frac{1-x}{x(x-2)}$$

Asymptotes are $x=0,\ x=2,\ y=0$

<u>x-intercepts</u> occur when $y=0$

$$0=\frac{1-x}{x(x-2)}$$

i.e. $1-x=0$

x-intercept is $(1,0)$

<u>y-intercepts</u> occur when $x=0$

$$y=\frac{1-0}{0}$$

There are no y-intercepts.

$$y=\frac{1-x}{x^2-2x}$$
$$\frac{dy}{dx}=\frac{(x^2-2x)\times -1-(2x-2)(1-x)}{(x^2-2x)^2}$$
$$=\frac{-x^2+2x+2x^2-4x+2}{(x^2-2x)^2}$$
$$=\frac{x^2-2x+2}{(x^2-2x)^2}$$

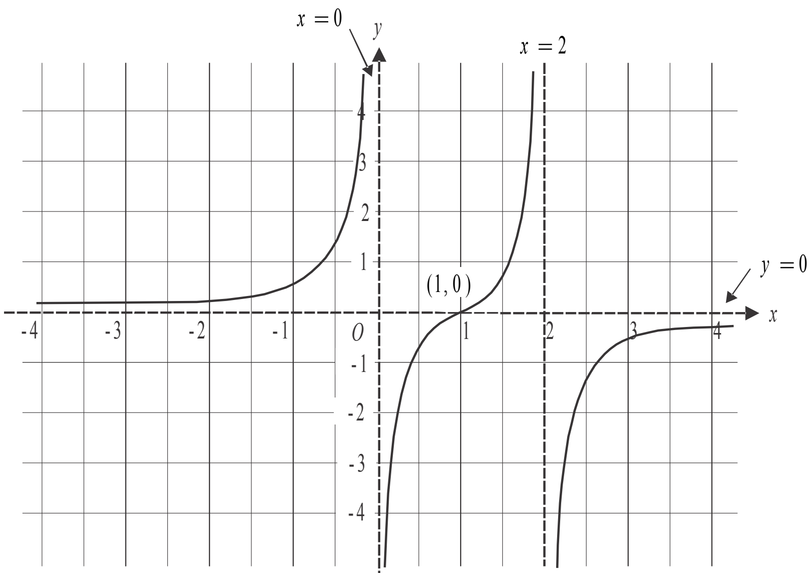
$\frac{dy}{dx}\neq 0$ since for $x^2-2x+2=0,$

$\Delta<0$

i.e. $(-2)^2-4\times 1\times 2<0$

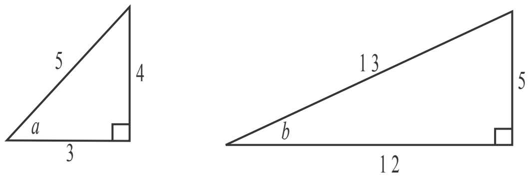
So there are no stationary points.

As $x\to-\infty,\ y\to 0^+$
As $x\to\infty,\ y\to 0^-$
As $x\to 0$ from the left, $y\to\infty$
As $x\to 0$ from the right, $y\to-\infty$
As $x\to 2$ from the left, $y\to\infty$
As $x\to 2$ from the right, $y\to-\infty$



(1 mark) correct asymptotes
(1 mark) correct x-intercept
(1 mark) correct shape of middle branch with no stationary point of inflection
(1 mark) correct shape of outer branches

Question 8 (3 marks)



Let $\arcsin\left(\frac{4}{5}\right)=a$

$$\sin(a)=\frac{4}{5}$$

so $\cos(a)=\frac{3}{5}$

Let $\arctan\left(\frac{5}{12}\right)=b$

$$\tan(b)=\frac{5}{12}$$

so $\sin(b)=\frac{5}{13}$

and $\cos(b)=\frac{12}{13}$

(1 mark)

Since $\quad x=\arcsin\left(\frac{4}{5}\right)-\arctan\left(\frac{5}{12}\right)$

then, $\quad x=a-b \qquad -(1)$

Also, $\sec(x)=\frac{1}{\cos(x)}$

From (1), $\cos(x)=\cos(a-b)$

$$\begin{aligned}&=\cos(a)\cos(b)+\sin(a)\sin(b) \quad \text{(formula sheet)}\\&=\frac{3}{5}\times\frac{12}{13}+\frac{4}{5}\times\frac{5}{13}\\&=\frac{36}{65}+\frac{20}{65}\\&=\frac{56}{65}\end{aligned}$$

(1 mark)

So $\sec(x)=\frac{65}{56}$

(1 mark)

Question 9 (4 marks)

$$(1+x^2)\frac{dy}{dx}-\frac{1}{x}=0$$

$$(1+x^2)\frac{dy}{dx}=\frac{1}{x}$$

$$\frac{dy}{dx}=\frac{1}{x(1+x^2)}$$

Let $\frac{1}{x(1+x^2)}=\frac{A}{x}+\frac{Bx+C}{1+x^2}$ **(1 mark)**

$$=\frac{A(1+x^2)+x(Bx+C)}{x(1+x^2)}$$

True iff $\quad 1=A(1+x^2)+x(Bx+C)$

Put $x=0,\quad 1=A$

Put $x=1,\quad 1=2+B+C$

So $\quad -1=B+C \qquad -(1)$

Put $x=2,\quad 1=5+2(2B+C)$

$-2=2B+C \qquad -(2)$

$(1)-(2)\quad 1=-B$

$B=-1$

In (1) $\quad C=0$

So $\frac{1}{x(1+x^2)}=\frac{1}{x}-\frac{x}{1+x^2}$ **(1 mark)**

If you have time, check i.e. $\frac{1}{x}-\frac{x}{1+x^2}=\frac{1+x^2-x^2}{x(1+x^2)}$

$=\frac{1}{x(1+x^2)}$ (all good)

$$\frac{dy}{dx}=\frac{1}{x}-\frac{x}{1+x^2}$$

$$y=\int\frac{1}{x}\,dx-\int\frac{x}{1+x^2}\,dx$$

$=\log_e(x)-\frac{1}{2}\log_e(1+x^2)+c,$ **(1 mark)**

$$=\log_e(x)-\log_e(1+x^2)^{\frac{1}{2}}+c,\quad x>0$$

Given $x=1,\ y=2,$

$$2=\log_e(1)-\log_e\left(\sqrt{2}\right)+c$$

so $\quad c=2+\log_e\left(\sqrt{2}\right)$

and so $y=\log_e(x)-\log_e(1+x^2)^{\frac{1}{2}}+2+\log_e\left(\sqrt{2}\right)$

$$y=\log_e\left(\frac{\sqrt{2}x}{\sqrt{1+x^2}}\right)+2$$

(1 mark)

Question 10 (7 marks)

a.

$$f(x) = \arcsin\left(\frac{x+1}{2}\right)$$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x+1}{2}\right)^2}} \times \frac{1}{2}$$ (Chain rule) **(1 mark)**

$$= \frac{1}{2\sqrt{1-\frac{(x^2+2x+1)}{4}}}$$

$$= \frac{1}{2\sqrt{\frac{4-(x^2+2x+1)}{4}}}$$

$$= \frac{1}{\sqrt{-x^2-2x+3}}$$

$$= \frac{1}{\sqrt{-(x+3)(x-1)}}$$

(1 mark)

b.

Let $y = \arcsin\left(\frac{x+1}{2}\right)$

Swap x and y for inverse.

$$x = \arcsin\left(\frac{y+1}{2}\right)$$

$$\sin(x) = \frac{y+1}{2}$$

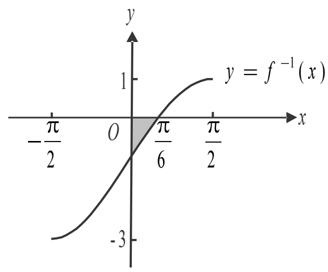
$$y = 2\sin(x) - 1$$

So $f^{-1}(x) = 2\sin(x) - 1$ as required.

(1 mark)

c.

Do a quick sketch of $y = f^{-1}(x)$.



<u>x-intercept</u> occurs when $y = 0$

$$0 = 2\sin(x) - 1$$

$$\sin(x) = \frac{1}{2}$$

$$x = \frac{\pi}{6}$$ **(1 mark)**

S is the region shaded.

$$\text{volume} = \pi\int_0^{\frac{\pi}{6}} y^2\, dx$$

(1 mark)

$$= \pi\int_0^{\frac{\pi}{6}} (2\sin(x) - 1)^2\, dx$$

$$= \pi\int_0^{\frac{\pi}{6}} (4\sin^2(x) - 4\sin(x) + 1)\, dx$$

$$= \pi\int_0^{\frac{\pi}{6}} (2 - 2\cos(2x) - 4\sin(x) + 1)\, dx \quad \text{(i.e. use } \cos(2x) = 1 - 2\sin^2(x)\text{)}$$

(1 mark)

$$= \pi\left[3x - \sin(2x) + 4\cos(x)\right]_0^{\frac{\pi}{6}}$$

$$= \pi\left\{\left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + 4\times\frac{\sqrt{3}}{2}\right) - (0 - 0 + 4\times 1)\right\}$$

$$= \pi\left(\frac{\pi}{2} + \frac{3\sqrt{3}}{2} - 4\right) \text{ cubic units}$$

(1 mark)