ATAR Notes

VCE SPECIALIST MATH 3/4 UNIT 3 RECAP

Wifi Details

Event code: 462204

Presented by:
Fiona Yu

LECTURE SCHEDULE

Start	End	Duration	Details
10:15 am	11:00 am	45 minutes	Content block 1
11:00 am	11:15 am	15 minutes	Break 1
11:15 am	12:00 am	45 minutes	Content block 2
12:00 pm	12:15 pm	15 minutes	Break 2
12:15 pm	1:00 pm	45 minutes	Content block 3

HOUSEKEEPING

- If you have any questions, ask me during the break!
- If the mic drops out, please tell me!
- There are toilets on every level just look for the black cartoon human on the wall

OVERVIEW

1st content block:

→Vectors

2nd content block:

→Circular Functions + complex numbers

3rd content block:

→Complex numbers

OVERVIEW OF SPECIALIST 3&4

Typical Unit 3 topics:

- Vectors

  → vectors in 3D, linear dependence, proofs

- Functions and Graphs

  →reciprocal circular functions, trig identities, rational functions

- Complex Numbers

  →Polar form, de Moivre's Theorem, solutions and factors over C

- Calculus

  →Further differentiation and applications

TODAY’S SESSION

- Don’t expect yourself to understand everything!
- Specialist is really challenging and the hardest part of it is *understanding it* (it scales so high for a reason)
- From today’s lesson, I hope you can take away something even if it is just one thing. (whether this be a mathematical concept or a behavioural change)

HOW TO STUDY FOR SPESH

- Set weekly goals and break them up into daily goals

(Example: I want to finish chapter 4 of my Spesh textbook this week. On Monday I will do exercise 4A and 4B and on Wednesday…etc)

- After you finish up a topic (e.g. complex numbers), do VCAA questions for it. (This is ultimately where many students meet their downfall)

- Remember, textbook questions are for understanding the concept. VCAA questions are for application of your knowledge. You need to do both.

VECTORS

Recall:

- Vectors have both a magnitude and direction
- Scalars have magnitudes only

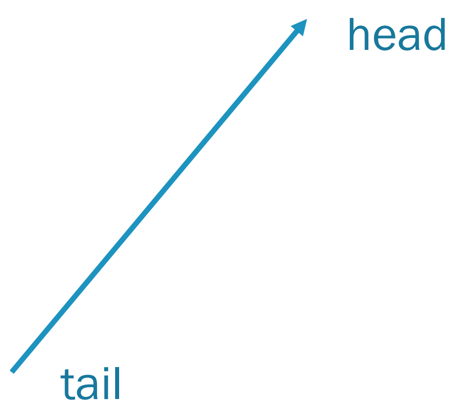
Vectors, including:

- addition and subtraction of vectors and their multiplication by a scalar, and position vectors
- linear dependence and independence of a set of vectors and geometric interpretation
- magnitude of a vector, unit vector, and the orthogonal unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$
- resolution of a vector into rectangular components
- scalar (dot) product of two vectors, deduction of dot product for $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ system; its use to find scalar and vector resolutes
- parallel and perpendicular vectors

^^ the part of the study design we are covering for vectors today

REPRESENTATION OF VECTORS

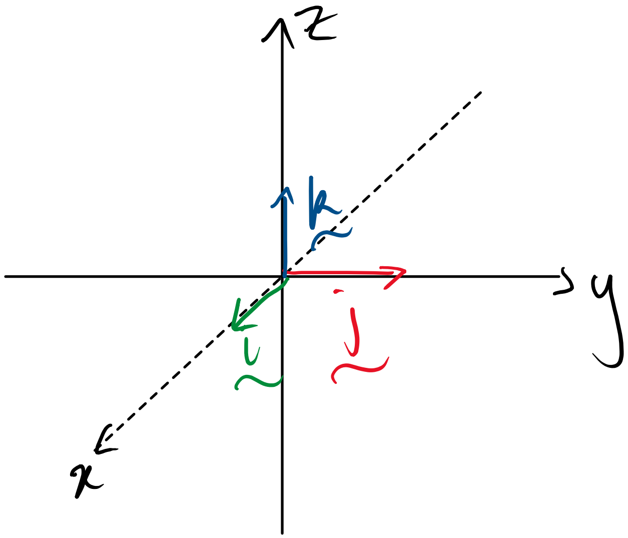
- Vectors are depicted as **arrows** – with a **head** and a **tail**.
- Vectors can be treated as either *position* vectors and *free vectors*, depending on context
- **Position vectors** denoted like $\overrightarrow{OP}$ - can't be shifted around.
- **Free vectors** are the same, regardless of where the 'tail' is.



CARTESIAN COORDINATE SYSTEM

- We can deal with vectors in a **Cartesian coordinate system**, using a Cartesian representation.

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$$



PROPERTIES OF VECTORS

Length (magnitude)

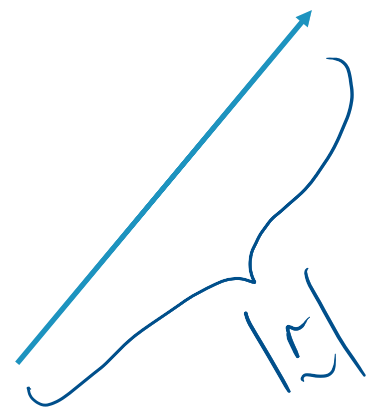
- The length of a vector $\vec{r}$ is denoted $|\vec{r}|$. **This is not the absolute value!**

- If we are dealing with the Cartesian representation of a vector, $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, then we have

$$|\vec{\boldsymbol{r}}| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2} \quad \mathbf{(2d)}$$

$$|\vec{\boldsymbol{r}}| = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2} \quad \mathbf{(3d)}$$

- Visually, this is the length of the 'arrow'



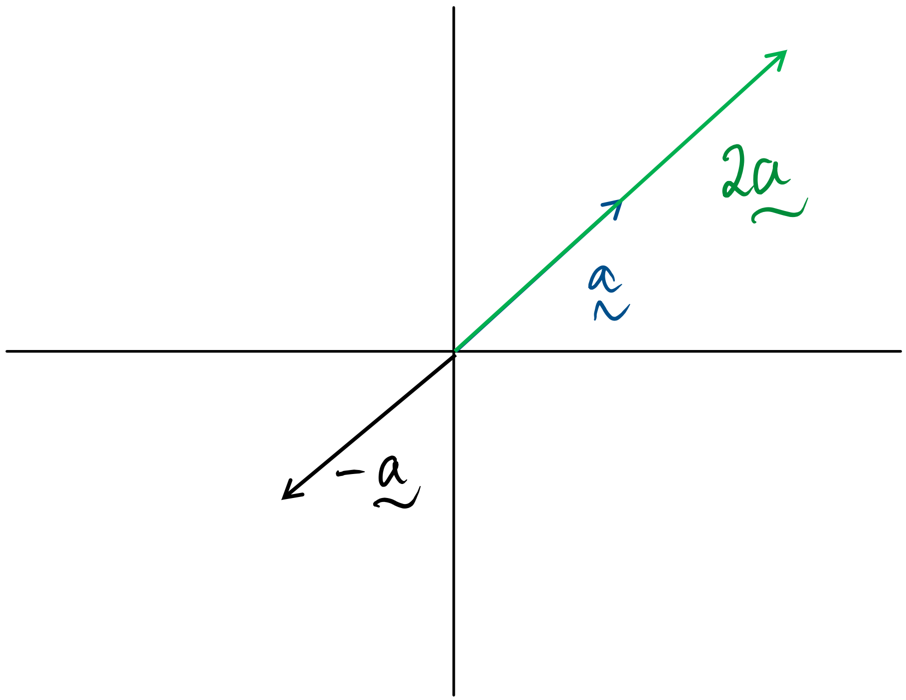
UNIT VECTORS

- Unit vectors are a special type of vector that have **magnitude 1.**
- **If we want to specify <u>only a direction</u>,** we use unit vectors (because we can rescale them to any desired length)
- Given any vector $\vec{r}$, the corresponding unit vector is denoted $\hat{r}$

$$\hat{\boldsymbol{r}} = \frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|}$$

SCALAR MULTIPLICATION

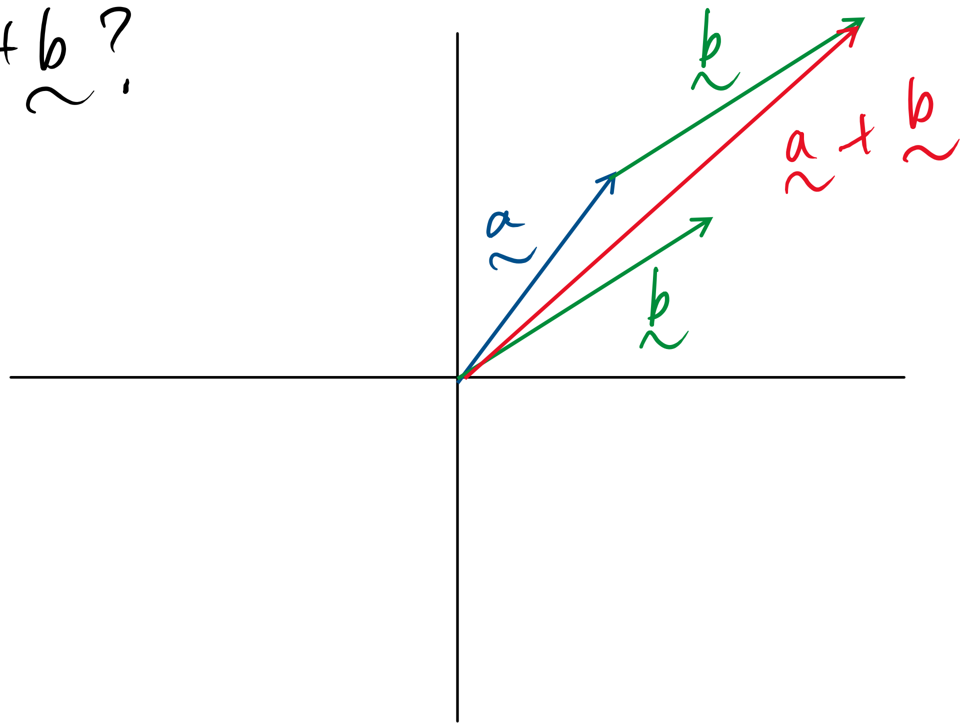
We can multiply a vector by a **scalar** to change its length. A **negative** scalar reverses the direction of the vector.



VECTOR ADDITION

Visually, we add vectors **head** to **tail.** Vector subtraction is simply multiplying by (-1) and adding.

What's $\underset{\sim}{a} + \underset{\sim}{b}$?

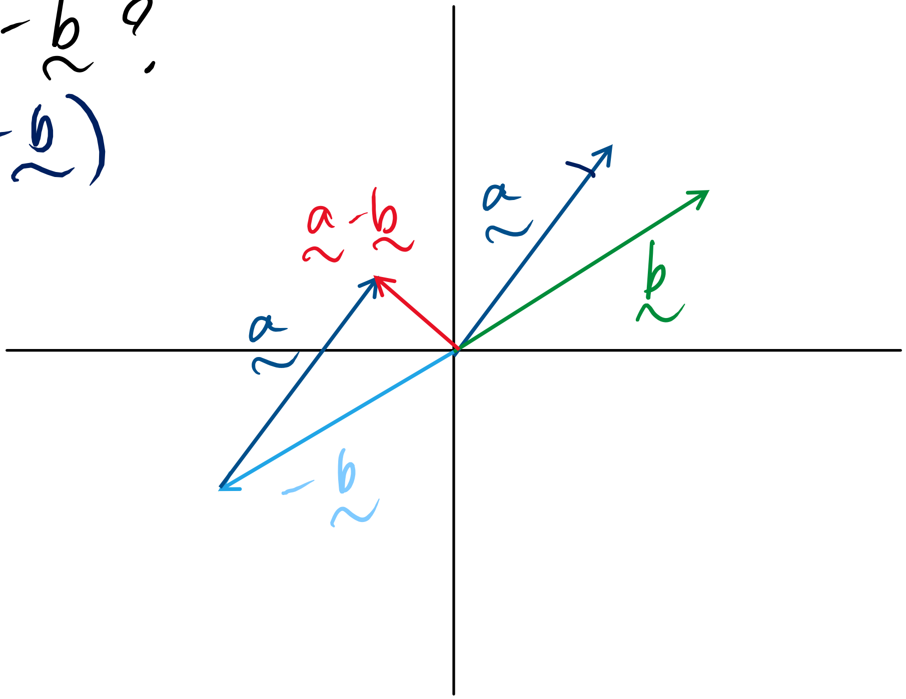


VECTOR ADDITION

Visually, we add vectors **head** to **tail.** Vector subtraction is simply multiplying by (-1) and adding.

What's $\underset{\sim}{a} - \underset{\sim}{b}$?

$\underset{\sim}{a} + (-\underset{\sim}{b})$

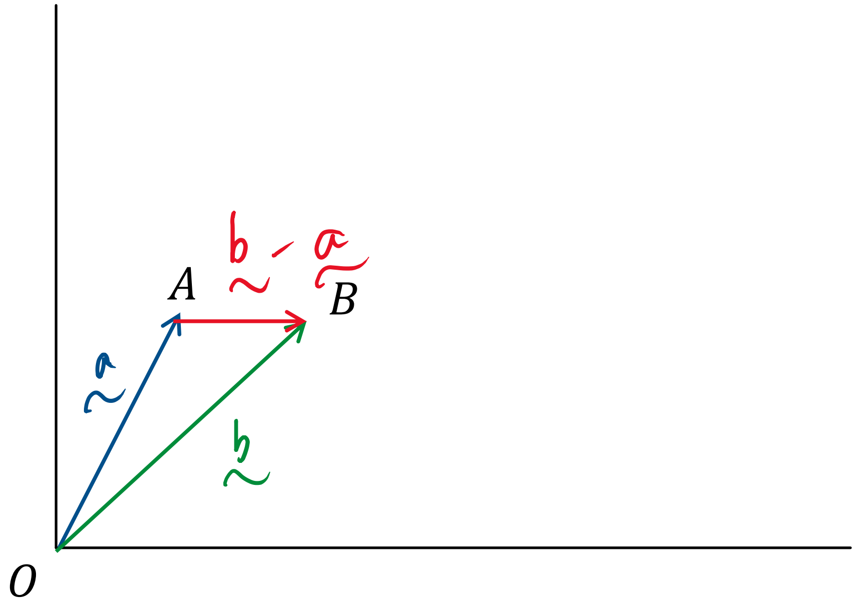


VECTORS

Last point minus first point rule

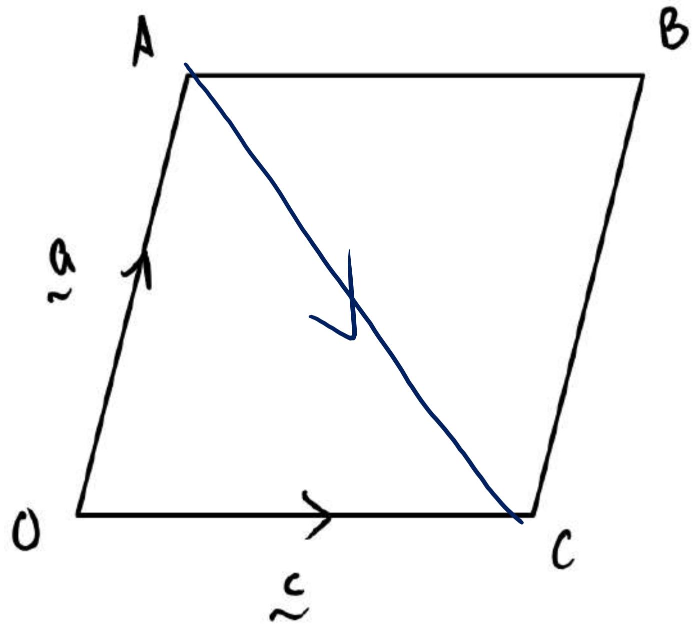
- If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, then we can calculate:

$$\overrightarrow{\boldsymbol{AB}} = \overrightarrow{\boldsymbol{OB}} - \overrightarrow{\boldsymbol{OA}} = \vec{\boldsymbol{b}} - \vec{\boldsymbol{a}}$$



VECTORS

For instance, we can use this rule to find representations of diagonals in a parallelogram:



$$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$$

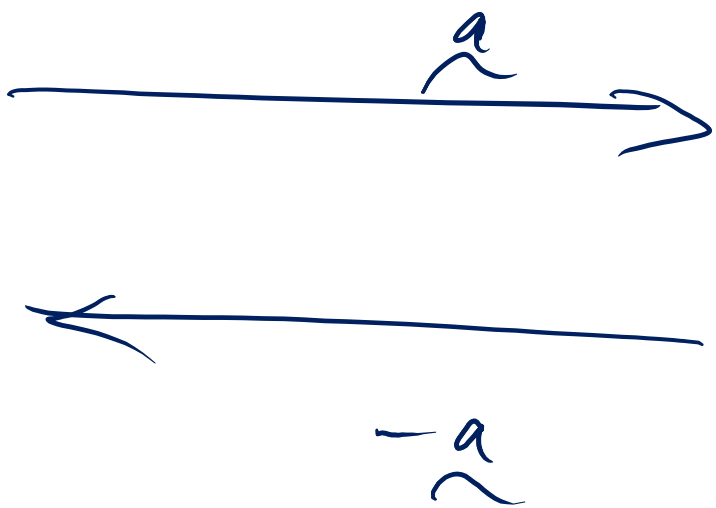
$$\overrightarrow{CA} = \underset{\sim}{a} - \underset{\sim}{c}$$

PARALLEL VECTORS

- Two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel if

$$\underset{\sim}{a} = k\underset{\sim}{b}$$

Where $k \in R$



DOT PRODUCT/SCALAR PRODUCT

- This is a certain way of 'multiplying' vectors.
- The dot product is defined as:

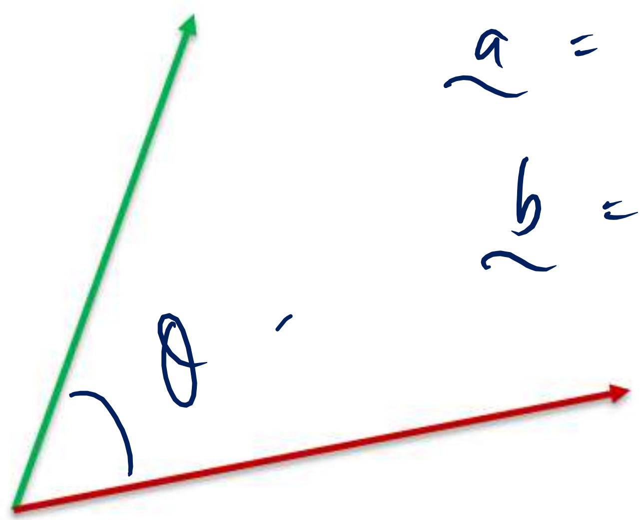
$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}| \cos(\boldsymbol{\theta})$$

- Where θ is the *tail-to-tail* angle between $\vec{a}$ and $\vec{b}$, $0 < \theta \leq \pi$.

$\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= 2 + 2 + 1$

$= 5$



$\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

DOT PRODUCT

Important results for the scalar product

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |a||b| \cos(\theta)$

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (k\vec{b}) = (k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a}, \vec{b}$ **are orthogonal (perpendicular)**
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

These properties are very useful!

DOT PRODUCT

- If we have $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then we have

$$\boldsymbol{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

- The dot product formula is often used to find the angles between two vectors

VCAA 2008

If the vectors $\underset{\sim}{a} = m\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m\underset{\sim}{i} + m\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular, then

A. $m = 0$

B. $m = -6$ or $m = 2$

C. $m = -2$ or $m = 6$

D. $m = -2$ or $m = 0$

E. $m = -1$ or $m = 1$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = m^2 + 4m - 12 = 0$

$(m+6)(m-2) = 0$

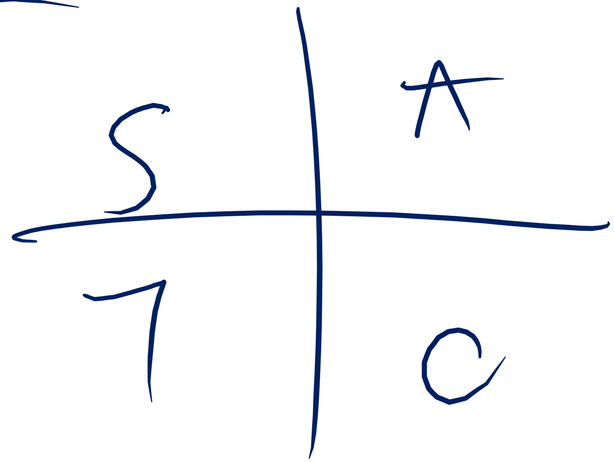
$m = -6$ & $m = 2$

DOT PRODUCT

- Recall that the dot product formula is

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right| \left|\underset{\sim}{b}\right| \cos(\theta)$$

Where $\theta \in [0, \pi]$



Implications

If θ is an acute angle then $\underset{\sim}{a} \cdot \underset{\sim}{b} > 0$

If θ is an obtuse angle then $\underset{\sim}{a} \cdot \underset{\sim}{b} < 0$

Knowing details like this can ultimately be the difference between getting a low 40s and a high 40s study score.

VCAA 2017 NHT

Consider the vectors $\underset{\sim}{a} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + c\underset{\sim}{j} + \underset{\sim}{k}$.

Find the value of c, $c \in R$, if the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{\pi}{3}$.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -2 - 2c + 3 = 1 - 2c$

$|\underset{\sim}{a}| = \sqrt{14}$, $|\underset{\sim}{b}| = \sqrt{5 + c^2}$

$1 - 2c = \frac{1}{2}\sqrt{70 + 14c^2}$

$(1-2c)^2 = \frac{1}{4}(70 + 14c^2)$

$4c^2 - 4c + 1 = \frac{1}{4}(70 + 14c^2)$

$16c^2 - 16c + 4 = 70 + 14c^2$

$2c^2 - 16c - 66 = 0$

$c^2 - 8c - 33 = 0$

$(c - 11)(c + 3) = 0$

$c = 11$, $c = -3$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 1 - 2c$

$\therefore c = -3$ as

$\underset{\sim}{a} \cdot \underset{\sim}{b} > 0$

LINEAR DEPENDENCE

- A set of vectors is said to be either **linearly dependent** or **linearly independent**
- Often quite a confusing concept for students! (you can honestly get away with memorising a formula)

- A set of vectors is said to be **linearly dependent** if at least one of its members can be expressed as a linear combination of the other set of vectors.

LINEAR DEPENDENCE

- *The set* $\{\vec{a}, \vec{b}, \vec{c}\}$ *are* ***linearly dependent*** *if there exists* k_1, k_2, k_3, ***not all zero****, such that*

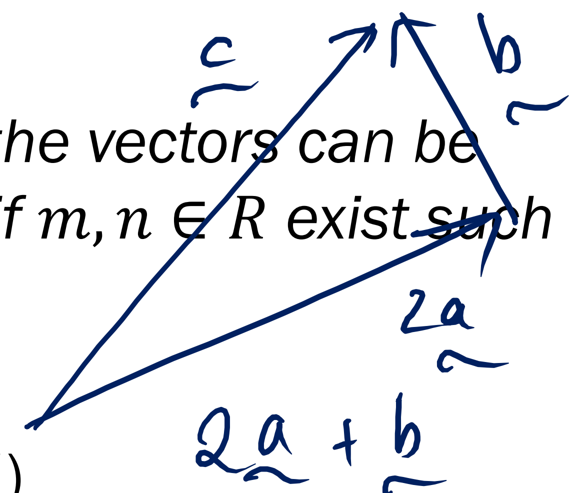
$$k_1\vec{a} + k_2\vec{b} + k_3\vec{c} = \vec{0}$$

- However, the following alternative definition works for Specialist (this is what you would use in your working out)

$\{\vec{a}, \vec{b}, \vec{c}\}$ *are* ***linearly dependent*** *if one of the vectors can be expressed in terms of the other two, i.e. if* $m, n \in R$ *exist such that*

$$\vec{c} = m\vec{a} + n\vec{b}$$

(memorise this^^^!!)



LINEAR DEPENDENCE

Notice

- 2 vectors are dependent only if they are parallel $\underset{\sim}{a} = k\underset{\sim}{b}$
- In 2 dimensions, >2 vectors $\Rightarrow$ dependent
- In 3 dimensions, >3 vectors $\Rightarrow$ dependent
- 3 coplanar vectors $\Rightarrow$ dependent

LINEAR DEPENDENCE

Matrix method (use only on CAS, don't use as apart of working out!)

Consider the vectors forming the set $\{\vec{a}, \vec{b}, \vec{c}\}$

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \qquad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k},$$

$$\vec{\mathrm{c}} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

- **Set is linearly dependent if** $\mathbf{det}\begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \\ \boldsymbol{c_1} & \boldsymbol{c_2} & \boldsymbol{c_3} \end{pmatrix} = \mathbf{0}$
- **Set is linearly independent if** $\mathbf{det}\begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \\ \boldsymbol{c_1} & \boldsymbol{c_2} & \boldsymbol{c_3} \end{pmatrix} \neq \mathbf{0}$

VCAA 2008

Consider the vectors $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{c} = m\underset{\sim}{i} + n\underset{\sim}{k}$ where m and n are non-zero real constants.

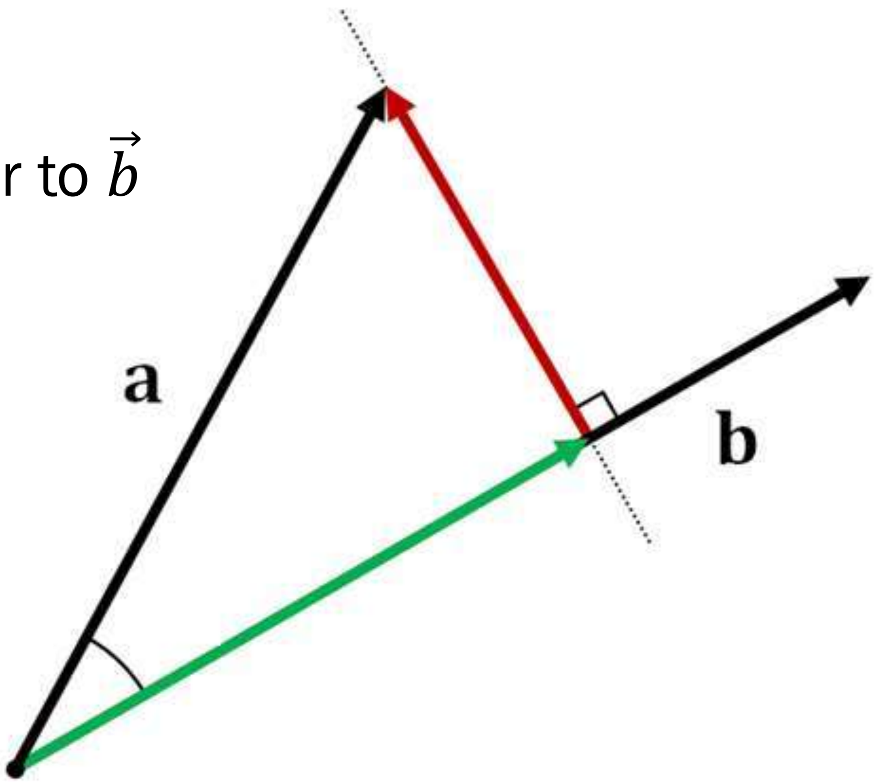
Find $\dfrac{m}{n}$ so that $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ form a **linearly dependent** set of vectors.

3 marks

VECTOR RESOLUTES

Given two vectors $\vec{a}$ and $\vec{b}$, we may want to express $\vec{a}$ in terms of:

- A component parallel to $\vec{b}$
- A component perpendicular to $\vec{b}$



VECTOR RESOLUTES

Projection of $\vec{a}$ parallel to $\vec{b}$

$$\overrightarrow{\boldsymbol{a}_{\parallel}} = (\vec{\boldsymbol{a}} \cdot \widehat{\boldsymbol{b}})\widehat{\boldsymbol{b}} = \frac{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}}{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{b}}}\vec{\boldsymbol{b}}$$

Projection of $\vec{a}$ perpendicular to $\vec{b}$

$$\overrightarrow{\boldsymbol{a}_{\perp}} = \vec{\boldsymbol{a}} - \overrightarrow{\boldsymbol{a}_{\parallel}} = \vec{\boldsymbol{a}} - \frac{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}}{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{b}}}\vec{\boldsymbol{b}}$$

CIRCULAR FUNCTIONS

In Specialist, you will be covering two types of circular functions.

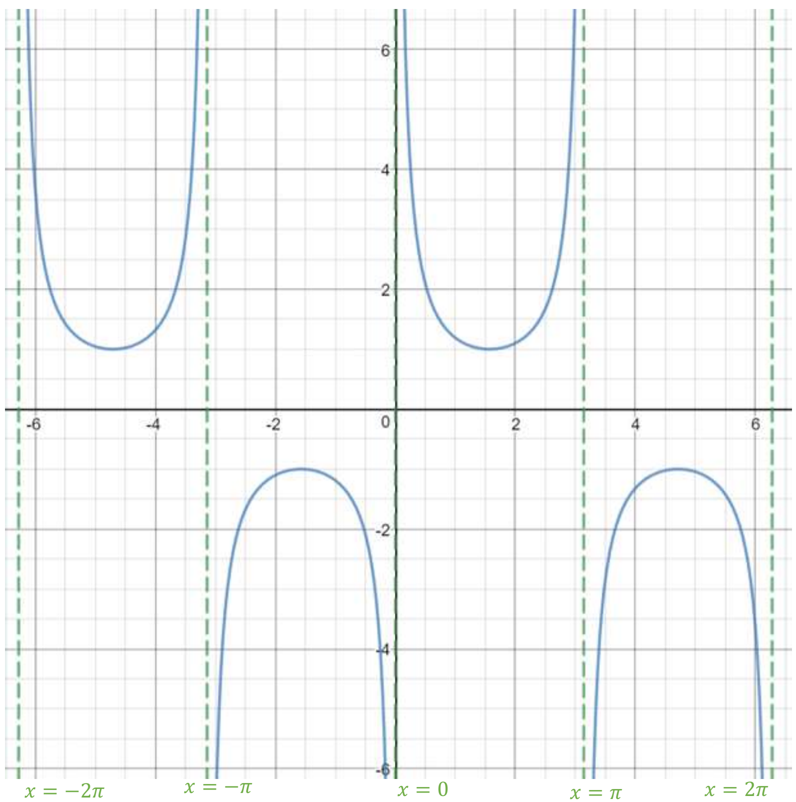
- Reciprocal circular functions
- Inverse circular functions

You will need to be well acquainted with these functions (much like how you know your exponentials, logarithms, square root, hyperbola and trunci for Methods)

You will also need to know quite a few formulas (compound and double angle formula)

RECIPROCAL CIRCULAR FUNCTIONS

$$y = \text{cosec}(x) = \frac{1}{\sin(x)}$$



Domain:

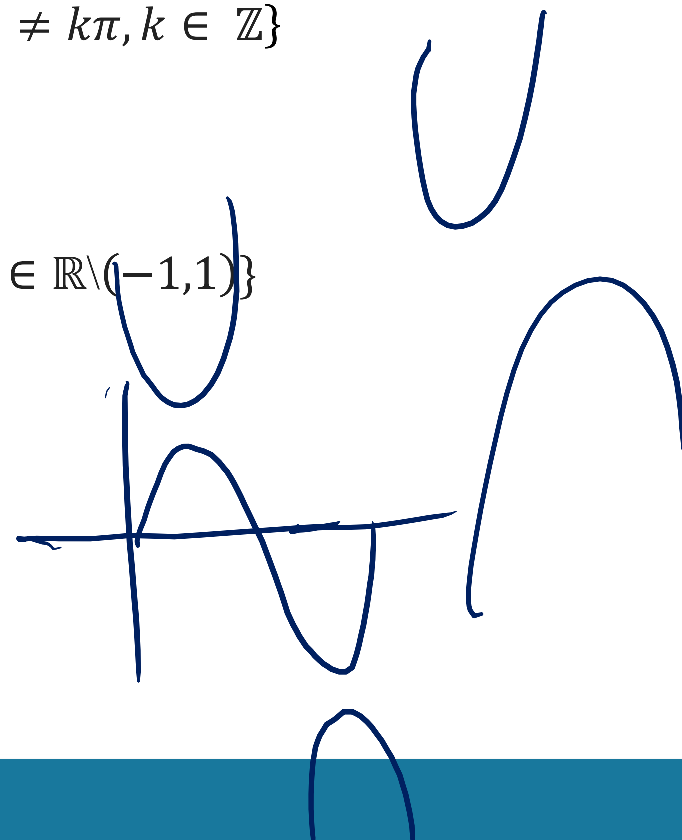
$\{x \in \mathbb{R}: x \neq k\pi, k \in \mathbb{Z}\}$

Range:

$\{y \in \mathbb{R}: y \in \mathbb{R}\backslash(-1,1)\}$

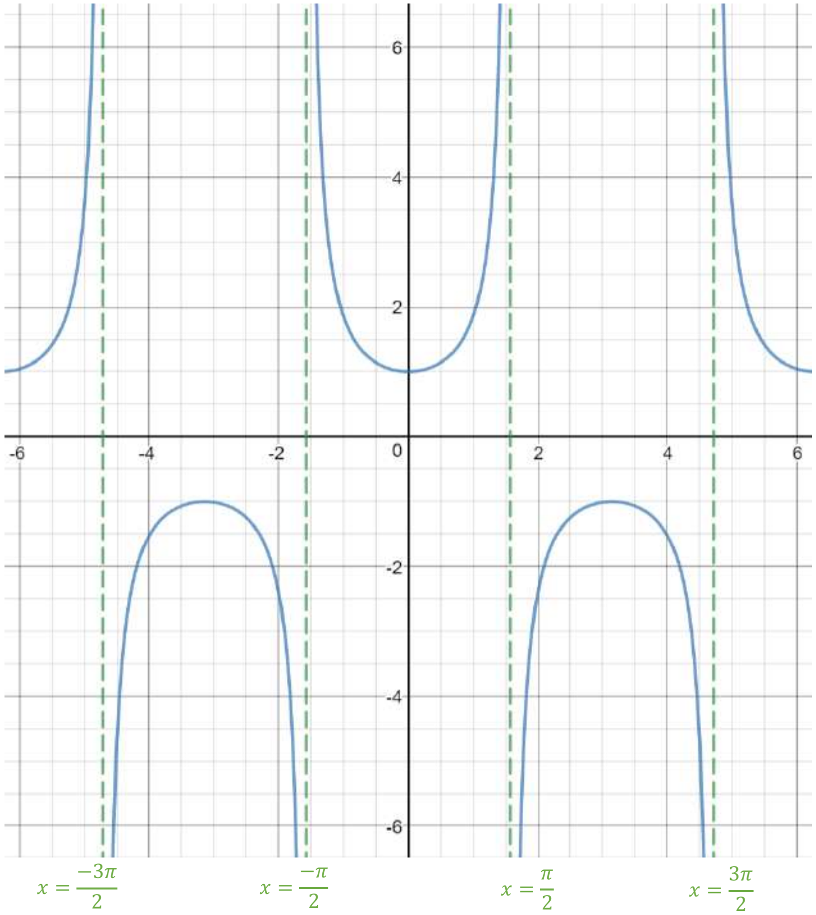
Period:

2π



RECIPROCAL CIRCULAR FUNCTIONS

$$y = \sec(x) = \frac{1}{\cos(x)}$$



Domain:

$$\left\{x \in \mathbb{R}: x \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}\right\}$$

Range:

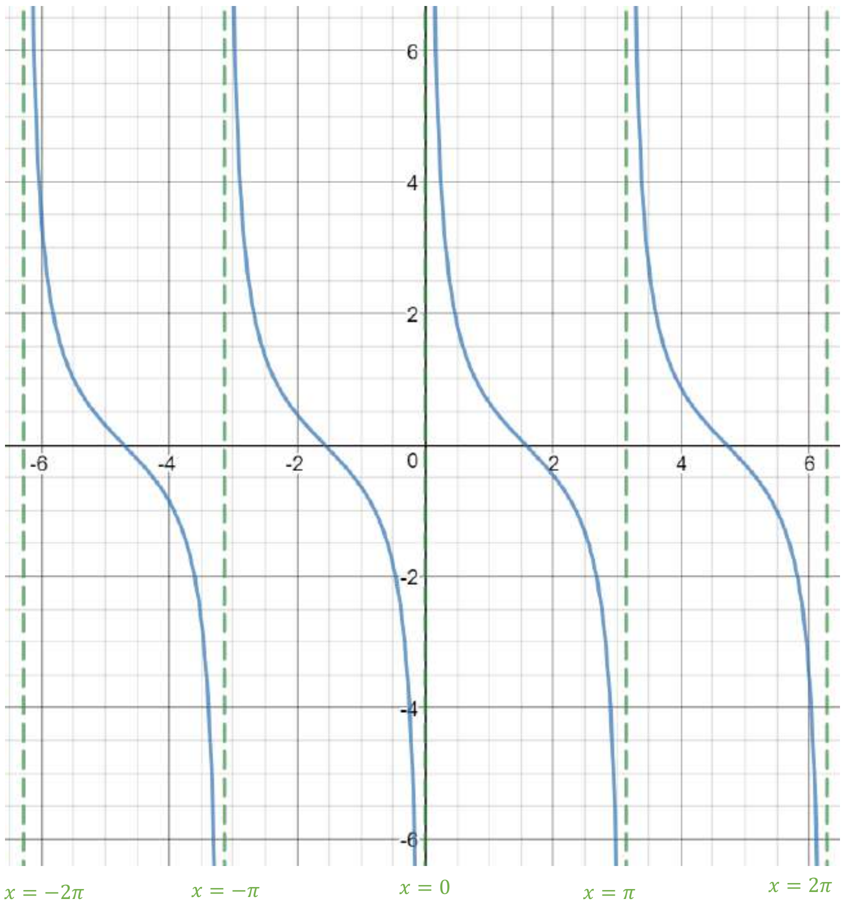
$\{y \in \mathbb{R}: y \in \mathbb{R}\backslash(-1,1)\}$

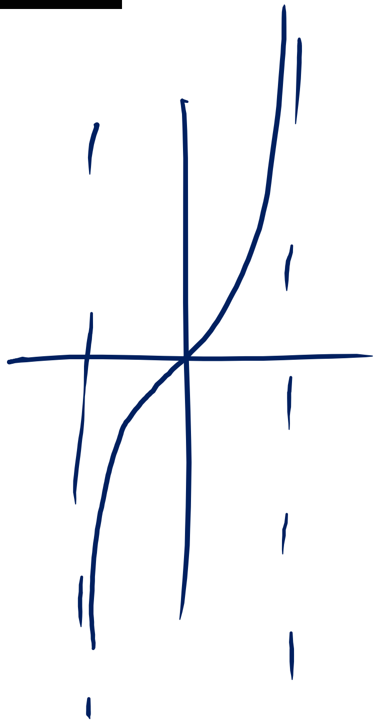
Period:

2π

RECIPROCAL CIRCULAR FUNCTIONS

$$y = \cot(x) = \frac{1}{\tan(x)}$$





Properties

Domain:
$\{x \in \mathbb{R}: x \neq k\pi, k \in \mathbb{Z}\}$

Range:
$\{y \in \mathbb{R}: -\infty < y < \infty\}$

Note:

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan(x)} = \cot(x)$$

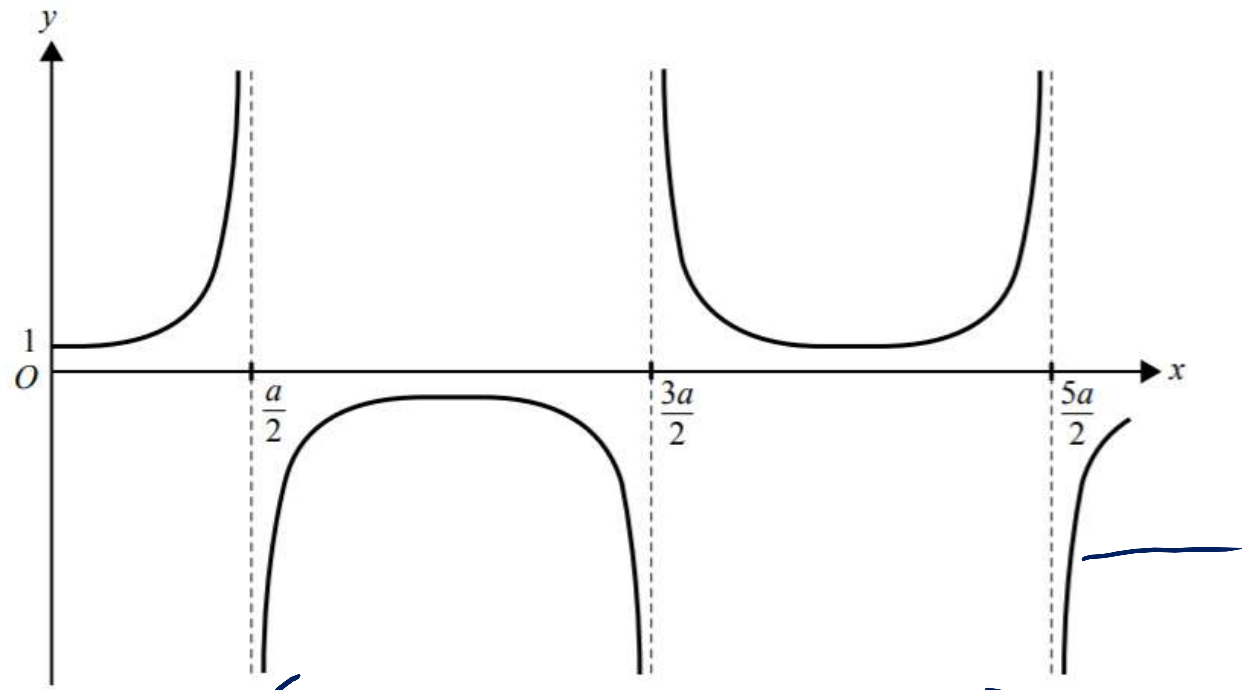
$\cos(nx)$

$\sin(nx)$

period $= \frac{2\pi}{n}$

$\operatorname{cosec}(nx)$

$\sec(nx)$



A rule for the function whose graph is shown above, where a is a positive constant, could be

A. $y = \operatorname{cosec}\left(\frac{2\pi}{a}\left(x - \frac{a}{2}\right)\right)$

B. $y = \operatorname{cosec}\left(\frac{2\pi}{a}\left(x + \frac{a}{4}\right)\right)$

C. $y = \operatorname{cosec}\left(\frac{\pi}{a}\left(x - \frac{a}{2}\right)\right)$

D. $y = \operatorname{cosec}\left(\frac{\pi}{a}x + \frac{a}{2}\right)$

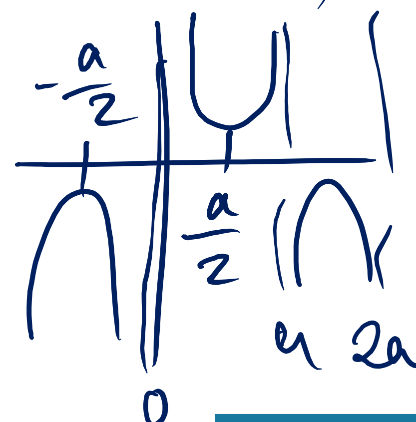
E. $y = \operatorname{cosec}\left(\frac{\pi}{a}\left(x + \frac{a}{2}\right)\right)$

period $= 2a$

$\frac{2\pi}{n} = 2a$

$n = \frac{\pi}{a}$

$\operatorname{cosec}\left(\frac{\pi}{a}x\right)$



COMPOUND FORMULA

- These are on the formula sheet
- Don’t recommend to memorise it but can if you want to/are able to

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$
$\tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$

DOUBLE ANGLE FORMULA

- These are on the formula sheet

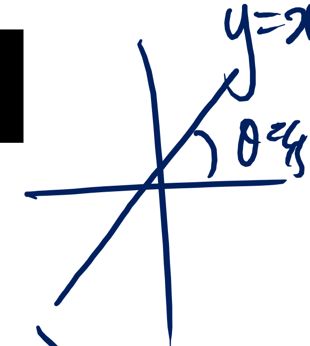
- Recommend that you memorise it

$\cos(2x)$ $= \cos^2(x) - \sin^2(x)$ $= 1 - 2\sin^2(x)$ $= 2\cos^2(x) - 1$	$\sin(2x)$ $= 2\sin(x)\cos(x)$	$\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$

- These formula become really useful when we move onto Calculus (fun!!!!!)

VCAA 2015 (MODIFIED)

$\theta - \alpha = \frac{\pi}{4}$



Find the acute angle between the two linear equations

$m = \tan\theta$

$2 = \tan\theta$

$$y = 2x - 6$$

$$y = \frac{x}{3} + \sqrt{6}$$

$m = \tan(\theta)$

Give your answer in the form $k\pi$, where $k \in R$.

$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

$\theta = \tan^{-1}(2)$, $\alpha = \tan^{-1}\left(\frac{1}{3}\right)$

$\tan\theta$

$= \tan(\tan^{-1}(2))$

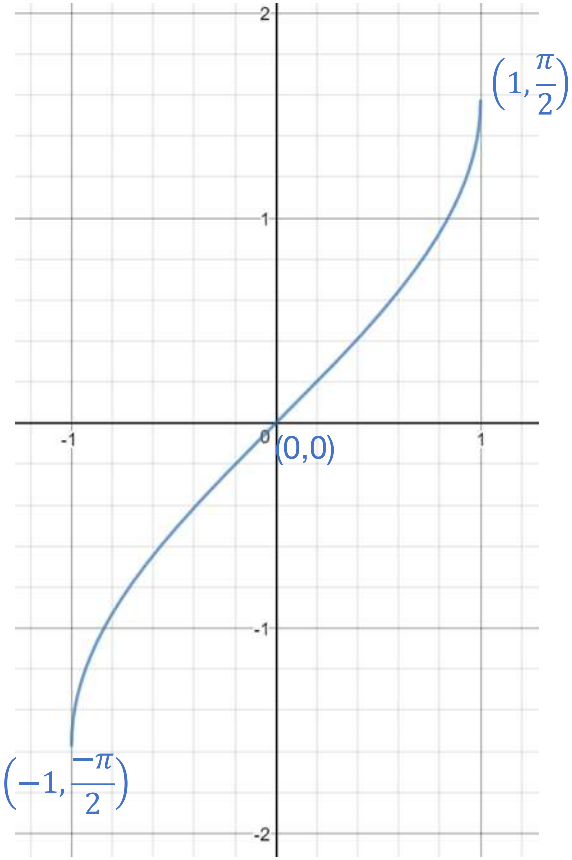
$\theta - \alpha$

$\left[\tan^{-1}(2) - \tan^{-1}\left(\frac{1}{3}\right)\right]$

$\tan(\theta - \alpha) = \frac{2 - \frac{1}{3}}{1 + 2 \times \frac{1}{3}} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$

INVERSE CIRCULAR FUNCTIONS

$$y = \sin^{-1}(x) \;\; or \; y = \arcsin(x)$$



Domain:

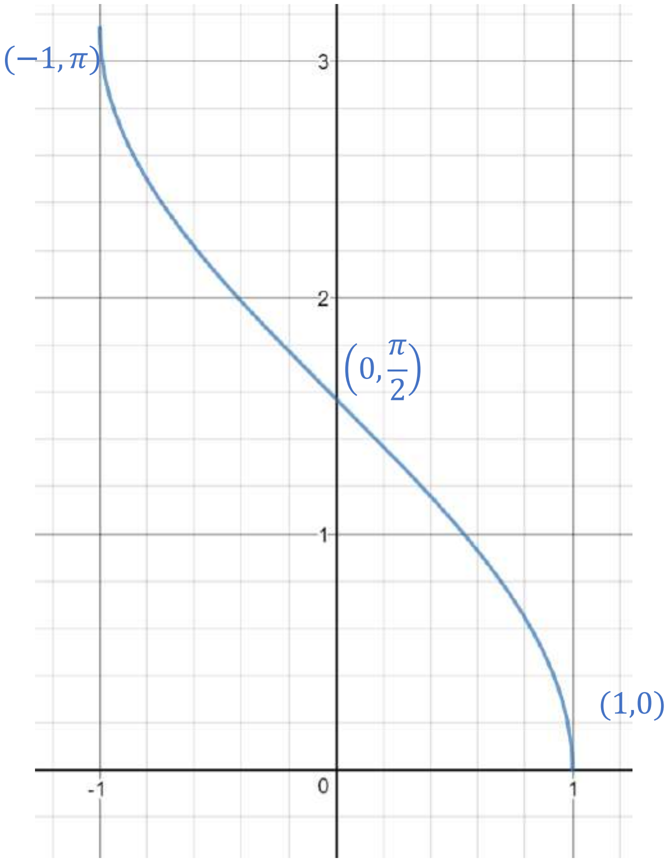
$[-1,1]$

Range:

$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

INVERSE CIRCULAR FUNCTIONS

$$y = \cos^{-1}(x) \;\; or \; y = \arccos(x)$$



Domain:

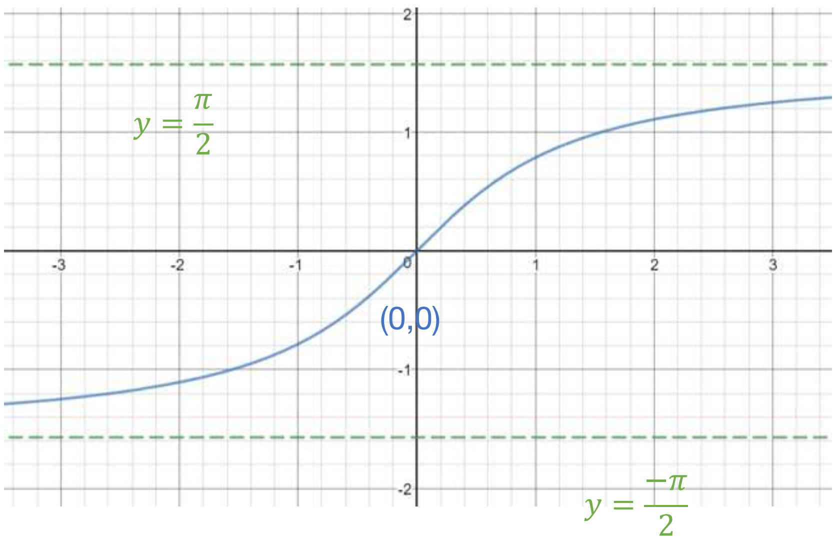
$[-1,1]$

Range:

$[0, \pi]$

INVERSE CIRCULAR FUNCTIONS

$$y = \tan^{-1}(x)\ \ or\ \arctan(x)$$



Domain:

R

Range:

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

INVERSE & ORIGINAL

Note: $= x$

$\sin(\sin^{-1}(x))$ ← will always hold true

$\sin^{-1}(\sin(x)) = x$ ← will NOT always hold true

(same applies for cos and tan)

DOMAIN OF INVERSE

Finding domain of $\sin^{-1}$ and $\cos^{-1}$ that have been transformed

Domain of $\sin^{-1}(x)$ and $\cos^{-1}(x)$ is $[-1,1]$.

Therefore, to find the domain of $\sin^{-1}(ax+b)$ or $\cos^{-1}(ax+b)$,

$\sqrt{f(x)}$

$f(x) \geq 0$

Solve, $-1 \leq ax+b \leq 1$

$$-1-b \leq ax \leq 1-b$$

$$\frac{-1}{a}(1+b) \leq x \leq \frac{1}{a}(1-b)$$

$\therefore$ the domain is $\left[-\frac{1}{a}(1+b), \frac{1}{a}(1-b)\right]$

Domain of $\tan^{-1}(x)$ is $\mathbb{R}$

RANGE OF INVERSE

Finding range of $\sin^{-1}$ and $\cos^{-1}$ that have been transformed

Range of $\sin^{-1}(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$-\frac{\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}$

To find the range of $a\sin^{-1}(x) + c$...

$-\frac{a\pi}{2} \leq a\sin^{-1}(x) \leq \frac{a\pi}{2}$

Multiply the range by a and add c

$-\frac{a\pi}{2} + c \leq a\sin^{-1}(x) + c \leq \frac{a\pi}{2} + c$

$\therefore$ the range of $a\sin^{-1}(x) + c$ is $\left[-\frac{a\pi}{2} + c, \frac{a\pi}{2} + c\right]$

Range of $\cos^{-1}(x)$ is $[0, \pi]$.

To find the range of $a\cos^{-1}(x) + c$...

Multiply the range by a and add c

$\therefore$ the range of $a\cos^{-1}(x) + c$ is $[c, a\pi + c]$

RANGE OF INVERSE

Finding range of $\tan^{-1}$ that has been transformed

Range of $\tan^{-1}(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

To find the range of $a\tan^{-1}(x) + c$...

Multiply the range by a and add c

$\therefore$ the range of $a\tan^{-1}(x) + c$ is $\left(-\frac{a\pi}{2} + c, \frac{a\pi}{2} + c\right)$

VCAA 2009

$0 \le \cos^{-1}(x-b) \le \pi$

$0 \le a\cos^{-1}(x-b) \le a\pi$

Consider the function f with rule $f(x) = a\cos^{-1}(x-b)$.

Given that f has domain $[2, 4]$ and range $[0, 6\pi]$, it follows that

$a\pi = 6\pi$

$a = 6$

~~**A.**~~ $a = 6, b = -3$

~~**B.**~~ $a = 3, b = 6$

~~**C.**~~ $a = -3, b = 6$

D. $a = 6, b = 3$

E. $a = -6, b = 3$

$-1 \le x - b \le 1$

$\underbrace{-1+b}_{2} \le x \le \underbrace{1+b}_{4}$

$2b = 6, \; b = 3$

SOLVING

Solve for x

$$\sec(2x) = 2, x \in [0, \pi]$$

$$\frac{1}{\cos(2x)} = 2$$

$$\cos(2x) = \frac{1}{2}$$

let $a = 2x \longrightarrow x = a/2$

$$\cos(a) = \frac{1}{2}, \quad a \in [0, 2\pi]$$

ref angle $= \frac{\pi}{3}$

$$a = \frac{\pi}{3}, \frac{5\pi}{3} \quad \Big| \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$$

COMPLEX NUMBERS

- Revolves around the idea that $\sqrt{-1} = i, \therefore i^2 = -1$

Complex numbers, including:

- C, the set of numbers z of the form $z = x + yi$ where x, y are real numbers and $i^2 = -1$, real and imaginary parts, complex conjugates, modulus
- use of an argand diagram to represent points, lines, rays and circles in the complex plane
- equality, addition, subtraction, multiplication and division of complex numbers
- polar form (modulus and argument); multiplication and division in polar form, including their geometric representation and interpretation, proof of basic identities involving modulus and argument
- De Moivre's theorem, proof for integral powers, powers and roots of complex numbers in polar form, and their geometric representation and interpretation
- n^{th} roots of unity and other complex numbers and their location in the complex plane
- factors over C of polynomials with integer coefficients; and informal introduction to the fundamental theorem of algebra
- factorisation of polynomial functions of a single variable over C, for example, $z^8 + 1$, $z^2 - i$, $z^3 - (2 - i)z^2 + z - 2 + i$
- solution over C of corresponding polynomial equations by completing the square, factorisation and the conjugate root theorem.

COMPLEX UMBERS

A complex number $\boldsymbol{z = x + yi}$ has both a real part and an imaginary part

- Real part is denoted $\mathbf{Re}(\boldsymbol{z}) = \boldsymbol{x}$
- Imaginary part is denoted $\mathbf{Im}(\boldsymbol{z}) = \boldsymbol{y}$

Examples

1. $\sqrt{-25}$ $\longrightarrow$ $5i$
2. $\mathrm{Re}(3 - 4i)$

$= 3$

OPERATIONS ON $\mathbb{C}$

- The properties of algebra hold for complex numbers
- Make sure you brush up on your algebra!

Addition:

$$(\boldsymbol{a}+\boldsymbol{b}\boldsymbol{i})+(\boldsymbol{c}+\boldsymbol{d}\boldsymbol{i})=(\boldsymbol{a}+\boldsymbol{c})+(\boldsymbol{b}+\boldsymbol{d})\boldsymbol{i}$$

Real multiplication:

$$\boldsymbol{k}(\boldsymbol{a}+\boldsymbol{b}\boldsymbol{i})=(\boldsymbol{k}\boldsymbol{a})+(\boldsymbol{k}\boldsymbol{b})\boldsymbol{i}$$

Complex multiplication:

$$(\boldsymbol{a}+\boldsymbol{b}\boldsymbol{i})(\boldsymbol{c}+\boldsymbol{d}\boldsymbol{i})=(\boldsymbol{a}\boldsymbol{c}-\boldsymbol{b}\boldsymbol{d})+(\boldsymbol{a}\boldsymbol{d}+\boldsymbol{b}\boldsymbol{c})\boldsymbol{i}$$

$$\Rightarrow \textbf{expand in the usual way}$$

OPERATIONS ON $\mathbb{C}$

- Analogise the addition/subtraction of complex numbers to 'like terms'

Examples

Let $z_1 = 3 + 4i$ and $z_2 = 5 - 7i$

1. $z_1 + z_2$
2. $z_1 \times z_2$

COMPLEX CONJUGATE

Given $z = a + bi$

The complex conjugate of z is represented by $\overline{z}$.

$$\overline{z} = a - bi$$

Example

Find the complex conjugate of $\underline{6 - 7i}$

$6 + 7i$

This is important for later…

COMPLEX CONJUGATE

Useful properties of the complex conjugate

Given $z = x + yi$

- $z\bar{z} = x^2 + y^2 = |z|^2$
- $z + \bar{z} = 2x$
- $z - \bar{z} = 2y$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \times \overline{z_2}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

OPERATIONS ON $\mathbb{C}$

To simplify

$$\frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$

We use a technique called 'realising' (very similar to rationalising)

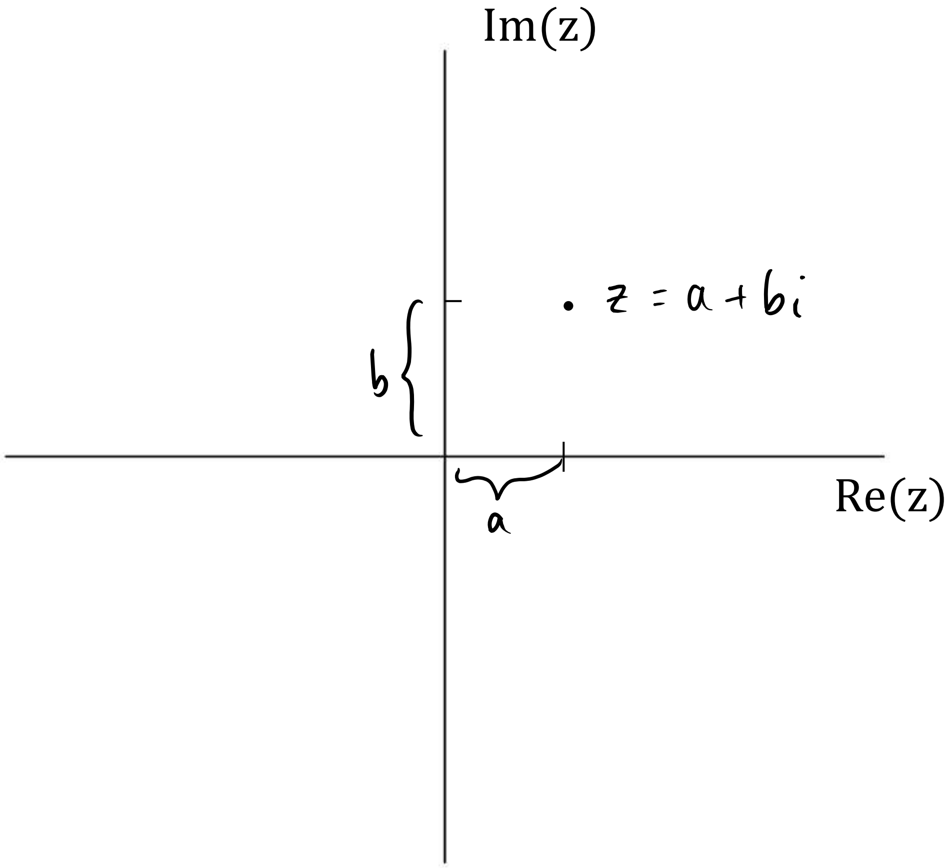
Example

Simplify $\frac{3}{2+3i}$ $\times \frac{2-3i}{2-3i} = \frac{6-9i}{4+9}$

ARGAND DIAGRAM

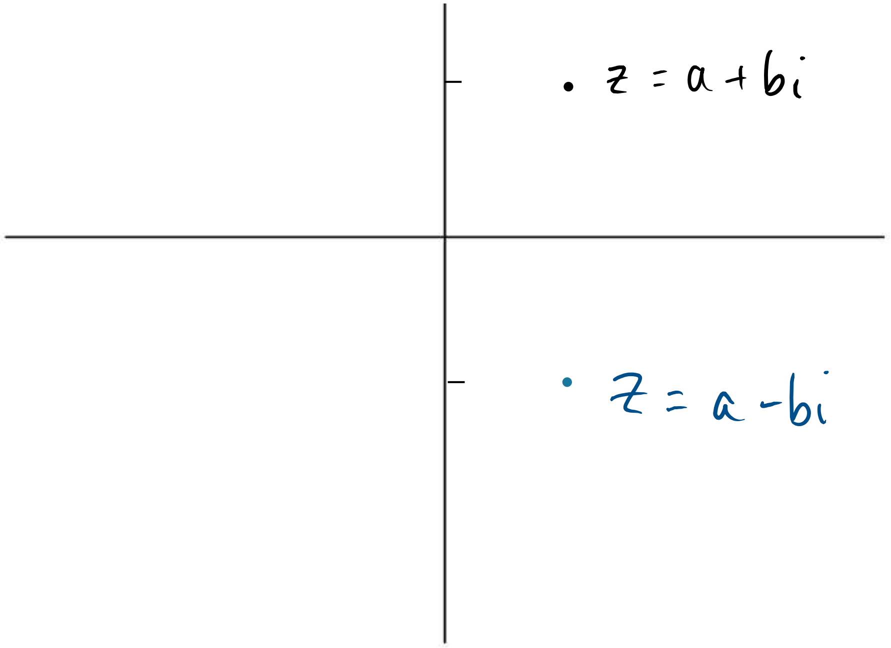
- A complex number $z = x + yi$ can be represented on something called an argand diagram.
- $z = a + bi$, this becomes

(a, b) on the Argand plane



COMPLEX CONJUGATE

- You need to understand the visual change that taking the conjugate has on the number in the Argand plane
- Given $z = a + bi, \bar{z} = a - bi$
- It has the effect of reflecting the point across the horizontal axis.



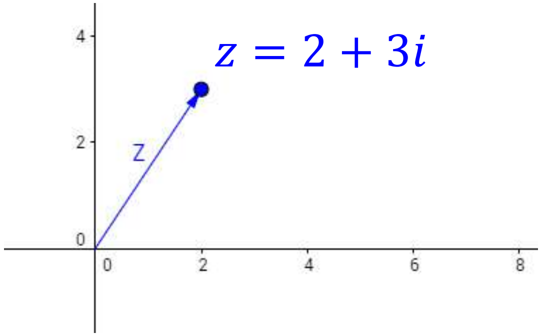
COMPLEX MODULUS

- Visually, the modulus is the distance between the origin and the complex number on the Argand plane

More generally,

Given $z = a + bi, |z| = \sqrt{a^2 + b^2}$

Example



POLAR FORM

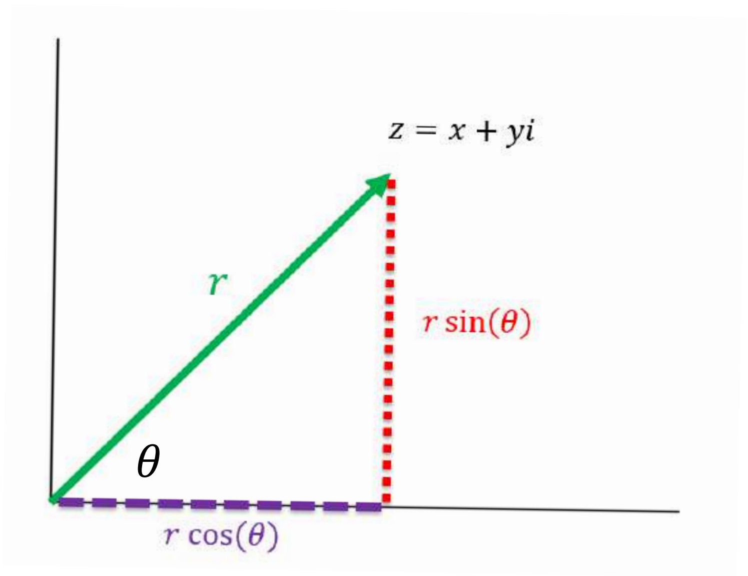
- Earlier, we had been dealing with complex numbers in cartesian form (distinct real and imaginary parts)

$$z = a + bi$$

- Cartesian form of complex numbers are useful for addition and subtraction. But what if we want to multiply, divide and take powers of complex numbers?
- Imagine evaluating $(2 - 3i)^8$ by hand. (Big yikes)
- But with polar form of complex numbers, we can multiple, divide and take powers and roots of complex numbers!

POLAR FORM

- The polar form of a complex number is $z = r\text{cis}(\theta)$
- Given $z = x + yi, r = |z| = \sqrt{x^2 + y^2}$
- $\theta = \text{Arg}(z)$ (principal argument of z)



$$r\text{cis}(\theta) = \underbrace{r\cos(\theta)}_{x} + \underbrace{r\sin(\theta)}_{y}\, i$$

ARGUMENT

- The argument (measured in radians for spesh) is effectively the angle θ, measured anticlockwise from the positive $\mathrm{Re}(z)$ axis.
- $\arg(z)\ vs.\ \mathrm{Arg}(\mathrm{z})$?
- Consider $\mathrm{cis}\left(\frac{\pi}{6}\right)$ and $\mathrm{cis}\left(\frac{13\pi}{6}\right)$

ARGUMENT

- Clearly $\operatorname{cis}\left(\frac{\pi}{6}\right) = \operatorname{cis}\left(\frac{13\pi}{6}\right)$
- Therefore, by extension there are infinitely many ways you can represent a complex number in polar form.
- To avoid this, we introduce the concept of 'Principal argument'
- The principal argument is just the argument but with a restricted range essentially.
- We restrict it to $\theta \in (-\pi, \pi]$
- Because we restrict the **principal** argument $\operatorname{Arg}(z)$, it becomes **unique** (i.e. there is only **one** allowed value for any complex number $z \in \mathbb{C}$).

COVERSION TO POLAR

Given $z = x + yi$, convert this to polar form ($z = r\text{cis}(\theta)$).

Method

1. Find $r = |z| = \sqrt{x^2 + y^2}$
2. Find $\theta = \text{Arg}(z)$ $\in (-\pi, \pi]$
3. Put it altogether to achieve

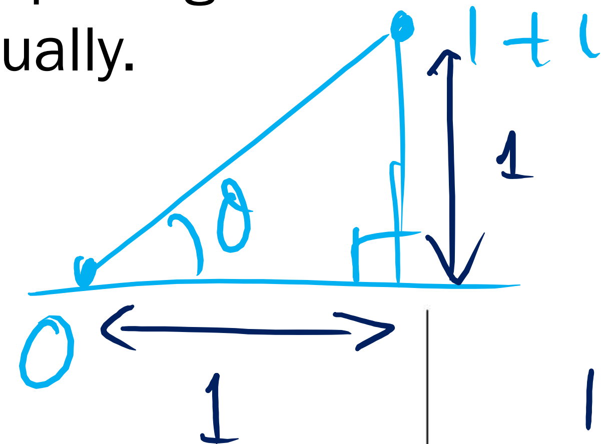
$$z = r \underline{\text{cis}(\theta)}$$

$(\cos\theta + i\sin\theta))$

CALCULATING ARGUMENT

$\tan\theta = 1$

- The best way to calculate the principal argument of a complex number is to look at it visually.

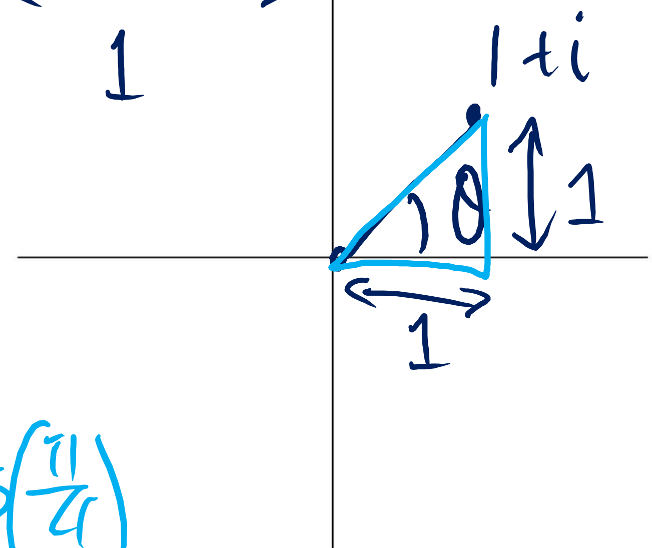


Convert the following to polar form

Example 1:

$1 + i$

$= r \operatorname{cis} \theta$



$r = |1+i| = \sqrt{2}$

$\theta = \frac{\pi}{4}$

$\therefore 1+i = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

CONVERSION TO POLAR

Example 2

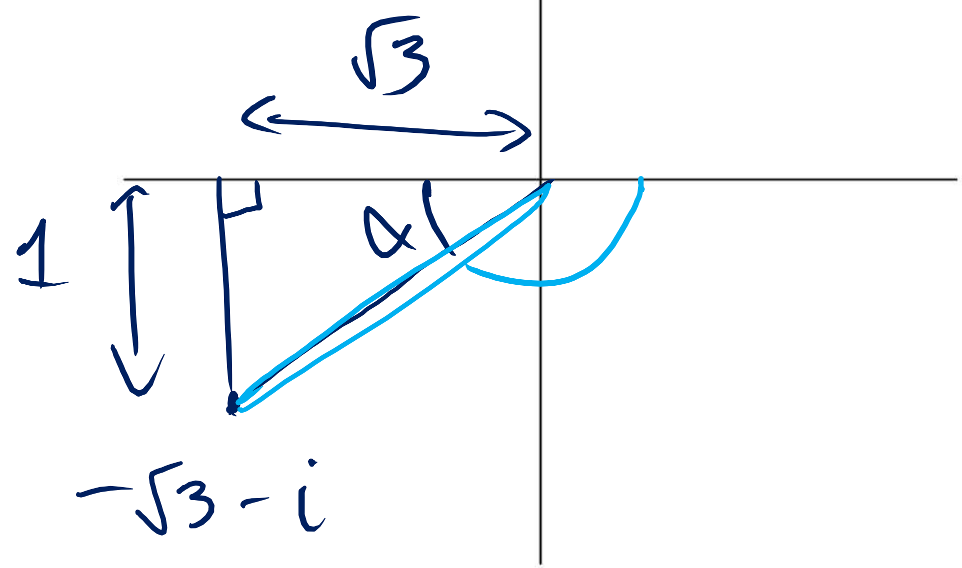
$\operatorname{Arg}(z) \in (-\pi, \pi]$

$z = -\sqrt{3} - i$

$\tan(\alpha) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \alpha = \frac{\pi}{6}$

$\operatorname{Arg}(z) = -\pi + \frac{\pi}{6}$

$= -\frac{5\pi}{6}$



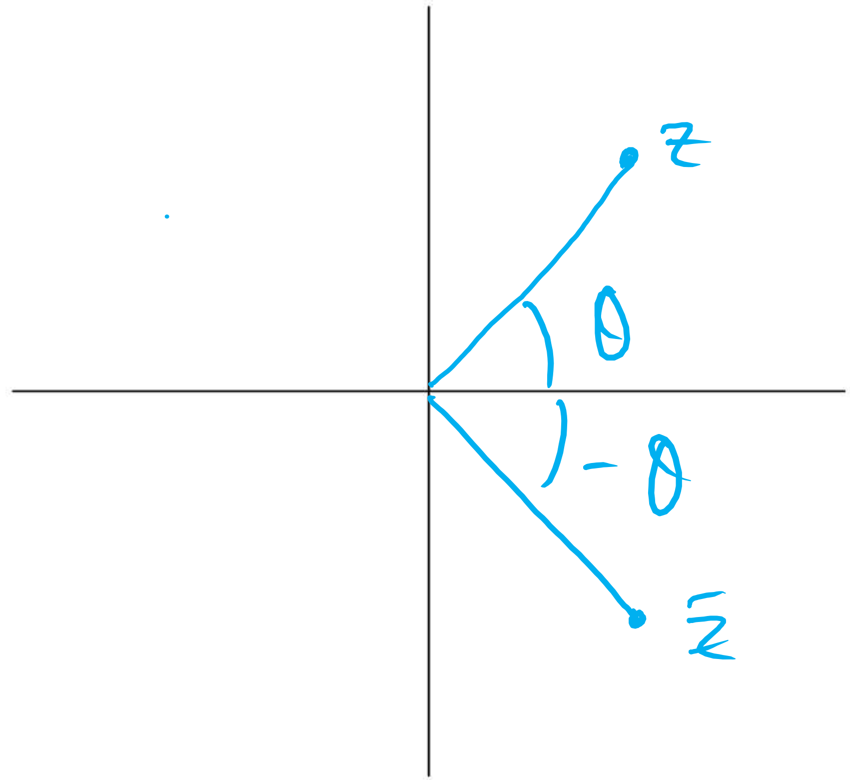
$\therefore -\sqrt{3} - i = 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

COMPLEX CONJUGATE

- The conjugate of $z = r\mathrm{cis}(\theta)$ is $z = r\mathrm{cis}(-\theta)$

$z = x + yi$

$\bar{z} = x - yi$



POLAR FORM

- Recognise that

$$\mathrm{cis}(\theta) = \underbrace{\cos(\theta)}_{x} + \underbrace{\sin(\theta)\, i}_{y}$$

- Hence, trigonometry is very closely related to complex numbers.
- Recall that $\underline{\cos(-\theta) = \cos(\theta)\ \&\ \sin(-\theta) = -\sin(-\theta)}$
- Hence,

$$\cos(\theta) - \sin(\theta)\, i = \mathrm{cis}(-\theta)$$

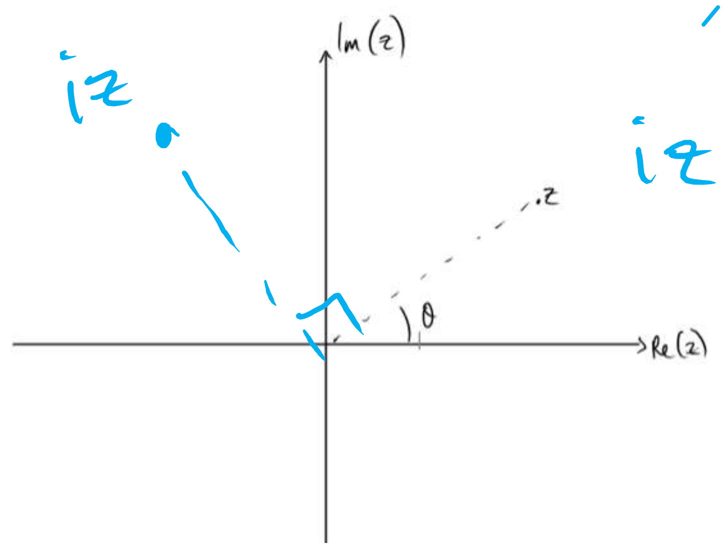
OPERATIONS IN POLAR FORM

- Adding/subtracting →say goodbye to polar and hello to cartesian form
- Multiplying:
$$r_1\text{cis}(\theta_1)\times r_2\text{cis}(\theta_2) = r_1 r_2\text{cis}(\theta_1 + \theta_2)$$
- Dividing:
$$\frac{r_1\text{cis}(\theta_1)}{r_2\text{cis}(\theta_2)} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$$

- $\frac{1}{z} = \frac{1}{r}cis(-\theta)$
- $\bar{z} = rcis(-\theta)$

MULTIPLICATION BY i

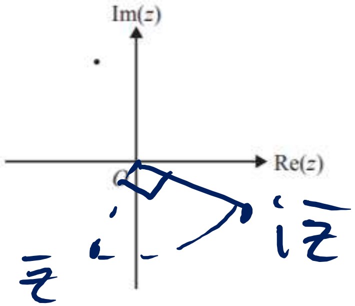
- What happens if we multiply a complex number by i?
- Since $i = \text{cis}\left(\frac{\pi}{2}\right)$
- Then $\underbrace{r\text{cis}(\theta)}_{z}\times i = r\text{cis}(\theta)\times\text{cis}\left(\frac{\pi}{2}\right) = r\text{cis}\left(\theta + \frac{\pi}{2}\right)$
- This is a rotation of 90° anticlockwise ☺



VCAA 2010

$i \times \bar{z}$

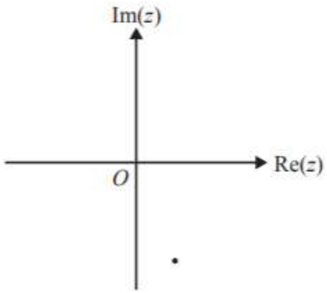
A particular complex number z is represented by the point on the following argand diagram.



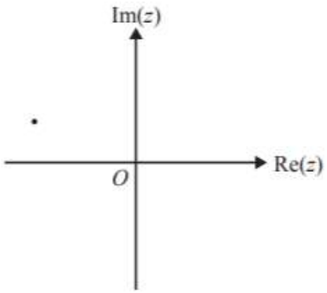
All axes below have the **same scale** as those in the diagram above.

The complex number $i\bar{z}$ is best represented by

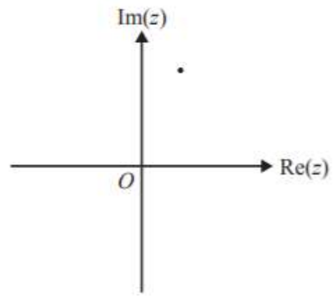
A.



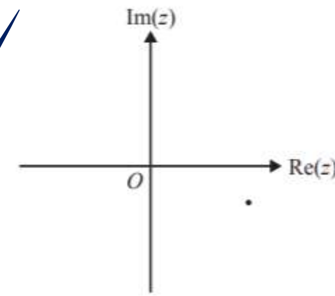
B.



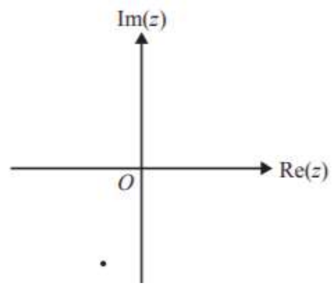
C.



D.



E.



DE MOIVRE'S THEOREM

- This theorem allows us to take powers of a complex number

$$z = r\text{cis}(\theta)$$
$$z^2 = \big(r\text{cis}(\theta)\big)\times\big(r\text{cis}(\theta)\big) = r^2\text{cis}(2\theta)$$
$$z^3 = \big(r\text{cis}(\theta)\big)\times\big(r\text{cis}(\theta)\big)\times\big(r\text{cis}(\theta)\big) = r^3\text{cis}(3\theta)$$
$$\ldots$$
$$\boldsymbol{z^n = \cdots = r^n\mathbf{cis}(n\theta)}$$

ROOTS OF COMPLEX NUMBERS

- We use the polar form of complex numbers to obtain the powers of them (e.g. z^7)
- We can also use the polar form of complex numbers to obtain the roots of them too (e.g. $z^{\frac{1}{2}}, z^{\frac{1}{3}}$)
- For any number, x, we have two square roots: $+\sqrt{x}$ and $-\sqrt{x}$
- What are the cube roots of 1? The cube roots must satisfy the equation

$$z^3 = 1$$

$P(z) = z^3 - 1$

$z^3 - 1 = 0$

$$z = 1, z = -\frac{1}{2} + \frac{3}{2}i, z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

FUNDAMENTAL THM OF ALGEBRA

- You are expected to know the 'idea' of this theorem

'A polynomial of nth degree will have n number of roots'

ROOTS OF COMPLEX NUMBERS

- So how do we calculate what these roots are?
- We employ De Moivre's theorem

(My) Method

- Set up an equation in the form $z^n = a$
- Write a in polar form and set $z = r\text{cis}(\theta), \therefore z^n = r^n\text{cis}(n\theta)$
- Equate the modulus
- Equate $n\theta = \text{Arg}(a) + 2k\pi$, $k \in Z$
- Sub in values of k until you get the desired number of roots/solutions
- Make sure your final answers have arguments in the restricted interval $(-\boldsymbol{\pi}, \boldsymbol{\pi}]$!

ROOTS OF COMPLEX NUMBERS

$P(z) = z^3 - 1$, $P(z) = 0$

Example

Solve $z^3 = 1$ on $\mathbb{C}$ giving your answers in cartesian form.

$1 = 1\,\text{cis}(0)$ $\quad Arg(z) \in (-\pi, \pi]$

let $z = r\,\text{cis}(\theta)$

$z^3 = r^3\text{cis}(3\theta)$

$r^3\text{cis}(3\theta) = 1\,\text{cis}(0)$

$r^3 = 1$, $r = 1$

$3\theta = 0 + 2k\pi$, $k \in \mathbb{Z}$

$k = 0$: $3\theta = 0$
$\theta = 0$

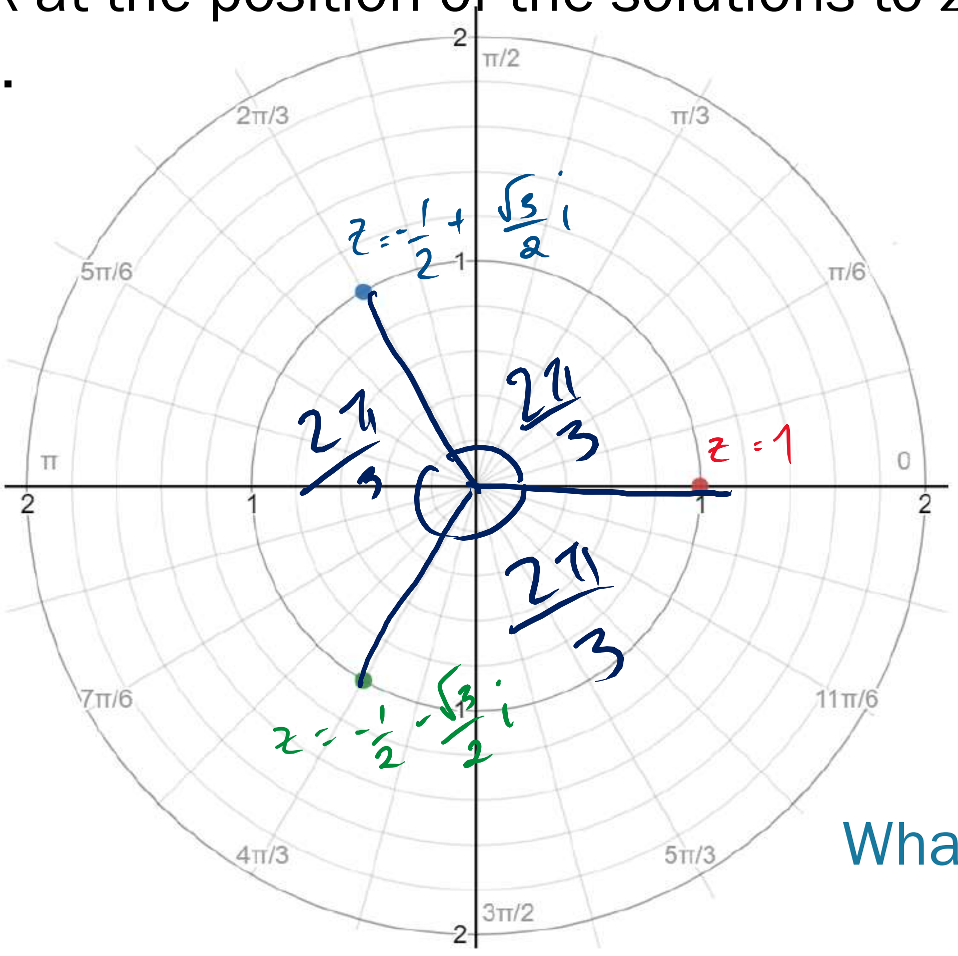
$k = 1$: $3\theta = 2\pi$
$\theta = \frac{2\pi}{3}$

$k = -1$: $3\theta = -2\pi$
$\theta = \frac{-2\pi}{3}$

$\therefore z = \text{cis}(0), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}\left(\frac{-2\pi}{3}\right)$

ROOTS OF COMPLEX NUMBERS

Now let's look at the position of the solutions to $z^3 = 1$ on the Argand plane.



What do we notice?

ROOTS OF COMPLEX NUMBERS

For any equation in the form

$$z^n = a + bi$$

- There will be n number of solutions
- And each solution will be $\frac{2\pi}{n}$ radians apart from each other

$$z^3 + 2z + 1 = 0$$

~~sol are $\frac{2\pi}{3}$ rad apart~~

COMPLEX POLYNOMIAL

- In Methods, we learnt about polynomials over the real number plane.

- In Spesh, we learn about polynomials on the complex plane (don't worry, you won't have to sketch it. you just need to be able to factorise and solve these polynomials)

COMPLEX POLYNOMIALS

Recall the fundamental thm of algebra:

'A polynomial of nth degree will have n number of roots'

Conjugate factor theorem

- If a polynomial has real coefficients, and $(\mathrm{z}-\mathrm{a})$ is a factor, then $(\mathrm{z}-\bar{\mathrm{a}})$ must also be a factor.

- In other words, for real coefficient polynomials, roots must occur in conjugate pairs!

VCAA 2009

real coef

$\swarrow$ $\implies$ conjugate root thm

The polynomial equation $P(z) = 0$ has real coefficients, and has roots which include $\underbrace{z = -2 + i}$ and $\underbrace{z = 2}$.

$z = -2 - i$

The **minimum degree** of $P(z)$ would be

A. 1

B. 2

C. 3

D. 4

E. 5

SOLVING OVER $\mathbb{C}$

- Solving over $\mathbb{C}$ is pretty much the same as solving over $\mathbb{R}$ of polynomials like you were taught in Methods.
- Please remember methods for solving quadratics!

- However, please recognise that

$$\sqrt{x^2} = |x| = \pm x$$

- The idea of modulus is extremely ubiquitous in complex numbers and you will need it later on for calculus.

VCAA 2013 (MODIFIED)

Find the intersections of the following relations, expressing your answer(s) in the form $a + b\boldsymbol{i}$

circle

$$z\bar{z} = 4$$

$$\underbrace{|z+\bar{z}| = |z-\bar{z}|}_{\text{line}} \quad (2)$$

Where $z \in C$.

let $z = x + yi$

$z + \bar{z} = 2x$

$z - \bar{z} = 2yi$

(2): $|2x| = |2yi|$

$\sqrt{x^2} = \sqrt{y^2} \Rightarrow |x| = |y|$

$\pm x = \pm y$

$y = x, \ y = -x$

SKETCHING IN THE ARGAND PLANE

- You will need to learn how to draw/interpret points, rays, lines and circles on the Argand plane.

- Note that regions in the complex plane is no longer covered in the study design so you won't be expected to sketch regions (they're gross anyways).

NOTE

- For relations in the complex plane, you *only* deal with equalities.

c. On the Argand diagram ...w, shade the region defin... y

$\{z : A \ldots < \text{Arg}(z_1^4) \ldots \leq 2\},\ z \in C.$

2012

e. **i.** Shade the region $\{z : | z - \ldots | \leq 1,\ z \in C\}$ on the argand diagram in **part c.**

2011

b. Find all elements of $\{ \ldots |z + \ldots$ ...ressing your answer(s) in the form a ...

3 marks

CIRCLES

- A circle with radius r and centred at a is of the form

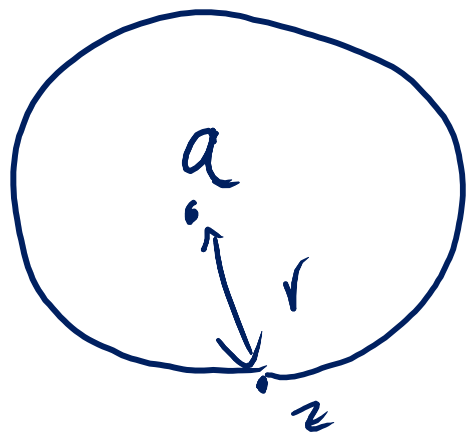
$$|z-a|=r$$

$x^2+y^2=r^2$

- For example…

$|z|=1$ specifies the unit circle

- $|z-i|=2$ is a circle centred at $i \equiv (0,1)$, of radius 2 (**<u>not</u>** $\sqrt{\mathbf{2}}$)



Alternatively,

$$z\bar{z}=r^2$$

defines a circle centred at $(0,0)$ with radius r.

$$(z-a)(\bar{z}-\bar{a})=r^2$$

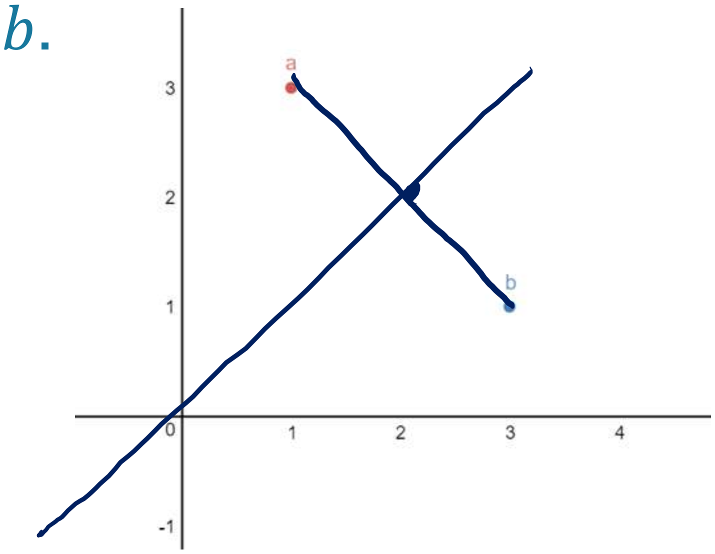
defines a circle centred at a with radius r.

LINES

This takes on the form of

$$|z - a| = |z - b|$$

What does this look like? The line is the perpendicular bisector of the line segment joining a to b.



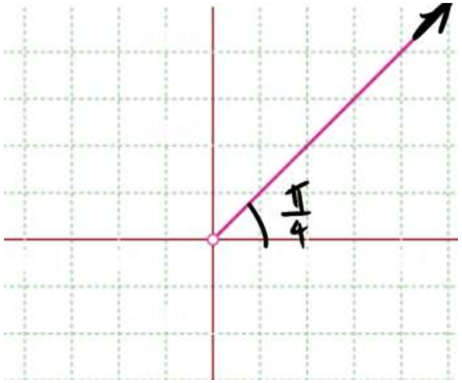
Simply knowing this is a line is not good enough, you have to be able to understand it is the perpendicular bisector

RAYS

$\mathrm{Arg}(z - (0+0i))$

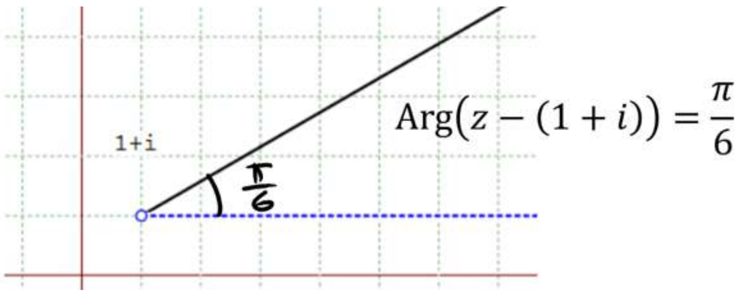
- Rays are defined by equations of the form $\mathrm{Arg}(z) = \theta$.
- $\mathrm{Arg}(z) = \theta$ is a ray beginning at the origin, angled at θ relative to the positive $\mathrm{Re}(z)$ axis.
- **Always an open hole at the origin – because $\mathbf{Arg}(\mathbf{0} + \mathbf{0}\boldsymbol{i})$ undefined**

$$\mathrm{Arg}(z) = \frac{\pi}{4}$$



RAYS

- More generally, we can consider equations of the form $\mathrm{Arg}(\mathrm{z} - \mathrm{a}) = \theta$
- Ray begins at point a (open circle here!)
- Ray is at angle θ relative to the positive $\mathrm{Re}(z)$ axis



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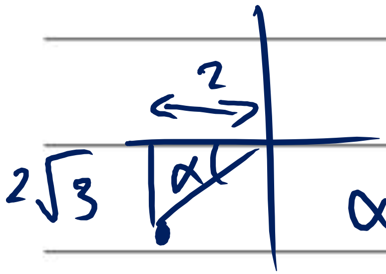
Question 4 (10 marks)

a. Express $-2-2\sqrt{3}i$ in polar form. 1 mark

$-2-2\sqrt{3}i = r\,\text{cis}\,\theta$

$r = |-2-2\sqrt{3}i| = 4$



$\theta = -\pi + \alpha \;,\; \frac{-2\pi}{3}$

$\alpha = \frac{\pi}{3}$

$\therefore -2-2\sqrt{3}i = 4\,\text{cis}\left(\frac{-2\pi}{3}\right)$

b. Show that the roots of $z^2 + 4z + 16 = 0$ are $z = -2-2\sqrt{3}i$ and $z = -2+2\sqrt{3}i$. 1 mark

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c. Express the roots of $z^2 + 4z + 16 = 0$ in terms of $2 - 2\sqrt{3}i$. 1 mark

$z = -2 - 2\sqrt{3}i, \ -2 + 2\sqrt{3}i$

$-(2 - 2\sqrt{3}i), \ -\overline{(2 - 2\sqrt{3}i)}$

d. Show that the cartesian form of the relation $|z| = |z - (2 - 2\sqrt{3}i)|$ is $x - \sqrt{3}y - 4 = 0$. 2 marks

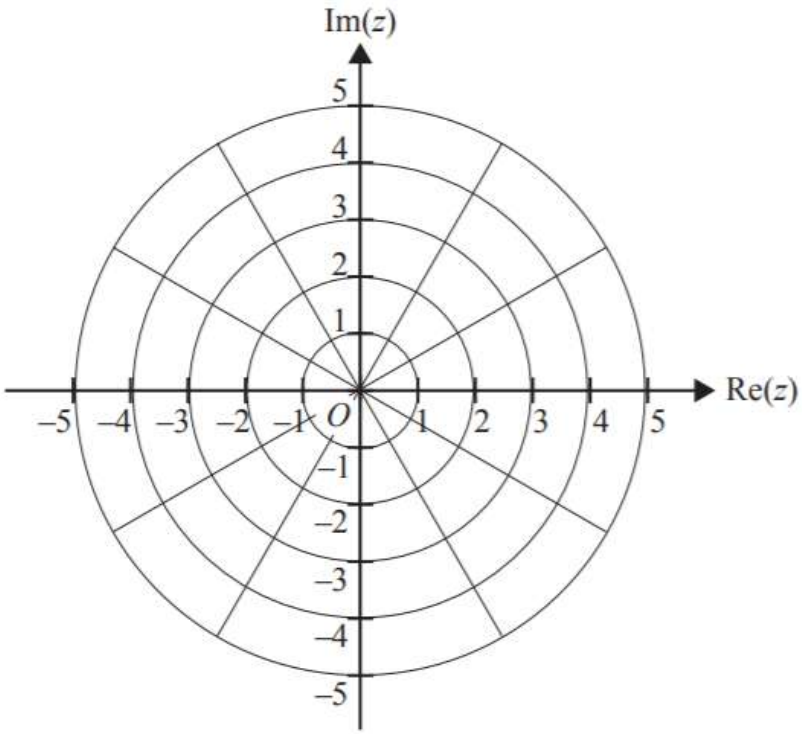
Let $z = x + yi$

$|(x-2) + (y + 2\sqrt{3})i|$

$\sqrt{x^2 + y^2} = \sqrt{(x-2)^2 + (y + 2\sqrt{3})^2}$

$\cancel{x^2} + \cancel{y^2} = \cancel{x^2} - 4x + 4 + \cancel{y^2} + 4\sqrt{3}y + 12$

e. Sketch the line represented by $x-\sqrt{3}y-4=0$ and plot the roots of $z^2+4z+16=0$ on the Argand diagram below. 2 marks



f. The equation of the line passing through the two roots of $z^2+4z+16=0$ can be expressed as $|z-a|=|z-b|$, where $a, b \in C$.

Find b in terms of a. 1 mark

COMPLEX NUMBER REVIEW

- What did we cover today?

Complex numbers, including:

- C, the set of numbers z of the form $z = x + yi$ where x, y are real numbers and $i^2 = -1$, real and imaginary parts, complex conjugates, modulus
- use of an argand diagram to represent points, lines, rays and circles in the complex plane
- equality, addition, subtraction, multiplication and division of complex numbers
- polar form (modulus and argument); multiplication and division in polar form, including their geometric representation and interpretation, proof of basic identities involving modulus and argument
- De Moivre's theorem, proof for integral powers, powers and roots of complex numbers in polar form, and their geometric representation and interpretation
- n^{th} roots of unity and other complex numbers and their location in the complex plane
- factors over C of polynomials with integer coefficients; and informal introduction to the fundamental theorem of algebra
- factorisation of polynomial functions of a single variable over C, for example, $z^8 + 1$, $z^2 - i$, $z^3 - (2 - i)z^2 + z - 2 + i$
- solution over C of corresponding polynomial equations by completing the square, factorisation and the conjugate root theorem.

ATARNotes

QUESTIONS?