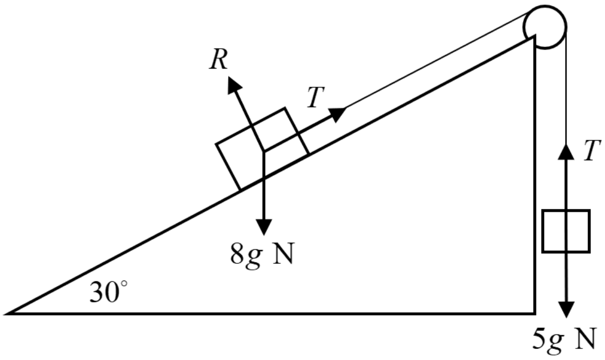
Victorian Certificate of Education – 2018 VCAA Examinations

SPECIALIST MATHEMATICS

Written Examination 1

PROVISIONAL SOLUTIONS

Question 1a (1 mark)



Question 1b (3 marks)

Define positive acceleration as up the plane.

8 kg mass: $8a = T - 8g\sin(30^\circ)$

5 kg mass: $5a = 5g - T$

Therefore, $13a = 5g - 4g$

$= g$

Hence, the 8 kg mass accelerates up the plane at

$\frac{g}{13}$ ms^{-2}.

Question 2a (1 mark)

$|1+i| = \sqrt{1+1} = \sqrt{2}$

$\tan(\text{Arg}(1+i)) = 1$

$\text{Arg}(1+i) = \frac{\pi}{4} \quad (\text{Re}(1+i) = \text{Im}(1+i) > 0)$

Hence, $1+i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$, as required.

Question 2b (3 marks)

$\left|\sqrt{3}-i\right| = \sqrt{3+1} = 2$

$\tan\left(\text{Arg}\left(\sqrt{3}-i\right)\right) = \frac{-1}{\sqrt{3}}$

$\text{Arg}\left(\sqrt{3}-i\right) = -\frac{\pi}{3} \quad \left(\text{Re}\left(\sqrt{3}-i\right) > 0\right)$

Therefore, $\sqrt{3}-i = 2\,\text{cis}\left(-\frac{\pi}{6}\right)$.

$$\frac{\left(\sqrt{3}-i\right)^{10}}{(1+i)^{12}} = \frac{\left[2\,\text{cis}\left(-\frac{\pi}{6}\right)\right]^{10}}{\left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right]^{12}}$$

$$= \frac{2^{10}\,\text{cis}\left(\frac{\pi}{3}\right)}{2^{6}\,\text{cis}(\pi)}$$

$$= 16\,\text{cis}\left(-\frac{2\pi}{3}\right)$$

$$= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -8 - 8\sqrt{3}i$$

Question 3 (4 marks)

$$\frac{d}{dx}\left[2x^2\right]\sin(y)+2x^2\frac{d}{dy}[\sin(y)]\frac{dy}{dx}+\frac{d}{dx}[x]y$$
$$+x\frac{d}{dy}[y]\frac{dy}{dx}=0$$

$$4x\sin(y)+2x^2\cos(y)\frac{dy}{dx}+y+x\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left(2x^2\cos(y)+x\right)=-4x\sin(y)-y$$

$$\frac{dy}{dx}=\frac{-4x\sin(y)-y}{2x^2\cos(y)+x}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{6},\frac{\pi}{6}\right)}=\frac{\frac{-2\pi}{3}\cdot\frac{1}{2}-\frac{\pi}{6}}{\frac{\pi^2}{18}\cdot\frac{\sqrt{3}}{2}+\frac{\pi}{6}}$$
$$=\frac{\frac{-1}{2}}{\frac{\pi\sqrt{3}+6}{36}}$$
$$=\frac{-18}{\pi\sqrt{3}+6}$$

Question 4 (4 marks)

$\mathrm{E}(X)=2, \mathrm{Var}(X)=2, \mathrm{E}(Y)=2, \mathrm{Var}(Y)=4.$

$$\begin{cases}a\mathrm{E}(X)+b\mathrm{E}(Y)=10\\ a^2\mathrm{Var}(X)+b^2\mathrm{Var}(Y)=44\end{cases}$$

$$\Rightarrow\begin{cases}2a+2b=10\\ 2a^2+4b^2=44\end{cases}$$

Therefore, $2a^2+4(5-a)^2=44.$

$$2a^2+4\left(a^2-10a+25\right)=44$$
$$6a^2-40a+56=0$$
$$3a^2-20a+28=0$$
$$(a-2)(3a-14)=0$$

$a=2$ only $\quad (a,b\in\mathbb{Z}).$

$b=3.$

Question 5 (4 marks)

$$f(x)=\frac{x+1}{x^2-4}$$

The denominator of f vanishes at $x=\pm 2.$

As $x\to\pm\infty$, $f(x)\to 0.$

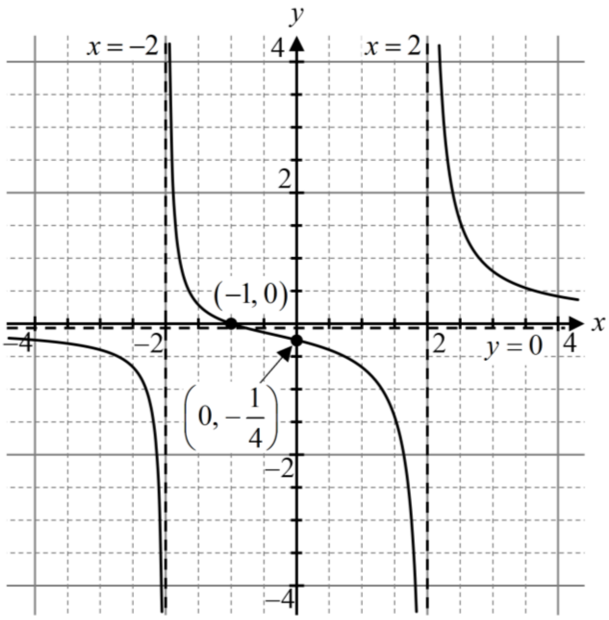
As $x\to -2^-$, $f(x)\to -\infty.$

As $x\to -2^+$, $f(x)\to \infty.$

As $x\to 2^-$, $f(x)\to -\infty.$

As $x\to 2^+$, $f(x)\to \infty.$

$f(0)=-\frac{1}{4}$ and $f(x)=0$ when $x=-1.$



Question 6 (3 marks)

$$\underset{\sim}{\mathrm{r}}(t)=\sin(t)\underset{\sim}{\mathrm{i}}+\cos(t)\underset{\sim}{\mathrm{j}}+t^2\underset{\sim}{\mathrm{k}}$$

$$\underset{\sim}{\mathrm{p}}(t)=2\dot{\underset{\sim}{\mathrm{r}}}(t)$$
$$=2\left(\cos(t)\underset{\sim}{\mathrm{i}}-\sin(t)\underset{\sim}{\mathrm{j}}+2t\underset{\sim}{\mathrm{k}}\right)$$
$$=2\cos(t)\underset{\sim}{\mathrm{i}}-2\sin(t)\underset{\sim}{\mathrm{j}}+4t\underset{\sim}{\mathrm{k}}$$

$$\Delta\underset{\sim}{\mathrm{p}}=\underset{\sim}{\mathrm{p}}(\pi)-\underset{\sim}{\mathrm{p}}\left(\frac{\pi}{2}\right)$$
$$=\left(-2\underset{\sim}{\mathrm{i}}+4\pi\underset{\sim}{\mathrm{k}}\right)-\left(-2\underset{\sim}{\mathrm{j}}+2\pi\underset{\sim}{\mathrm{k}}\right)$$
$$=-2\underset{\sim}{\mathrm{i}}+2\underset{\sim}{\mathrm{j}}+2\pi\underset{\sim}{\mathrm{k}}\ \text{ kg ms}^{-1}$$

Question 7 (3 marks)

$$\cot(2x)+\frac{1}{2}\tan(x)=\frac{1-\tan^2(x)}{2\tan(x)}+\frac{1}{2}\tan(x)$$
$$=\frac{1-\tan^2(x)+\tan^2(x)}{2\tan(x)}$$
$$=\frac{1}{2}\cot(x)$$

Hence, $a=\frac{1}{2}.$

Question 8a (1 mark)

$\frac{dQ_{\text{in}}}{dt} = 0 \cdot 5 = 0$

Let $V(t)$ = volume of solution.

$V(t) = \int_0^t (5-3)\,dw + 16$

$= [2w]_0^t + 16$

$= 2t + 16$

$\frac{dQ_{\text{out}}}{dt} = \frac{Q}{V(t)} \cdot 3$

$= \frac{3Q}{16+2t}$

Hence, $\frac{dQ}{dt} = \frac{-3Q}{16+2t}$, as required.

Question 8b (3 marks)

$\int_{\frac{1}{2}}^{Q} \frac{-1}{3w}\,dw = \int_0^t \frac{1}{16+2w}\,dw$

$\left[\frac{-1}{3}\log_e(w)\right]_{\frac{1}{2}}^{Q} = \left[\frac{1}{2}\log_e(16+2w)\right]_0^t$

$(Q > 0 \text{ and } t \geq 0)$

$\frac{-1}{3}\log_e(Q) + \frac{1}{3}\log_e\left(\frac{1}{2}\right) = \frac{1}{2}\log_e(16+2t)$

$-\frac{1}{2}\log_e(16)$

$\log_e\left[\left(\frac{1}{2Q}\right)^{\frac{1}{3}}\right] = \log_e\left[\left(\frac{16+2t}{16}\right)^{\frac{1}{2}}\right]$

$\frac{1}{2Q} = \frac{(16+2t)^{\frac{3}{2}}}{64}$

Hence, $Q(t) = \frac{32}{(16+2t)^{\frac{3}{2}}}$.

Question 9a (2 marks)

$x = \sec(t)$ and $y = \frac{1}{\sqrt{2}}\tan(t)$.

$\begin{cases} \sec^2(t) = x^2 \\ \tan^2(t) = 2y^2 \end{cases}$

Therefore, $x^2 - 2y^2 = \sec^2(t) - \tan^2(t)$.

Hence, $x^2 - 2y^2 = 1$, as required.

Question 9b (1 mark)

$x^2 - 2(x-1)^2 = 1$

$x^2 - 2(x^2 - 2x + 1) = 1$

$-x^2 + 4x - 3 = 0$

$(x-1)(x-3) = 0$

$x = 1, 3$

Question 9c (2 marks)

Let V = required volume.

$y^2 = \frac{x^2-1}{2}$

$V = \pi\int_1^3 \left(\frac{x^2-1}{2} - (x-1)^2\right)dx$

$= \pi\int_1^3 \left(\frac{x^2-1}{2} - (x^2 - 2x + 1)\right)dx$

$= \pi\int_1^3 \left(\frac{-x^2}{2} + 2x - \frac{3}{2}\right)dx$

$= \pi\left[\frac{-x^3}{6} + x^2 - \frac{3x}{2}\right]_1^3$

$= \pi\left[-\frac{9}{2} + 9 - \frac{9}{2} - \left(-\frac{1}{6} + 1 - \frac{3}{2}\right)\right]$

$= \frac{2\pi}{3}$ units^3

Question 10 (5 marks)

$\underset{\sim}{r}(t) = \frac{t^3}{3}\underset{\sim}{i} + \left(\arcsin(t) + t\sqrt{1-t^2}\right)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = t^2\underset{\sim}{i} + \left(\frac{1}{\sqrt{1-t^2}} + \sqrt{1-t^2} + \frac{-t^2}{\sqrt{1-t^2}}\right)\underset{\sim}{j}$

$= t^2\underset{\sim}{i} + \left(\frac{2-2t^2}{\sqrt{1-t^2}}\right)\underset{\sim}{j}$

$= t^2\underset{\sim}{i} + 2\sqrt{1-t^2}$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{t^4 + 4(1-t^2)}$

$= \sqrt{t^4 - 4t^2 + 4}$

$= \sqrt{(t^2-2)^2}$

$= |t^2 - 2|$

Since $0 < t < 1$, $|\dot{\underset{\sim}{r}}(t)| = 2 - t^2$.

Hence, $a = -1$, $b = 0$, $c = 2$.

END OF PROVISIONAL SOLUTIONS