Neap:

Trial Examination 2018

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

Question	Answer	Question	Answer
1	C	11	E
2	E	12	D
3	A	13	A
4	D	14	A
5	B	15	E
6	B	16	C
7	D	17	E
8	B	18	B
9	C	19	D
10	C	20	A

Question 1 C

$-1 \le \sin(x) \le 1$ and so, $-\frac{3}{2} \le \sin(x) - \frac{1}{2} \le \frac{1}{2}$.

Given that $0 \le \left|\sin(x) - \frac{1}{2}\right| \le \frac{3}{2}$, the maximum value of g is $\frac{3}{2}$.

Alternatively, a CAS could be used to graph of $y = g(x)$ and the maximum value obtained from the graph.

Question 2 E

The range of $y = \cos^{-1}(x)$ is $0 \le y \le \pi$.

From the graph, the range of $y = a\cos^{-1}\left(\frac{x}{5} - 1\right)$ is $0 \le y \le 10$.

So, $a\pi = 10 \Rightarrow a = \frac{10}{\pi}$.

Question 3 A

The range of f is $2 \le y < \infty$.

Interchange x and y and solve for y:

$$x = \sec\left(\frac{y}{2}\right) + 1$$

$$\frac{1}{x-1} = \cos\left(\frac{y}{2}\right)$$

$$\cos^{-1}\left(\frac{1}{x-1}\right) = \frac{y}{2}$$

$$\Rightarrow f^{-1}(x) = 2\cos^{-1}\left(\frac{1}{x-1}\right), 2 \le x < \infty$$

Question 4 D

$z^3 - 3z^2 + z - 3 = (z-3)(z-i)(z+i)$ either by using a CAS or by using by-hand factorisation (see below).

$$\begin{aligned} z^3 - 3z^2 + z - 3 &= z^2(z-3) + 1(z-3) \\ &= (z-3)(z^2+1) \\ &= (z-3)(z-i)(z+i) \end{aligned}$$

So, a linear factor of $P(z)$ is $z + i$.

Question 5 B

The set, S, consists of all points in the complex plane that are equidistant from 0 and $-a$.

In the Cartesian plane, this set corresponds to the perpendicular bisector of the line segment joining (0, 0) and $(-a, 0)$.

The midpoint of the line segment is $\left(-\frac{a}{2}, 0\right)$ and so the equation of the perpendicular bisector is $x = -\frac{a}{2}$; that is, $\text{Re}(z) = -\frac{a}{2}$.

Question 6 B

Using de Moivre's theorem obtains $r^n \text{cis}(n\theta) = r_1 \text{cis}(\alpha)$.

Comparing moduli, obtains $r^n = r_1 \Rightarrow r = (r_1)^{\frac{1}{n}}$.

Comparing arguments, obtains $\text{cis}(n\theta) = \text{cis}(\alpha)$.

$n\theta = \alpha + 2k\pi$ where $k \in Z$.

$\theta = \frac{1}{n}(\alpha + 2k\pi)$

So, $(r_1)^{\frac{1}{n}} \text{cis}\left(\frac{1}{n}(\alpha + 2k\pi)\right)$.

Question 7 D

One approach is to use the expand command of a CAS.

$$\frac{x-3}{(x-1)^2(x^2+1)} = \frac{-3x}{2(x^2+1)} - \frac{1}{2(x^2+1)} + \frac{3}{2(x-1)} - \frac{1}{(x-1)^2}$$

The *RHS* can be expressed as:

$\frac{-3x-1}{2(x^2+1)} + \frac{3}{2(x-1)} - \frac{1}{(x-1)^2}$, which is of the form $\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$.

Note the following:

If the factor in the denominator is $(ax+b)^n$, then the corresponding term(s) in the partial fraction decomposition is $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$.

Here $(x-1)^2$ corresponds to $\frac{3}{2(x-1)} - \frac{1}{(x-1)^2}$.

If the factor in the denominator is $ax^2 + bx + c$, then the term in the partial fraction decomposition is $\frac{Ax+B}{ax^2+bx+c}$.

Here (x^2+1) corresponds to $\frac{-3x-1}{2(x^2+1)}$.

Question 8 B

$$\int_0^{\frac{\pi}{6}} \cot\left(\frac{\pi}{2} - 2x\right) dx = \int_0^{\frac{\pi}{6}} \tan(2x) dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sin(2x)}{\cos(2x)} dx$$

If $u = \cos(2x)$, then $\frac{du}{dx} = -2\sin(2x)$ and so $\sin(2x) = -\frac{1}{2}\frac{du}{dx}$.

When $x = 0$, $u = 1$ and when $x = \frac{\pi}{6}$, $u = \frac{1}{2}$.

So we obtain $-\int_1^{\frac{1}{2}} \frac{1}{2u} du = \int_{\frac{1}{2}}^1 \frac{1}{2u} du$.

Alternatively, the substitution $u = \sin\left(\frac{\pi}{2} - 2x\right)$ can also be used.

Question 9 C

$A = 4\pi r^2 \Rightarrow r = \sqrt{\frac{A}{4\pi}}$

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4\pi}{3}\left(\frac{A}{4\pi}\right)^{\frac{3}{2}}$$

$$= \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}}$$

$$\frac{dV}{dA} = \frac{1}{4}\sqrt{\frac{A}{\pi}}$$

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$= \frac{1}{4}\sqrt{\frac{A}{\pi}}\frac{dA}{dt}$$

Question 10 C

At (0, 0), $\frac{dy}{dx} = 0$, and so **A** is not correct.

At (–1, 1), $\frac{dy}{dx} = 0$, and so **B**, **D** and **E** are not correct.

Question 11 E

Let the arc length be L.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{\cos(\sqrt{x})}{2\sqrt{x}}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{\cos^2\sqrt{x}}{4x}$$

Substituting this into the arc length formula gives $L = \int_a^b \sqrt{1 + \frac{\cos^2\sqrt{x}}{4x}}\, dx$.

Question 12 D

If the number of people who have been infected is P, then the number of people who have not been infected is $N - P$.

So, $\frac{dP}{dt} = k(P(N - P)) = kP(N - P)$.

Question 13 A

The most efficient approach is to use a CAS differential equation solver feature.

Solving $\frac{dy}{dx} = \sqrt{4 - y^2}$ with $y\left(\frac{\pi}{6}\right) = 1$ gives $\sin^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6} = x - \frac{\pi}{6}$ (or equivalent).

Solving $\sin^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6} = x - \frac{\pi}{6}$ for y gives $y = 2\sin(x)$.

A by-hand approach requires solving $\frac{dx}{dy} = \frac{1}{\sqrt{4 - y^2}}$ to obtain $x = \sin^{-1}\left(\frac{y}{2}\right) + c$, using the given condition to find the value of c and then re-arranging to express y in terms of x.

Question 14 A

The particle's direction of motion is given by the unit vector in the direction of the velocity vector.

$$\underset{\sim}{r}'(t) = \sec^2(t)\underset{\sim}{i} + 2\tan(t)\sec^2(t)\underset{\sim}{j}$$

$$\begin{aligned}\underset{\sim}{r}'\left(\frac{\pi}{4}\right) &= \sec^2\left(\frac{\pi}{4}\right)\underset{\sim}{i} + 2\tan\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right)\underset{\sim}{j} \\ &= 2\underset{\sim}{i} + 4\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\hat{\underset{\sim}{r}}'\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2^2 + 4^2}}(2\underset{\sim}{i} + 4\underset{\sim}{j}) \\ &= \frac{1}{\sqrt{5}}\underset{\sim}{i} + \frac{2}{\sqrt{5}}\underset{\sim}{j}\end{aligned}$$

Question 15 E

$\overrightarrow{RS} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\overrightarrow{RT} = 2\underset{\sim}{i} + (m-1)\underset{\sim}{j} + n\underset{\sim}{k}$

As R, S and T are collinear, $\overrightarrow{RT} = \lambda\overrightarrow{RS}$ where λ is a scalar.

So $2\underset{\sim}{i} + (m-1)\underset{\sim}{j} + n\underset{\sim}{k} = \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$.

Equating the $\underset{\sim}{i}$ coefficients we obtain $\lambda = 2$.

Equating the $\underset{\sim}{j}$ coefficients we obtain $m - 1 = -2\lambda$ and so $m = -3$.

Equating the $\underset{\sim}{k}$ coefficients we obtain $n = \lambda$ and so $n = 2$.

Question 16 C

The resultant force is:

$$\begin{aligned}\underset{\sim}{F}_1 + \underset{\sim}{F}_2 &= 6\underset{\sim}{i} - 8\underset{\sim}{j} - 10\underset{\sim}{i} + 24\underset{\sim}{j}\\ &= -4\underset{\sim}{i} + 16\underset{\sim}{j} \text{ (N)}\end{aligned}$$

$$\begin{aligned}\underset{\sim}{a} &= \frac{1}{4}(-4\underset{\sim}{i} + 16\underset{\sim}{j})\\ &= -\underset{\sim}{i} + 4\underset{\sim}{j} \text{ (ms}^{-2}\text{)}\end{aligned}$$

$$\begin{aligned}|\underset{\sim}{a}| &= \sqrt{(-1)^2 + (4)^2}\\ &= \sqrt{17} \text{ (ms}^{-2}\text{)}\end{aligned}$$

Question 17 E

Let R newtons represent the reading on the scales.

$$\begin{aligned}R - 80g &= 80a\\ R &= 80(g+1)\end{aligned}$$

Question 18 B

In the horizontal direction: $T\cos(\theta) = F$ and so $T > F$.

In the vertical direction: $N + T\sin(\theta) = W$ and so $N < W$.

Question 19 D

An approximate 95% confidence interval for μ is $\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$.

So in this instance an approximate 95% confidence interval for μ is

$\left(15.8 - 1.96 \times \frac{\sqrt{6.1}}{4}, 15.8 + 1.96 \times \frac{\sqrt{6.1}}{4}\right)$ that is, (14.6, 17.0).

Question 20 A

$$\begin{aligned}\text{E}(D) &= 80 - 2 \times 54\\ &= -28\end{aligned}$$

$$\begin{aligned}\text{var}(D) &= 7^2 + 4 \times 5^2\\ &= 149\end{aligned}$$

SECTION B

Question 1 (9 marks)

a. The range of f is $0 < y \le 1$. A1

b. $f''(x) = \dfrac{2x^2 - 1}{(x^2 + 1)^{\frac{5}{2}}}$ A1

c. Attempting to solve $f''(x) = 0$ for x. M1

So, $x = \dfrac{\sqrt{2}}{2}$.

$f\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\sqrt{6}}{3}$

So, the coordinates are $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{6}}{3}\right)$. A1

d. Let the volume be V_x.

$V_x = \pi \displaystyle\int_0^{\sqrt{3}} \frac{1}{1 + x^2}\, dx$ M1

So, $V_x = \dfrac{\pi^2}{3}$. A1

e. Let the volume be V_y.

When $x = 0$, $y = 1$ and when $x = \sqrt{3}$, $y = \dfrac{1}{2}$.

$y^2 = \dfrac{1}{1 + x^2} \Rightarrow x^2 = \dfrac{1}{y^2} - 1$ M1

$V_y = \pi \displaystyle\int_{\frac{1}{2}}^{1} \left(\frac{1}{y^2} - 1\right) dy$ M1

So, $V_y = \dfrac{\pi}{2}$. A1

Question 2 (11 marks)

a. **Method 1:**

Considering $(x - c)(x + q) + r \equiv (x - a)(x - b)$. M1

Putting $x = c$ gives $r = (c - a)(c - b)$. A1

Considering the coefficient of x we obtain $q - c = -(a + b)$.

So, $q = c - a - b$. A1

Method 2:

Use the proper fraction command of a CAS. M1

$q = c - a - b$ and $r = (c - a)(c - b)$ A1 A1

b. **i.** The vertical asymptote is $x = c$. A1

ii. The non-vertical asymptote is $y = x + c - a - b$. A1

c. $f'(x) = 1 - \dfrac{r}{(x-c)^2}$ M1

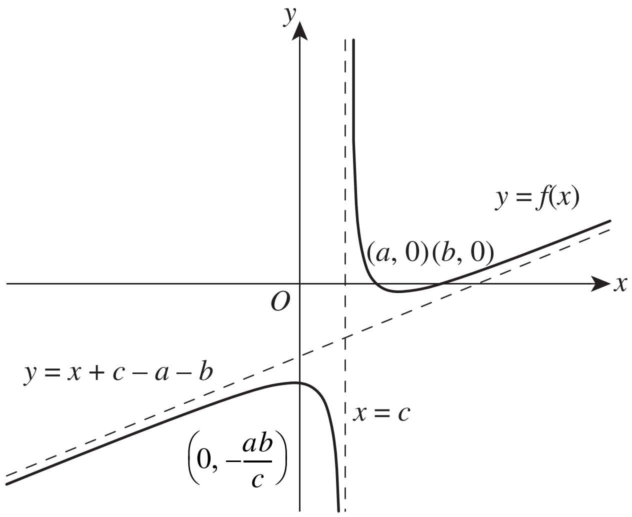
Solving $f'(x) = 0 \Rightarrow x = c \pm \sqrt{r}$.

As $r = (c-a)(c-b)$ then $x = c \pm \sqrt{(c-a)(c-b)}$. A1

As $c > b > a > 0$, then $(c-a)(c-b) > 0$, and so there are two real roots. A1

Hence the graph of $y = f(x)$ has two stationary points.

d.



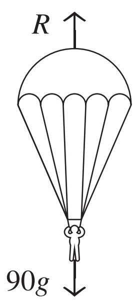
two correct branches with correct asymptotic behaviour A1

the non-vertical asymptote crossing the positive y-axis A1

(a, 0), (b, 0) and $\left(0, -\dfrac{ab}{c}\right)$ A1

Question 3 (12 marks)

a.



A1

b. At terminal velocity, $\Sigma F = 0$.

So, $k(60)^2 = 90g$. A1

Hence, $k = \dfrac{g}{40}$.

c. $90v\frac{dv}{dx} = 90g - \frac{g}{40}v^2$ A1

$$\int \frac{90v}{90g - \frac{g}{40}v^2}\,dv = \int dx$$ M1

$-\frac{1800}{g}\log_e\left|90g - \frac{g}{40}v^2\right| = x + c$ A1

$90g - \frac{g}{40}v^2 = Ae^{-\frac{gx}{1800}}$

$$v^2 = \frac{40}{g}\left(90g - Ae^{-\frac{gx}{1800}}\right)$$ A1

When $x = 0$, $v = 0$ and so, $A = 90g$. M1

Hence, $v^2 = 3600\left(1 - e^{-\frac{gx}{1800}}\right)$.

d. $v = 60\sqrt{1 - e^{-g}}$

So, $v = 59.9983$ m/s (correct to four decimal places). A1

e. $90\frac{dv}{dt} = 90g - 90v$ A1

Let t_1 be the time taken for the parachutist's velocity to decrease to 20 m/s.

$$\int_{59.9983}^{20} \frac{dv}{g - v} = \int_0^{t_1} dt$$ M1

Attempting to solve the equation for t_1. M1

So, $t_1 = 1.59$ s (correct to two decimal places). A1

Question 4 (10 marks)

a. $v_x = \int 0\,dt \Rightarrow v_x = c_x$

When $t = 0$, $v_x = V\cos(\theta)$ and so $c_x = V\cos(\theta)$.
So $v_x = V\cos(\theta)$. M1

$x = \int (V\cos(\theta))dt \Rightarrow x = V\cos(\theta)t + d_x$

When $t = 0$, $x = 0$ and so, $d_x = 0$.
Hence, $x = V\cos(\theta)t$. A1

$v_y = \int -g\,dt \Rightarrow v_y = -gt + c_y$

When $t = 0$, $v_y = V\sin(\theta)$ and so, $c_y = V\sin(\theta)$.
So $v_y = V\sin(\theta) - gt$. M1

$y = \int (V\sin(\theta) - gt)dt \Rightarrow y = V\sin(\theta)t - \frac{1}{2}gt^2 + d_y$

When $t = 0$, $y = 0$ and so, $d_y = 0$.

Hence, $y = V\sin(\theta)t - \frac{1}{2}gt^2$. A1

Given that $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$ we obtain $\underset{\sim}{r} = (V\cos(\theta)t)\underset{\sim}{i} + \left(V\sin(\theta)t - \frac{1}{2}gt^2\right)\underset{\sim}{j}$.

b. The parametric equations are $x = V\cos(\theta)t$ and $y = V\sin(\theta)t - \frac{1}{2}gt^2$. A1

Substituting $t = \frac{x}{V\cos(\theta)}$ into $y = V\sin(\theta)t - \frac{1}{2}gt^2$ gives

$y = V\sin(\theta)\left(\frac{x}{V\cos(\theta)}\right) - \frac{1}{2}g\left(\frac{x}{V\cos(\theta)}\right)^2$. M1

So, $y = \tan(\theta)x - \frac{gx^2}{2V^2\cos^2(\theta)}$.

c. As the particle just clears the first wall we can take (2, 2) as a point on the path.

Using $y = \tan(\theta)x - \frac{gx^2}{2V^2\cos^2(\theta)}$ we obtain $2 = 2\tan(55°) - \frac{4g}{2V^2\cos^2(55°)}$. M1

Attempting to solve for V.

So, $V = 8.341$ ms^{-1} (correct to three decimal places) A1

d. To find the position of the second wall we need to find the other value of x for which $y = 2$ and $V = 8.341$.

Attempting to solve $2 = 2\tan(55°) - \frac{gx^2}{2(8.341)^2\cos^2(55°)}$ for x. M1

So the second wall is 4.7 m (correct to one decimal place) from O. A1

Question 5 (8 marks)

a. $\bar{X} \sim \mathrm{N}\left(35, \frac{64}{100}\right)$

$\Pr(\text{type I error}) = \Pr(\bar{X} > 36.5)$ M1

$= 0.0304$ (correct to four decimal places) A1

b. **i.** $\Pr(\text{type II error}) = \Pr(\text{accept } H_0 | H_1 \text{ is true})$

$= \Pr(\bar{X} \le 36.5 | \mu = 37.9)$ M1

$= \Pr(\bar{X} \le 36.5)$ when $\bar{X} \sim \mathrm{N}\left(37.9, \frac{64}{100}\right)$ A1

$= 0.0401$ (correct to four decimal places) A1

ii. $\frac{\bar{x} - 37.9}{0.8} = -\left(\frac{\bar{x} - 35}{0.8}\right)$ A1

Attempting to solve this equation for $\bar{x}$. M1

$\bar{x} = 36.45$ A1

Question 6 (10 marks)

a. Substituting $z = x + yi$ into the quadratic equation gives:

$(x + yi)^2 + b(x + yi) + 1 = 0$ M1

$x^2 - y^2 + 2xyi + bx + byi + 1 = 0$ A1

Considering the real part we obtain $x^2 - y^2 + bx + 1 = 0$ and considering the imaginary part we obtain $2xy + by = 0$ and so $(2x + b)y = 0$. A1

b. $(2x + b)y = 0 \Rightarrow y = 0$ or $b = -2x$ A1

Substituting $y = 0$ into $x^2 - y^2 + bx + 1 = 0$ gives $x^2 + bx + 1 = 0$. M1

$x^2 + bx + 1 = 0 \Rightarrow b = -\frac{x^2 + 1}{x}, x \ne 0$ A1

c. **i.** When $b = -2x$, we obtain $x^2 - y^2 + 2x^2 + 1 = 0$.
So $x^2 + y^2 = 1$. A1

ii. a circle of radius 1 and centre (0, 0) A1

d. **i.** When $b = -\frac{x^2 + 1}{x}, x \ne 0$, we obtain $x^2 - y^2 + \left(-\frac{x^2 + 1}{x}\right)x + 1 = 0 \Rightarrow y^2 = 0$.

So, $y = 0$, $x \ne 0$. A1

ii. The x-axis with the point (0, 0) excluded. A1