Year 2018

VCE

Specialist Mathematics

Trial Examination 1

Solutions

KILBAHA MULTIMEDIA PUBLISHING	TEL: (03) 9018 5376
PO BOX 2227	FAX: (03) 9817 4334
KEW VIC 3101	kilbaha@gmail.com
AUSTRALIA	http://kilbaha.com.au

- While every care has been taken, no guarantee is given that these answers are free from error. Please contact us if you believe you have found an error.

Question 1

Method 1

$(z-ai)(z+ai)=z^2+a^2$ is a factor

$(z^2+a^2)(z^2+bz+c)$

$=z^4+bz^3+(c+a^2)z^2+ba^2z+ca^2=z^4+6z^3+41z^2+96z+400$ M1

$\Rightarrow\ b=6 \quad 41=c+a^2\ ,\quad ba^2=96\quad ,\quad a^2=16\ \Rightarrow\ a=\pm 4\quad ,\ c=25$

$(z^2+16)(z^2+6z+25)=0$ A1

$(z^2+16)\left((z+3)^2-16i^2\right)=0$ M1

$(z+4i)(z-4i)(z+3+4i)(z+3-4i)=0$

$z=\pm 4i\ ,\ -3\pm 4i$ A1

Method II

Let $P(z)=z^4+6z^3+41z^2+96z+400=0$

$P(ai)=(ai)^4+6(ai)^3+41(ai)^2+96ai+400=0$

$=a^4-6a^3i-41a^2+96ai+400=0$ M1

$=a^4-41a^2+400+(96a-6a^3)i=0$

the real part must be zero

$a^4-41a^2+400=0$

$(a^2-25)(a^2-16)=0$

$a=\pm 5\ ,\ \pm 4$

and the imaginary part must also be zero

$96a-6a^3=0$

$6a(16-a^2)=0$

$a=0\ ,\ \pm 4$

the only common solution which satisfy both are $a=\pm 4$ A1

so $(z-4i)(z+4i)=z^2-16i^2=z^2+16=0$ is a factor

$z^4+6z^3+41z^2+96z+400=0$

$(z^2+16)(z^2+6z+25)=0$

$(z^2+16)(z^2+6z+9+16)=0$ M1

$(z^2+16)\left((z+3)^2-16i^2\right)=0$

$(z+4i)(z-4i)(z+3+4i)(z+3-4i)=0$

$z=\pm 4i\ ,\ -3\pm 4i$ A1

Question 2

a. $f(x)=\sqrt{\arcsin\left(\frac{3x}{4}\right)}$ domain $0\le\frac{3x}{4}\le1$

$0\le x\le\frac{4}{3}$ or dom $f=\left[0,\frac{4}{3}\right]$ A1

$f(0)=\sqrt{\arcsin(0)}=0$, $f\left(\frac{4}{3}\right)=\sqrt{\arcsin(1)}=\sqrt{\frac{\pi}{2}}=\frac{\sqrt{2\pi}}{2}$

since it is a one-one function, the range is

$0\le y\le\sqrt{\frac{\pi}{2}}$ or range $f=\left[0,\frac{\sqrt{2\pi}}{2}\right]$ A1

b. $f(x)=\sqrt{\arcsin\left(\frac{3x}{4}\right)}=\left(\arcsin\left(\frac{3x}{4}\right)\right)^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}\left(\arcsin\left(\frac{3x}{4}\right)\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\arcsin\left(\frac{3x}{4}\right)\right)=\frac{1}{2}\left(\arcsin\left(\frac{3x}{4}\right)\right)^{-\frac{1}{2}}\frac{3}{\sqrt{16-9x^2}}$

$f'(x)=\frac{3}{2\sqrt{\left(16-9x^2\right)\sin^{-1}\left(\frac{3x}{4}\right)}}$ A1

hence $\int_0^{\frac{4}{3}}\frac{1}{\sqrt{\left(16-9x^2\right)\sin^{-1}\left(\frac{3x}{4}\right)}}\,dx=\frac{2}{3}\left[\sqrt{\arcsin\left(\frac{3x}{4}\right)}\right]_0^{\frac{4}{3}}$

$=\frac{2}{3}\left[f\left(\frac{4}{3}\right)-f(0)\right]=\frac{2}{3}\times\frac{\sqrt{2\pi}}{2}$

$=\frac{\sqrt{2\pi}}{3}$ so $b=2$, $c=3$ A1

Question 3

$\frac{dy}{dx}+\frac{4y^2}{16-9x^2}=0$, given that $y(0)=1$

$\frac{dy}{dx}=\frac{-4y^2}{16-9x^2}$, separating the variables

$-\int \frac{1}{y^2}\,dy=\int \frac{4}{16-9x^2}\,dx$ A1

by partial fractions

$\frac{4}{16-9x^2}=\frac{A}{4-3x}+\frac{B}{4+3x}$

$=\frac{A(4+3x)+B(4-3x)}{(4-3x)(4+3x)}=\frac{4(A+B)+3x(A-B)}{16-9x^2}$ M1

$(1)\ \ A+B=1 \quad (2)\ \ A-B=0 \quad \Rightarrow A=B=\frac{1}{2}$

$\frac{1}{y}=\int \frac{4}{16-9x^2}\,dx=\frac{1}{2}\int\left(\frac{1}{4+3x}+\frac{1}{4-3x}\right)dx$

$\frac{1}{y}=\frac{1}{2}\left[\frac{1}{3}\log_e\left(|4+3x|\right)-\frac{1}{3}\log_e\left(|4-3x|\right)\right]+c$

$\frac{1}{y}=\frac{1}{6}\log_e\left(\frac{|4+3x|}{|4-3x|}\right)+c$

To find c use $x=0$, $y=1$ $\Rightarrow 1=\frac{1}{6}\log_e(1)+c \quad \Rightarrow \quad c=1$ A1

$\frac{1}{y}=\frac{1}{6}\log_e\left(\frac{|4+3x|}{|4-3x|}\right)+1=\frac{\log_e\left(\frac{|4+3x|}{|4-3x|}\right)+6}{6}$

$y=\frac{6}{\log_e\left(\frac{|4+3x|}{|4-3x|}\right)+6}$ A1

Question 4

a. $y^2 = x^3 + 4x^2 = x^2(x+4) \Rightarrow y = -x\sqrt{x+4}$ and $y = x\sqrt{x+4}$,

these two functions represent the upper and lower parts of the curve in the second and third quadrants respectively, by symmetry

$$A = 2\int_{-4}^{0} -x\sqrt{x+4}\,dx = 2\int_{0}^{-4} x\sqrt{x+4}\,dx$$ A1

b. let $u = x+4$, $x = u-4$, $\frac{du}{dx} = 1$

terminals when $x = -4$, $u = 0$ when $x = 0$, $u = 4$

$$A = 2\int_{0}^{-4} x\sqrt{x+4}\,dx = 2\int_{4}^{0}(u-4)\sqrt{u}\,du$$

$$A = 2\int_{4}^{0}\left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$$ M1

$$A = 2\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right]_4^0 = 4\left[u^{\frac{3}{2}}\left(\frac{u}{5} - \frac{4}{3}\right)\right]_4^0$$ A1

$$A = 4\left[u^{\frac{3}{2}}\left(\frac{3u-20}{15}\right)\right]_4^0 = 4\left[\left(0 - 4^{\frac{3}{2}}\left(\frac{12-20}{15}\right)\right)\right]$$

$$A = \frac{256}{15} \text{ units}^2$$ A1

c. $y^2 = x^3 + 4x^2$ using implicit differentiation

$$2y\frac{dy}{dx} = 3x^2 + 8x \quad , \quad \frac{dy}{dx} = \frac{3x^2 + 8x}{2y}$$ A1

now when $x = 2$ $y = x\sqrt{x+4} = 2\sqrt{6}$ so $\frac{dy}{dx} = \frac{12+16}{4\sqrt{6}} = \frac{7}{\sqrt{6}}$

also given $\frac{dy}{dt} = -7$, then $\frac{dx}{dt} = \frac{dx}{dy}.\frac{dy}{dt} = \frac{\sqrt{6}}{7} \times^{-} 7 = -\sqrt{6}$

the particle is moving inwards at $\sqrt{6}$ ms^{-1} A1

alternatively $y = x\sqrt{x+4}$ using the product rule

$$\frac{dy}{dx} = \sqrt{x+4} + \frac{x}{2\sqrt{x+4}} = \frac{2x+8+x}{2\sqrt{x+4}} = \frac{3x+8}{2\sqrt{x+4}}$$

when $x = 2$ $\frac{dy}{dx} = \frac{6+8}{2\sqrt{6}} = \frac{7}{\sqrt{6}}$

Question 5

a. Let D be the weights of medium sized dogs, $D \stackrel{d}{=} N\left(10, 2.5^2\right)$

and let C be the weights of cats, $C \stackrel{d}{=} N\left(5, 1.5^2\right)$.

Let T be the total weight of one dog and two actually different and independent cats

$T = D + C1 + C2$

$$E(T) = E(D) + E(C1) + E(C2)$$
$$= 10 + 5 + 5$$
$$= 20$$
$$\text{var}(T) = \text{var}(D) + \text{var}(C1) + \text{var}(C2)$$
$$= \left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2$$
$$= \frac{25}{4} + \frac{9}{2} = \frac{43}{4} = 10.75$$

$\text{E}(T) = 20$, $\text{var}(T) = 10.75$ A1

b.i. $H_0: \ \mu = 10$

$H_A: \ \mu > 10$ what we are trying to prove A1

ii. $\bar{x} = 11$, $\mu = 10$, $\sigma = 2.5$, $n = 25$

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{11 - 10}{\frac{2.5}{\sqrt{25}}} = \frac{1}{\frac{5}{2 \times 5}} = 2$$

$p = \Pr(Z \ge a)$ so $a = 2$

$\Pr(Z \le 1.65) = 0.95$ A1

Yes reject the null hypothesis, accept the alternative hypothesis, his dogs appear over-weight. A1

Question 6

$x^2y^2 + \frac{4}{\pi}\arctan(2x) = 2$ substitute $x = \frac{1}{2}$ $\Rightarrow$ $\frac{y^2}{4} + \frac{4}{\pi}\arctan(1) = 2$, $\frac{y^2}{4} + \frac{4}{\pi} \times \frac{\pi}{4} = 2$

$y^2 = 4$ so $y = \pm 2$ but in the fourth quadrant $y < 0$, so $y = -2$ A1

using implicit differentation, $\frac{d}{dx}\left(x^2y^2\right) + \frac{4}{\pi}\frac{d}{dx}\left(\arctan(2x)\right) = 0$ product rule in the first term

$2xy^2 + 2x^2y\frac{dy}{dx} + \frac{4}{\pi} \times \frac{2}{1 + 4x^2} = 0$ substitute $x = \frac{1}{2}$ and $y = -2$ M1

$$1 \times 4 + 2 \times \frac{1}{4} \times -2\frac{dy}{dx} + \frac{4}{\pi} \times \frac{2}{1 + 4 \times \frac{1}{4}} = 0 \quad \Rightarrow \quad 4 - \frac{dy}{dx} + \frac{4}{\pi} = 0$$

$\frac{dy}{dx} = 4 + \frac{4}{\pi} = \frac{4(\pi + 1)}{\pi}$ A1

Question 7

a. $\underset{\sim}{v}(t) = -2e^{-t}\,\underset{\sim}{i} + 2e^{2t}\,\underset{\sim}{j}$ ms^{-1} integrating

$\underset{\sim}{r}(t) = \int -2e^{-t}\,dt\,\underset{\sim}{i} + \int 2e^{2t}\,dt\,\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 2e^{-t}\underset{\sim}{i} + e^{2t}\underset{\sim}{j} + \underset{\sim}{c}$ A1

Now $\underset{\sim}{r}(0) = 2\underset{\sim}{i}$ $2\underset{\sim}{i} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{c} \Rightarrow \underset{\sim}{c} = -\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 2e^{-t}\underset{\sim}{i} + \left(e^{2t} - 1\right)\underset{\sim}{j}$ M1

The parametric equations are $x = 2e^{-t}$, $y = e^{2t} - 1$

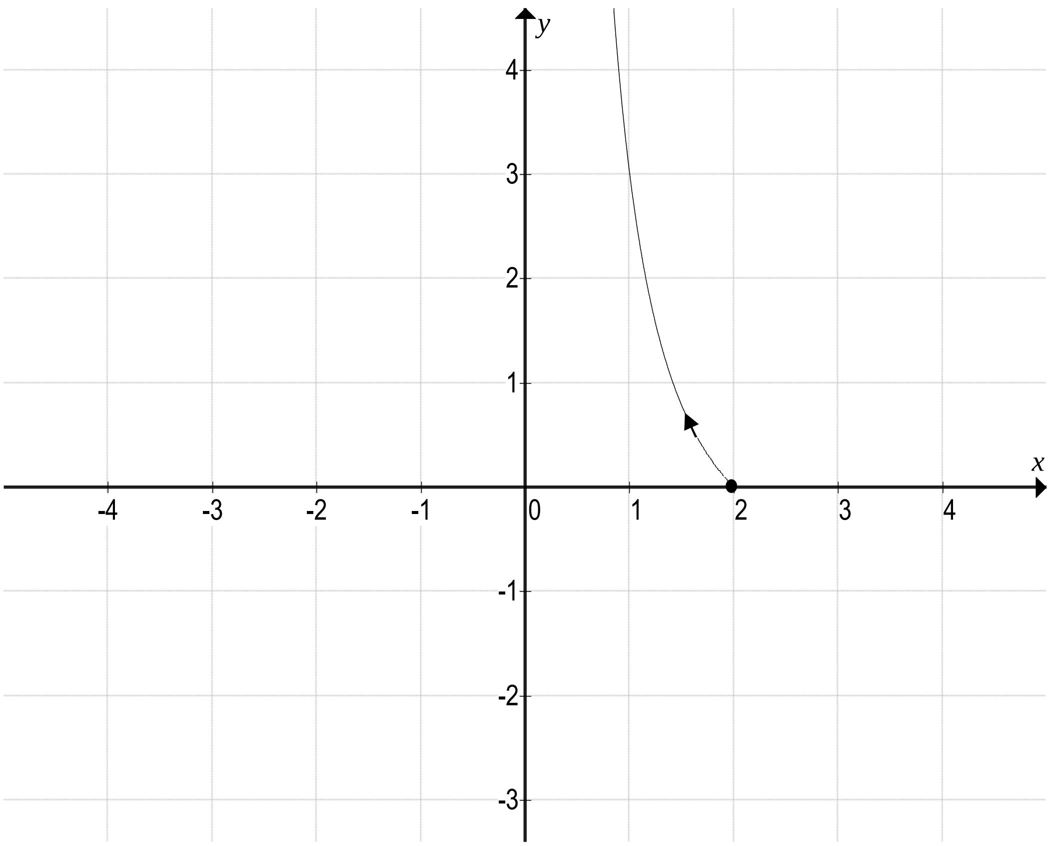
$e^{t} = \dfrac{2}{x} \Rightarrow y = \dfrac{4}{x^2} - 1$

But $t \ge 0$, so $0 < x \le 2$ and $y \ge 0$, so the particle moves on

part of the curve $y = \dfrac{4}{x^2} - 1$ A1

b. When $t = 0$, $x = 2$, $y = 0$,

graph on restricted domain, with direction of motion G1



Question 8

when $2\sec\left(\frac{x}{3}\right)=4 \Rightarrow \cos\left(\frac{x}{3}\right)=\frac{1}{2} \Rightarrow \frac{x}{3}=\frac{\pi}{3}$ then $x=\pi$ A1

$$V=\pi\int_0^{\pi}\left(16-4\sec^2\left(\frac{x}{3}\right)\right)dx$$

$$V=\pi\left[16x-12\tan\left(\frac{x}{3}\right)\right]_0^{\pi}$$ A1

$$V=\pi\left[\left(16\pi-12\tan\left(\frac{\pi}{3}\right)\right)-0\right]$$

$$V=\pi\left(16\pi-12\sqrt{3}\right)$$ A1

Question 9

a.

$$s=\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt$$

$x=5t^4+1$, $y=2t^5+3$

$\dot{x}=\frac{dx}{dt}=20t^3$ $\dot{y}=\frac{dy}{dt}=10t^4$

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=400t^6+100t^8=100t^6\left(t^2+4\right)$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=10t^3\sqrt{t^2+4} \quad \text{since } t\ge 0$$ A1

$$s=\int_0^{\sqrt{5}}10t^3\sqrt{t^2+4}\,dt$$

b.

$$s=\int_0^{\sqrt{5}}10t^3\sqrt{t^2+4}\,dt=\int_0^{\sqrt{5}}10t^2\sqrt{t^2+4}\,.\,t\,dt$$

let $u=t^2+4$ $\frac{du}{dt}=2t \Rightarrow t\,dt=\frac{1}{2}du$, $t^2=u-4$

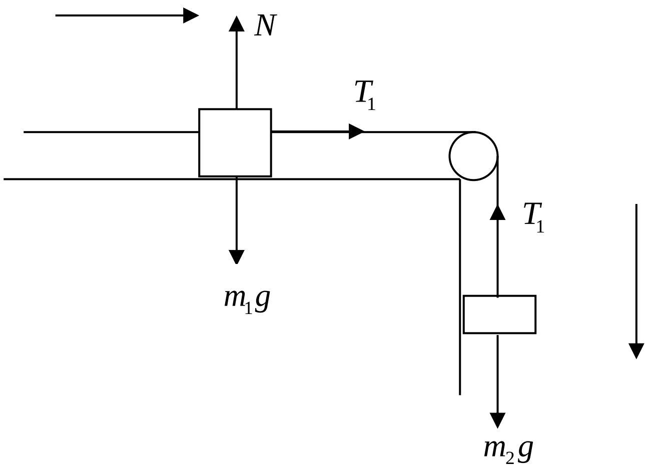
terminals when $t=0 \Rightarrow u=4$ when $t=\sqrt{5} \Rightarrow u=9$ M1

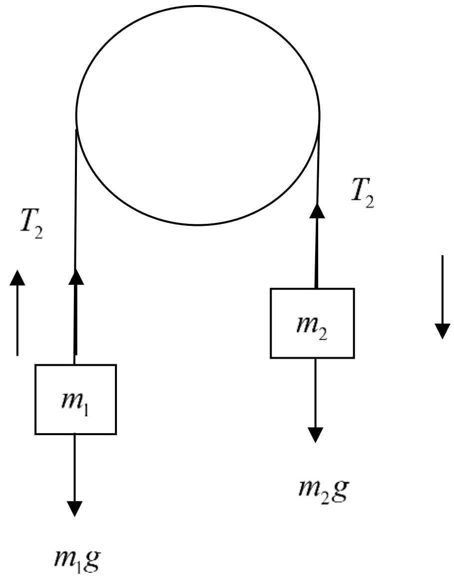
$$s=10\int_4^9(u-4)\frac{\sqrt{u}}{2}\,du=5\int_4^9\left(u^{\frac{3}{2}}-4u^{\frac{1}{2}}\right)du$$ A1

$$=5\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{8}{3}u^{\frac{3}{2}}\right]_4^9=\left[10u^{\frac{3}{2}}\left(\frac{3u-20}{15}\right)\right]_4^9=\frac{2}{3}\left(7\times 9^{\frac{3}{2}}+8\times 4^{\frac{3}{2}}\right)=\frac{2}{3}\left(7\times 27+8\times 8\right)$$

$$=\frac{506}{3}=168\tfrac{2}{3}$$ A1

Question 10





resolving parallel to the table

$(1)\ \ T_1 = 3m_1 a$

resolving downwards around the m_2 kg mass

$(2)\ \ m_2 g - T_1 = 3m_2 a$

$m_2 g - 3m_1 a = 3m_2 a$

$m_2 g = 3m_1 a + 3m_2 a$

$m_2 g = 3(m_1 + m_2)a$

$$a = \frac{m_2 g}{3(m_1 + m_2)}$$

for the pulley

around m_2 kg mass

$(3)\ \ m_2 g - T_2 = m_2 a$

around m_1 kg mass M2

$(4)\ \ T_2 - m_1 g = m_1 a$

adding to eliminate the tension T_1

$(3)+(4)\ \ m_2 g - m_1 g = a(m_1 + m_2)$

$$a = \frac{g(m_2 - m_1)}{m_1 + m_2}$$ A1

equating the accelerations

$$\frac{m_2 g}{3(m_1 + m_2)} = \frac{g(m_2 - m_1)}{m_1 + m_2}$$

$m_2 = 3(m_2 - m_1) = 3m_2 - 3m_1$

$3m_1 = 2m_2$

$$\frac{m_2}{m_1} = \frac{3}{2}$$ A1

END OF SUGGESTED SOLUTIONS