2018 VCAA Specialist Mathematics Exam 2 Solutions

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SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	B	D	A	D	C	C	E	D	A

11	12	13	14	15	16	17	18	19	20
C	A	B	C	E	B	E	D	E	B

Q1 E

Q2 $\sin^{-1}(cx+d)>0$, $0<cx+d\le 1$, $-\dfrac{d}{c}<x\le\dfrac{1-d}{c}$ B

Q3 $\dfrac{(2x+1)(x+1)}{(2x+1)^3(x-1)(x+1)}=\dfrac{1}{(2x+1)^2(x-1)}$ D

Q4 $\text{cosec}(-x)=\dfrac{1}{\sin(-x)}=-\dfrac{1}{\sin(x)}=-\dfrac{\cot(x)}{\cos(x)}=\dfrac{b}{a}$ A

Q5 $z+\dfrac{1}{z}=z+\dfrac{\bar{z}}{z\bar{z}}\in R$ if $z\bar{z}=|z|^2=1$ D

Q6 $O, z, iz, z+iz$ are the vertices of a square of side length of $|z|$, .: area of the triangle $=\dfrac{|z|^2}{2}$ C

Q7 Length $=\displaystyle\int_0^{2\pi}\sqrt{(2\cos(2t))^2+(-2\sin(t))^2}\,dt\approx 12.2$ C

Q8 Let $u=\tan(x)$, $\dfrac{du}{dx}=\sec^2(x)$

$\displaystyle\int_0^{\frac{\pi}{6}}\tan^2(x)\sec^2(x)dx=\int_0^{\frac{\pi}{6}}u^2\frac{du}{dx}dx=\int_0^{\frac{1}{\sqrt{3}}}u^2\,du$ E

Q9 $\sin(x+y)-\sin(x-y)=2\cos(x)\sin(y)$

$\dfrac{dy}{dx}=\dfrac{1}{\cos(x)\sin(y)}$, $\displaystyle\int\sin(y)dy=\int\sec(x)dx$ D

Q10 Gradient $=1$ when $x=0$; gradient $=-1$ when $y=0$ A

Q11 $\tilde{a}.\tilde{b}=ab\cos\theta$, $2m=\dfrac{\sqrt{3}}{2}(m^2+1)$, $\sqrt{3}m^2-4m+\sqrt{3}=0$

$m=\sqrt{3},\dfrac{1}{\sqrt{3}}$ C

Q12 $(\tilde{a}+\tilde{b}).(\tilde{a}+\tilde{b})=\tilde{a}.\tilde{a}+2\tilde{a}.\tilde{b}+\tilde{b}.\tilde{b}$

$|\tilde{a}+\tilde{b}|^2=|\tilde{a}|^2+2|\tilde{a}||\tilde{b}|\cos\theta+|\tilde{b}|^2$

.: $|\tilde{a}+\tilde{b}|^2=(|\tilde{a}|+|\tilde{b}|)^2$ if $\theta=0$

i.e. $|\tilde{a}+\tilde{b}|=|\tilde{a}|+|\tilde{b}|$ if $\tilde{a}//\tilde{b}$ A

Q13 $\tilde{v}=-3\sin(t)\tilde{i}+4\cos(t)\tilde{j}$

Speed $=\sqrt{9\sin^2(t)+16\cos^2(t)}=\sqrt{9+7\cos^2(t)}$

Min speed when $\cos(t)=0$, $t=\dfrac{\pi}{2}$ B

Q14 $\hat{b}=\dfrac{1}{\sqrt{14}}\tilde{b}$, $\tilde{a}.\hat{b}=\dfrac{1}{\sqrt{14}}\tilde{a}.\tilde{b}=\dfrac{1}{\sqrt{14}}(-3-6)=-\dfrac{9\sqrt{14}}{14}$ C

Q15 $20^2=4^2+2a(15)$, $a=12.8\text{ ms}^{-2}$, $P=8\times 12.8=102.4$ E

Q16 $F_2\sin 45°-4-3\sin 30°=0$, $F_2=\dfrac{11}{2\sin 45°}=\dfrac{11\sqrt{2}}{2}$ B

Q17 $2t-\dfrac{1}{2}\times 9.8t^2=-50$, $t\approx 3.4\text{ s}$ E

Q18 $s=\dfrac{67.31-58.42}{2\times 1.96}\approx 2.267857$, $\sigma\approx\sqrt{36}\times 2.267857\approx 13.61$ D

Q19 Population distribution: $\mu=66$ and $\sigma=\sqrt{\dfrac{16}{9}}=\dfrac{4}{3}$

Sample distribution:
The mean of the sample mean gestation periods $\bar{x}$ is $\mu=66$, and

$s=\dfrac{\sigma}{\sqrt{n}}=\dfrac{\frac{4}{3}}{\sqrt{5}}\approx 0.596285$

$\Pr(\bar{X}>65)\approx 0.9532$ E

Q20 Let X_M and X_S be random variables Mathematic score and Statistics score respectively.

$X_M>X_S$, $X_M-X_S>0$

$\text{E}(X_M-X_S)=\text{E}(X_M)-\text{E}(X_S)=71-75=-4$

$\text{var}(X_M-X_S)=\text{var}(X_M)+(-1)^2\,\text{var}(X_S)=10^2+7^2=149$

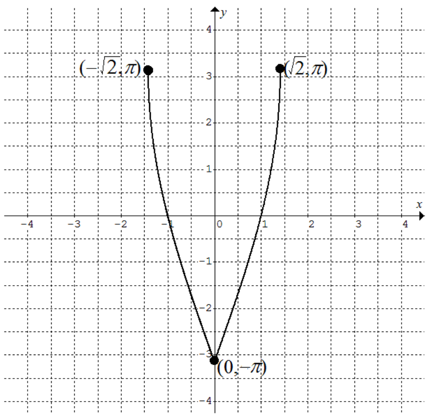
.: $\sigma=\sqrt{149}$

$\Pr(X_M>X_S)=\Pr(X_M-X_S>0)\approx 0.3716$ B

SECTION B

Q1a $f(x)=2\sin^{-1}(x^2-1)$, $-1\le x^2-1\le 1$, $0\le x^2\le 2$,
$-\sqrt{2}\le x\le\sqrt{2}$, $-\pi\le f(x)\le\pi$
D is $\left[-\sqrt{2},\sqrt{2}\right]$ and the range of f is $[-\pi,\pi]$

Q1b



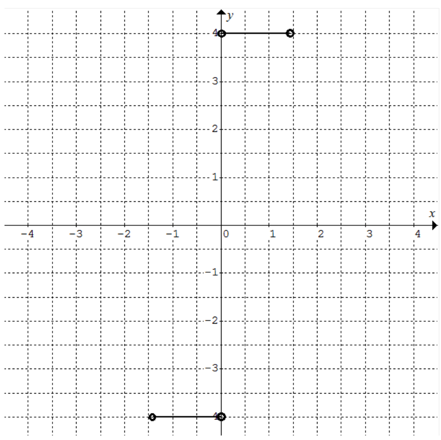
Q1c $f'(x)=\dfrac{4x}{\sqrt{1-(x^2-1)^2}}=\dfrac{4x}{\sqrt{(1+(x^2-1))(1-(x^2-1))}}$
$=\dfrac{4x}{\sqrt{x^2}\sqrt{2-x^2}}=\dfrac{4x}{|x|\sqrt{2-x^2}}=\dfrac{4}{\sqrt{2-x^2}}$ for $x>0$

Q1d For $x<0$, $f'(x)=\dfrac{4x}{|x|\sqrt{2-x^2}}=\dfrac{-4}{\sqrt{2-x^2}}$

Q1ei $f'(x)=\dfrac{4x}{|x|\sqrt{2-x^2}}$.: $g(x)=\dfrac{4x}{|x|}$
For $f'(x)$ to be defined, $x\ne 0$ and $2-x^2>0$
.: max domain of f' is $\left(-\sqrt{2},0\right)\cup\left(0,\sqrt{2}\right)$

Q1eii $g(x)=\begin{cases}-4 & -\sqrt{2}<x<0\\ 4 & 0<x<\sqrt{2}\end{cases}$

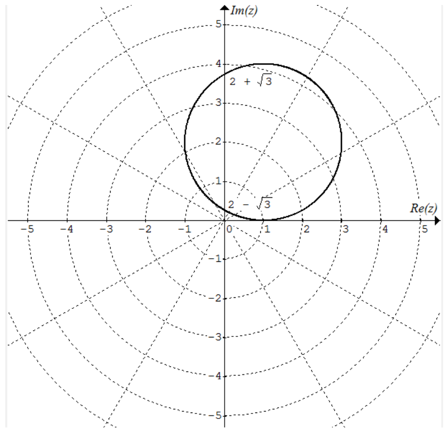
Q1eiii



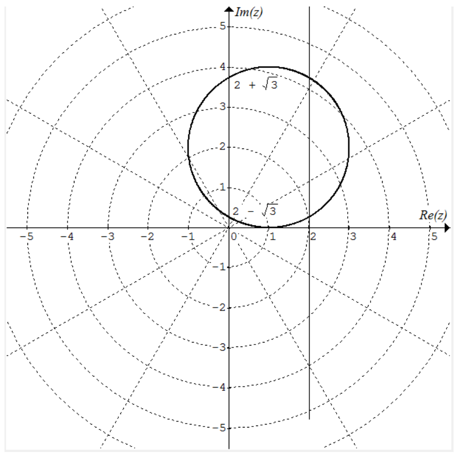
Q2a Centre $(1,2)$, radius 2

Q2b $|(x+1)+iy|=\sqrt{2}\,|x+i(y-1)|$, $(x+1)^2+y^2=2\left(x^2+(y-1)^2\right)$
$(x-1)^2+(y-2)^2=4$, centre $(1,2)$, radius 2

Q2c



Q2d The line is a perpendicular bisector of the section on the $\text{Re}(z)$ axis from 1 to 3, the line is $\text{Re}(z)=x=2$. The upper and lower points of intersection are $\left(2,2+\sqrt{3}\right)$ and $\left(2,2-\sqrt{3}\right)$ respectively.



Q2e The angle subtended by the arc at the centre of the circle is
$\theta=2\tan^{-1}\sqrt{3}=\dfrac{2\pi}{3}$
Segment area $=\dfrac{1}{3}\pi 2^2-\dfrac{1}{2}\times 2^2\sin\dfrac{2\pi}{3}=\dfrac{4\pi}{3}-\sqrt{3}$

Q3a $V=\int_0^h \pi x^2\,dy=\int_0^h \pi\left(y^2+\frac{1}{4}\right)dy=\pi\left[\frac{y^3}{3}+\frac{y}{4}\right]_0^h=\frac{\pi}{4}\left(\frac{4}{3}h^3+h\right)$

Q3b When $h=\frac{\sqrt{3}}{2}$, full $V=\frac{\pi}{4}\left(\frac{4}{3}\times\frac{3\sqrt{3}}{8}+\frac{\sqrt{3}}{2}\right)=\sqrt{3}\ \text{m}^3$

When $V=\frac{1}{2}\times\sqrt{3}$, $\frac{\pi}{4}\left(\frac{4}{3}h^3+h\right)=\frac{\sqrt{3}}{2}$, $h=0.68$ m

Q3ci $\frac{dV}{dt}=0.04-0.05\sqrt{h}$, $\frac{dV}{dh}\times\frac{dh}{dt}=\frac{dV}{dt}$

$\frac{\pi}{4}(4h^2+1)\frac{dh}{dt}=0.04-0.05\sqrt{h}$, $\frac{dh}{dt}=\frac{4-5\sqrt{h}}{25\pi(4h^2+1)}$

Q3cii When $h=0.25$, $\frac{dh}{dt}=\frac{4-5\sqrt{0.25}}{25\pi\left(4(0.25)^2+1\right)}\approx 0.0153\ \text{ms}^{-1}$

Q3d $\frac{dh}{dt}=\frac{4-5\sqrt{h}}{25\pi(4h^2+1)}$, $\frac{dt}{dh}=\frac{25\pi(4h^2+1)}{4-5\sqrt{h}}$,

$t=\int_0^{0.25}\frac{25\pi(4h^2+1)}{4-5\sqrt{h}}\,dh\approx 9.8$ s

Q3e
$t=25$, $h=0.4$
$t=30$, $h\approx 0.4+5\times\frac{4-5\sqrt{0.4}}{25\pi(4\times0.4^2+1)}\approx 0.43$ m

Q3f Let $\frac{dh}{dt}=\frac{4-5\sqrt{h}}{25\pi(4h^2+1)}=0$, $4-5\sqrt{h}=0$, $h=0.64$

Distance from the top $=\frac{\sqrt{3}}{2}-0.64\approx 0.23$ m

Q4a A: $x=t+1$, $y=t^2+2t$, ∴ $y=(x-1)^2+2(x-1)=x^2-1$
B: $x=t^2$, $y=t^2+3$, $y=x+3$

Q4b Sane x-coordinate when $t^2=t+1$, $t=\frac{1+\sqrt{1+4}}{2}=\frac{1+\sqrt{5}}{2}$

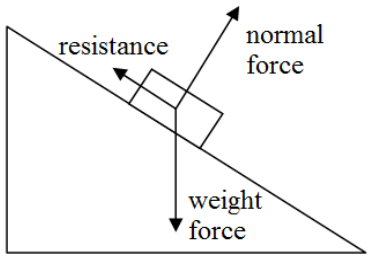
Same y-coordinate when $t^2+2t=t^2+3$, $t=\frac{3}{2}$
∴ A and B cannot be at the same point (same x and same y) at the same time. They will not collide.

Q4c Since $t\ge 0$, ∴ $x\ge 0$
Let $x^2-1=x+3$, $x\approx 2.562$ and $y\approx 5.562$, ∴ the two paths cross at $(2.562, 5.562)$.

Q4d $\dot{r}_A=\tilde{i}+(2t+2)\tilde{j}$, $|\dot{r}_A|=\sqrt{1^2+(2t+2)^2}$, and
$\dot{r}_B=2t\,\tilde{i}+2t\,\tilde{j}$, $|\dot{r}_B|=\sqrt{(2t)^2+(2t)^2}$, $t\ge 0$
Let $\sqrt{1^2+(2t+2)^2}>\sqrt{(2t)^2+(2t)^2}$, $0\le t<\frac{5}{2}$

Q4e Distance apart $=|\tilde{r}_B-\tilde{r}_A|=|(t^2-t-1)\tilde{i}+(3-2t)\tilde{j}|$
$=\sqrt{(t^2-t-1)^2+(3-2t)^2}<0.2$, ∴ $1.529<t<1.597$ approx
Period of time $\approx 1.597-1.529\approx 0.068$ h ≈ 4.1 min

Q5a



Q5bi $20a=20g\sin 30°-20v$

Q5bii $a=\frac{g}{2}-v$, $a=\frac{g-2v}{2}$

Q5c $v\frac{dv}{dx}=\frac{g-2v}{2}$, $\frac{dv}{dx}=\frac{g-2v}{2v}$, $\frac{dx}{dv}=\frac{2v}{g-2v}=\frac{g}{g-2v}-1$

Given $x=0$, $v=0$, $x=\int_0^v\left(\frac{4.9}{4.9-v}-1\right)dv=\left[-4.9\log_e(4.9-v)-v\right]_0^v$

∴ $x=-v+4.9\log_e\left(\frac{4.9}{4.9-v}\right)$

Q5d Let $-v+4.9\log_e\left(\frac{4.9}{4.9-v}\right)=15$, $v\approx 4.81\ \text{ms}^{-1}$ down the ramp

Q5ei $a=\frac{g-2v}{2}$, $\frac{dv}{dt}=\frac{g-2v}{2}$, $t=\int_0^{4.5}\frac{1}{4.9-v}\,dv$

Q5eii $t=\left[-\log_e(4.9-v)\right]_0^{4.5}=\log_e\frac{4.9}{0.4}\approx 2.51$ s

Q6a $H_0 : \mu = 150$; $H_1 : \mu < 150$

Q6b Standard deviation of $\overline{X} = \frac{15}{\sqrt{50}} = \frac{3}{\sqrt{2}}$ cm

Q6c p - value $= \Pr\left(\overline{X} < 145 \mid \mu = 150\right) \approx 0.0092$

Q6d Since p - value < 0.05, H_0 should be rejected at the 5% level of significance.

Q6e Let $\Pr\left(\overline{X} < \overline{h} \mid \mu = 150\right) = 0.05$ where $\overline{h}$ is the smallest value of the sample mean height that could be observed for H_0 to be not rejected.

$\Pr\left(Z < \frac{\overline{h} - 150}{\frac{3}{\sqrt{2}}}\right) = 0.05$, $\frac{\overline{h} - 150}{\frac{3}{\sqrt{2}}} = -1.6449$, $\overline{h} \approx 146.51$ cm

Q6f From part e, smallest $\overline{h} \approx 146.51$ for H_0 to be accepted at 5% level of significance.

$\Pr\left(\overline{X} > 146.51 \mid \mu = 145\right) \approx 0.24$

Q6g $\Pr(Z < z) = \frac{1 - 0.99}{2} = 0.005$, $z \approx -2.5758$

99% confidence interval for the mean height is

$\left(145 - 2.5758 \times \frac{3}{\sqrt{2}},\ 145 + 2.5758 \times \frac{3}{\sqrt{2}}\right)$, i.e. $(139.5, 150.5)$

Please inform mathline@itute.com re conceptual and/or mathematical errors