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YEAR 12 *Trial Exam Paper*

2018
SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips on how to approach the exam.

SECTION A – Multiple-choice questions

Question 1

Answer: C

Worked solution

The domain of the function $\sin^{-1}(x)$ is $[-1,1]$. This implies that $-1 \le \frac{1}{2x+1} \le 1$.

Looking at each boundary condition individually:

$$\frac{1}{2x+1} \ge -1$$
$$2x+1 \le -1$$
$$\Rightarrow \quad x \le -1$$

and

$$\frac{1}{2x+1} \le 1$$
$$2x+1 \ge 1$$
$$\Rightarrow x \ge 0$$

Therefore, the implied domain is $x \in (-\infty,-1] \cup [0,\infty)$.

Question 2

Answer: E

Worked solution

The range of the function is $[k, a\pi + k]$.

So if $f(x) < 0$, then $a\pi + k < 0$.

Therefore, $k < -a\pi$.

Tip

- *When questions involve inequations, check the ‘equal to’ condition in the question. By checking this condition, answers that don’t match the ‘equal to’ condition in the question can be eliminated.*

Question 3

Answer: B

Worked solution

Using the compound angle formulas:

$$\cos(\phi+\gamma)=\cos(\phi)\cos(\gamma)-\sin(\phi)\sin(\gamma)=-a \quad [1]$$
$$\cos(\phi-\gamma)=\cos(\phi)\cos(\gamma)+\sin(\phi)\sin(\gamma)=3b \quad [2]$$

Adding equation [1] and equation [2] gives:

$$2\cos(\phi)\cos(\gamma)=-a+3b$$

So, $\cos(\phi)\cos(\gamma)=\dfrac{-a+3b}{2}$.

Question 4

Answer: A

Worked solution

Using de Moivre's theorem:

$z^3=r^3\operatorname{cis}(3\theta)$ and $\overline{z}=r\operatorname{cis}(-\theta)$.

Then, using division in polar form:

$$\frac{z^3}{\overline{z}}=\frac{r^3\operatorname{cis}(3\theta)}{r\operatorname{cis}(-\theta)}=r^2\operatorname{cis}(3\theta--\theta)=r^2\operatorname{cis}(4\theta)$$

Question 5

Answer: E

Worked solution

The relation $|z-2+3i|=2$ represents the circle $(x-2)^2+(y+3)^2=4.$

Option A represents the perpendicular bisector between two points, which is a straight line.

Option B represents the equation of a circle with radius $\sqrt{2}.$

Option C represents the perpendicular bisector between the two points, which is a straight line.

Option D represents a vertical line.

Option E represents the equation of a circle with radius 2.

$$\begin{aligned}
(z-2+3i)(\overline{z}-2-3i)&=4\\
(x+yi-2+3i)(x-iy-2-3i)&=4\\
[(x-2)+i(y+3)][(x-2)-i(y+3)]&=4\\
(x-2)^2+(y+3)^2&=4
\end{aligned}$$

Tip

- *Remembering the different representations of the subsets of a plane such as rays, lines and circles is an efficient way to eliminate multiple-choice options without needing to do algebra.*

Question 6

Answer: C

Worked solution

Since all the coefficients of $P(z)=0$ are real, this means that the complex conjugates $z=-i$ and $z=2-i$ are also roots. Since three roots have already been stated, with the addition of the two complex conjugate roots the minimum degree of $P(z)$ is 5.

Question 7

Answer: D

Worked solution

Start off by finding the value of λ that makes the vectors linearly dependent.

Write each component of $\underset{\sim}{c}$ as a linear combination of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$\underset{\sim}{i}$ – component

$2m - n = 3 \qquad [1]$

$\underset{\sim}{j}$ – component

$-m + 3n = 1 \qquad [2]$

$\underset{\sim}{k}$ – component

$2m - 2n = \lambda \qquad [3]$

Rearranging [1] gives:

$n = 2m - 3$

Substituting *n* into [2] and solving for *m* and *n* gives:

$m = 2, n = 1$

Substituting *m* and *n* into [3] gives:

$\lambda = 2$

This value of λ makes the vectors linearly dependent, so option A is incorrect as the question is asking for values that make the vectors linearly independent.

Options B, C and E are all incorrect as they include $\lambda = 2$ in their possible values of λ .

Therefore, option D is correct as it provides values of λ that make the vectors linearly independent.

Tip

- *When attempting questions about linearly dependent/independent vectors, start off by showing linear dependence. Then take the time to carefully read your answer so that you are giving the correct response for the context of the question. That is, if the question is about linear independence, you give the response for linear independence and not linear dependence.*

Question 8

Answer: E

Worked solution

The vector resolute of $\underset{\sim}{a} = \underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$ perpendicular to $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ is given by:

$$\begin{aligned}\underset{\sim}{a} - (\underset{\sim}{a}.\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} &= (\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}) - \left((\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}).\frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})\right)\frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}) \\ &= \frac{1}{9}(-7\underset{\sim}{i} + 20\underset{\sim}{j} + 26\underset{\sim}{k})\end{aligned}$$

Tip

- *Ensure you are familiar with the language of the vector projections and which formula is associated with the projection so that the correct formula for the context of the question can be applied.*

Question 9

Answer: D

Worked solution

The objects will collide when they have the same position vector at the same time.

So, by equating the components of each vector and solving the equations simultaneously the value of b can be found.

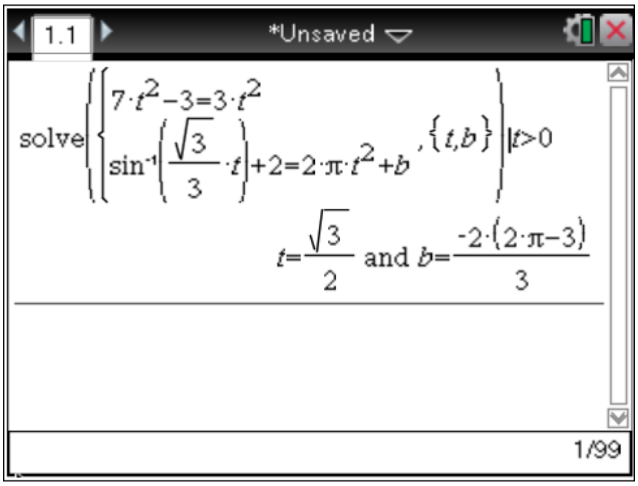
$\underset{\sim}{i}$ components:

$7t^2 - 3 = 3t^2$

$\underset{\sim}{j}$ components:

$\arcsin\left(\frac{\sqrt{3}}{3}t\right) + 2 = 2\pi t^2 + b$

Solving simultaneously:



So $b = 2 - \frac{4\pi}{3}$.

Tip

- *When solving equations using the calculator, restrict the domain of the variables to match the domain given in the question so that the calculator returns only those solutions in the specified domain.*

Question 10

Answer: D

Worked solution

Let $u = 2 - x \Rightarrow x = 2 - u.$

Then:

$$\frac{du}{dx} = -1 \Rightarrow dx = -du$$

And when

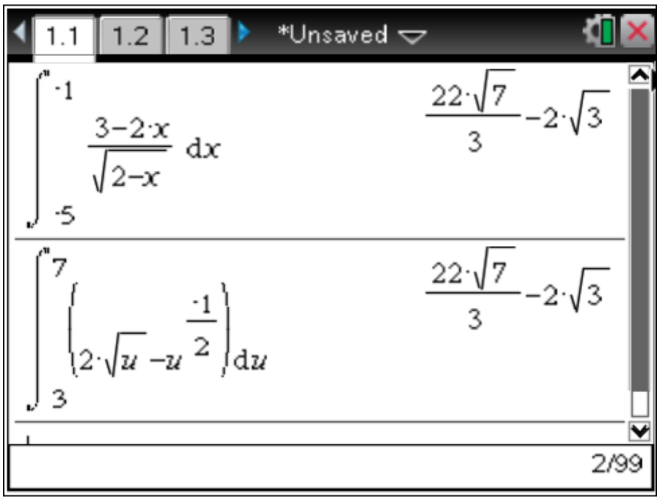
$x = -1 \Rightarrow u = 3$

$x = -5 \Rightarrow u = 7$

So

$$\begin{aligned}
\int_{-5}^{-1} \frac{3-2x}{\sqrt{2-x}}\,dx &= \int_7^3 \frac{3-2(2-u)}{\sqrt{u}}.-du \\
&= -\int_7^3 \frac{3-4+2u}{\sqrt{u}}\,du \\
&= -\int_7^3 u^{-\frac{1}{2}}(-1+2u)\,du \\
&= -\int_7^3 \left(2u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du \\
&= \int_3^7 \left(2u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du
\end{aligned}$$

Alternatively, the CAS calculator can be used to calculate the value of the given integral and then used to check the value of four of the options. Whichever option has the same value as the given integral will be the correct answer. If none of the four options have the correct answer, then the correct option will be the one that was not checked.



Tips

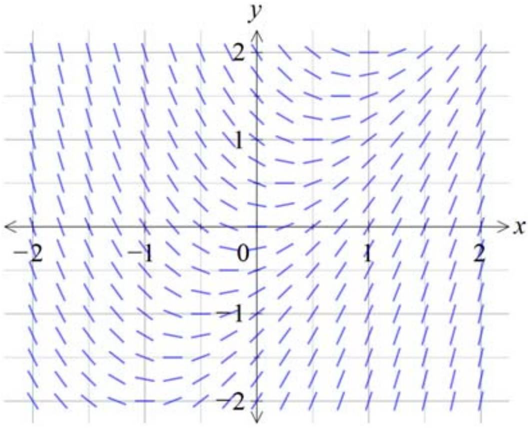
- *Don't forget to change the values of the terminals to match the substitution used.*
- *Remember to use properties of integral terminals such as* $-\int_a^b f(x)dx = \int_b^a f(x)dx$ *to get the correct final answer.*

Question 11

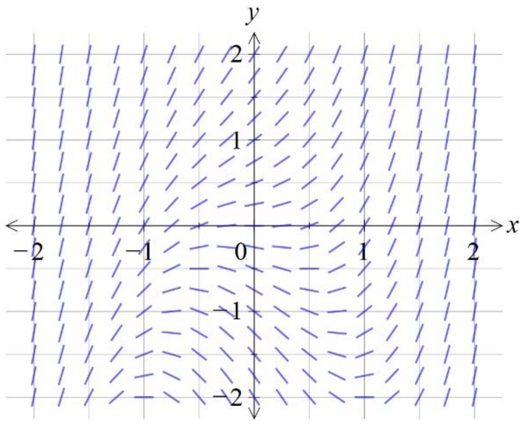
Answer: E

Worked solution

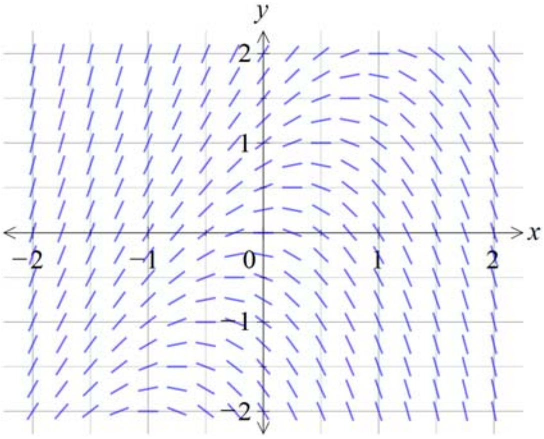
Option A has the slope field:



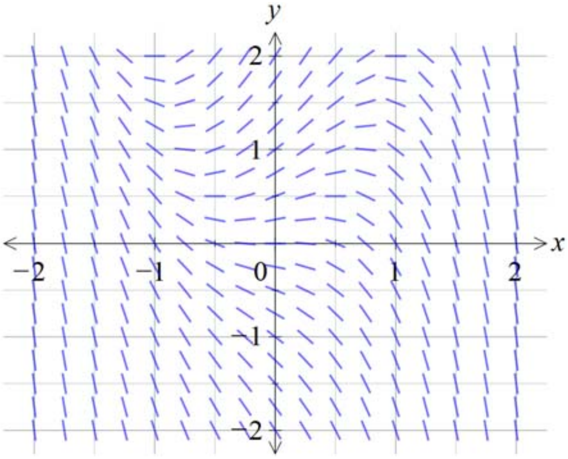
Option B has the slope field:



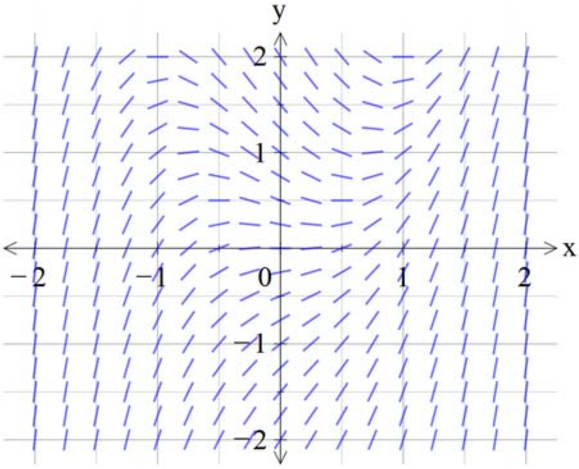
Option C has the slope field:



Option D has the slope field:



Option E has the slope field:



Therefore, option E is correct.

Alternatively, if we look at the behaviour of $\frac{dy}{dx}$ for $x = 0$ and $y > 0$, we can see that in the slope field given $\frac{dy}{dx} < 0$. This eliminates options B, C and D as $\frac{dy}{dx} > 0$ for these values of x and y.

If we then look at the behaviour of $\frac{dy}{dx}$ for $x = -1$ and $y > 0$, we can see that in the slope field given $\frac{dy}{dx} > 0$. This then eliminates option A as $\frac{dy}{dx} < 0$ for these values of x and y.

Tip

- *When graphing the slope field on the calculator, set the scale and size of the axes to the same as those used in the question. This will make it easier to see and compare the slope field to the one in the question. When doing this, only four options need to be checked. Either the correct option is one of the four or, if all four are wrong, the remaining option is the correct response.*

Question 12

Answer: C

Worked solution

Implicitly differentiating the function gives:

$$\cos(y) - x\sin(y)\frac{dy}{dx} + \sin(x)\frac{dy}{dx} + y\cos(x) = 0$$

Substituting the point $\left(\frac{\pi}{2}, \frac{\pi}{3}\right)$ and then simplifying gives:

$$\cos\left(\frac{\pi}{3}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{3}\right)\frac{dy}{dx} + \sin\left(\frac{\pi}{2}\right)\frac{dy}{dx} + \frac{\pi}{3}\cos\left(\frac{\pi}{2}\right) = 0$$

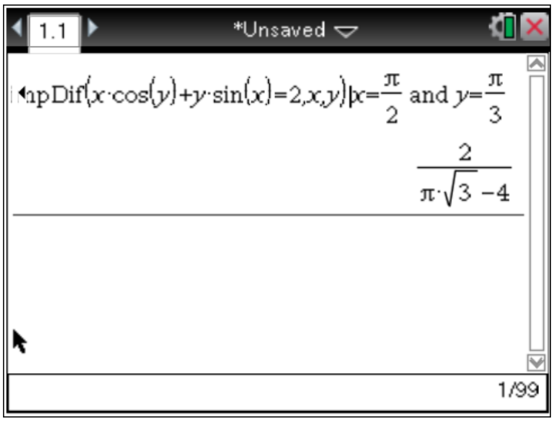
$$\frac{1}{2} - \frac{\sqrt{3}\pi}{4}\frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$\left(1 - \frac{\sqrt{3}\pi}{4}\right)\frac{dy}{dx} = -\frac{1}{2}$$

$$\left(\frac{4 - \sqrt{3}\pi}{4}\right)\frac{dy}{dx} = -\frac{1}{2}$$

$$\frac{dy}{dx} = \frac{2}{\pi\sqrt{3} - 4}$$

Using the CAS:



Question 13

Answer: C

Worked solution

Using Euler's method:

$$\frac{dy}{dx} = f(x),\ x_{n+1} = x_n + h\ ,\ y_{n+1} = y_n + hf(x_n)$$

$$x_0 = 2,\ y_0 = 1$$

$$x_1 = 2.1,\ y_1 = 1 + 0.1\left[\left(2+2\right)^2 - 1\right] = 2.5$$

$$x_2 = 2.2,\ y_2 = 2.5 + 0.1\left[\left(2.1+2\right)^2 - 1\right] = 4.08$$

$$x_3 = 2.3,\ y_3 = 4.08 + 0.1\left[\left(2.2+2\right)^2 - 1\right] = 5.745$$

So $y_3 = 5.745$, which is closest to 5.75, therefore select option C.

Question 14

Answer: C

Worked solution

Change in momentum is:

$$\begin{aligned}\Delta \underset{\sim}{\mathrm{p}} &= m\,\Delta \underset{\sim}{\mathrm{v}} \\ &= 3[(-3-2)\underset{\sim}{\mathrm{i}} + (4-(-3))\underset{\sim}{\mathrm{j}}] \\ &= -15\underset{\sim}{\mathrm{i}} + 21\underset{\sim}{\mathrm{j}}\end{aligned}$$

Question 15

Answer: A

Worked solution

From the diagram, the downward direction is positive, so an equation of motion for the body of mass m_2 is

$$m_2a = m_2g - T_2$$
$$\Rightarrow T_2 = m_2(g - a)$$

An equation of motion for the body of mass m_1 is

$$m_1a = T_2 + m_1g - T_1$$
$$\Rightarrow T_1 = T_2 + m_1(g - a)$$

Substituting T_2 into the equation and simplifying gives

$$T_1 = m_2(g - a) + m_1(g - a)$$
$$T_1 = (m_1 + m_2)(g - a)$$

Tips

- *Drawing free body diagrams of the forces in a connected particles question is a good way to visualise all the forces acting and in which direction they act.*
- *When a question assigns a particular direction of a force/acceleration to be positive, make sure to follow that direction in your calculations to get the correct answer.*

Question 16

Answer: B

Worked solution

From the information given, $a = 2x + 2$.

Expressing the acceleration as $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ gives

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x + 2$$
$$\frac{1}{2}v^2 = \int 2x + 2\,dx$$
$$\frac{1}{2}v^2 = x^2 + 2x + c$$

Applying the given conditions of $v = 3\sqrt{2}$ and $x = 2$ gives $c = 1$.

So

$$\frac{1}{2}v^2 = x^2 + 2x + 1$$
$$v^2 = 2x^2 + 4x + 2$$
$$v = \pm\sqrt{2x^2 + 4x + 2}$$

Checking the given conditions $v = 3\sqrt{2}$ and $x = 2$ means that the negative branch is rejected.

$$v = \sqrt{2x^2 + 4x + 2}$$
$$v = \sqrt{2(x^2 + 2x + 1)}$$
$$v = \sqrt{2(x+1)^2}$$
$$v = \sqrt{2}(x+1)$$

Tip

- *When taking the square root, make sure to check the conditions given to determine whether to eliminate the positive or negative branch.*
- *As this question can also be solved using a calculator, it is important to know how to solve differential equations with a given boundary on CAS.*

Question 17

Answer: E

Worked solution

The resultant force is R.

The equation of motion of the lift floor is

$$(m+M)a = R-(m+M)g$$
$$\Rightarrow \quad R = (m+M)a+(m+M)g$$
$$R = (m+M)(a+g)$$

Tip

- *Drawing a force diagram of all forces is a useful way of visualising all the forces with their directions, which will help in the derivation of the equations of motion.*

Question 18

Answer: D

Worked solution

The mean is given by

$$\begin{aligned}\mu &= E(W) = E(2X)+E(-3Y)\\ &= 2E(X)-3E(Y)\\ &= 2\times 9-3\times 7\\ &= -3\end{aligned}$$

The standard deviation is given by

$$\begin{aligned}\sigma &= \sqrt{\text{var}(W)} = \sqrt{\text{var}(2X)+\text{var}(-3Y)}\\ &= \sqrt{2^2\,\text{var}(X)+(-3)^2\,\text{var}(Y)}\\ &= \sqrt{4\times 4^2+9\times 2^2}\\ &= \sqrt{100}\\ &= 10\end{aligned}$$

Question 19

Answer: B

Worked solution

Does not exceed 202 g is the same as less than 202 g.

$$\text{So}\ \Pr(\bar{X} < 202) = \Pr\left(Z < \frac{202-200}{5/\sqrt{16}}\right)$$
$$= 0.945$$

Tip

- *In probability questions, pay close attention to the way the question is worded. For instance, in this question, 'does not exceed' is the same as 'less than'. So be aware of the wording so that you know whether you are trying to calculate the probability for a 'less than' or 'more than' interval.*

Question 20

Answer: B

Worked solution

A type I error is when the null hypothesis is rejected when it is true.

In this scenario, the null hypothesis is that the growth rate is the same.

Option B is the only option that meets the type I error guidelines.

'Concluding that the growth rate is higher than average when using the special fertiliser' represents rejecting the null hypothesis, and 'when in fact it is not' represents the null hypothesis is true. Therefore, option B is a type I error.

SECTION B

Question 1a.i.

Worked solution

$x=1$ and $y=-1$

Mark allocation: 1 mark

- 1 mark for writing both equations of the asymptotes

Question 1a.ii.

Worked solution

Stationary points occur when the derivative is equal to zero.

$$f'(x)=-\frac{2(x+1)}{(x-1)^3}=0, \quad \Rightarrow \quad x=-1$$

So $f(-1)=\dfrac{2(-1)}{(-1-1)^2}-1=-\dfrac{3}{2}$.

So the coordinates of the stationary point are $\left(-1,-\dfrac{3}{2}\right)$.

Mark allocation: 2 marks

- 1 mark for equating the first derivative to zero
- 1 mark for correct coordinate

Question 1a.iii.

Worked solution

A function is concave up when $f''(a)>0$.

$$f''(x)=\frac{4(x+2)}{(x-1)^4}$$

$$f''(-1)=\frac{4(-1+2)}{(-1-1)^4}=\frac{1}{4}$$

Since $f''(-1)=\dfrac{1}{4}>0$, the stationary point $\left(-1,-\dfrac{3}{2}\right)$ is concave up.

Mark allocation: 2 marks

- 1 mark for finding the second derivative, substituting $x=-1$ into it and evaluating
- 1 mark for concluding that the stationary point is concave up based on the evaluation of the second derivative

Tip

- *In 'show that questions', even though the result may be known or obvious, some form of working needs to be shown that proves/justifies the result.*

Question 1a.iv.

Worked solution

A point of inflection occurs when $f''(a) = 0$.

$$f''(x) = \frac{4(x+2)}{(x-1)^4} = 0$$

$\Rightarrow \quad x = -2$

So $f(-2) = \dfrac{2(-2)}{(-2-1)^2} - 1 = -\dfrac{13}{9}$.

So, there is a point of inflection at $\left(-2, -\dfrac{13}{9}\right)$.

Mark allocation: 2 marks

- 1 mark for equating the second derivative to 0 and finding $x = -2$
- 1 mark for the correct point of inflection coordinate

Explanatory note

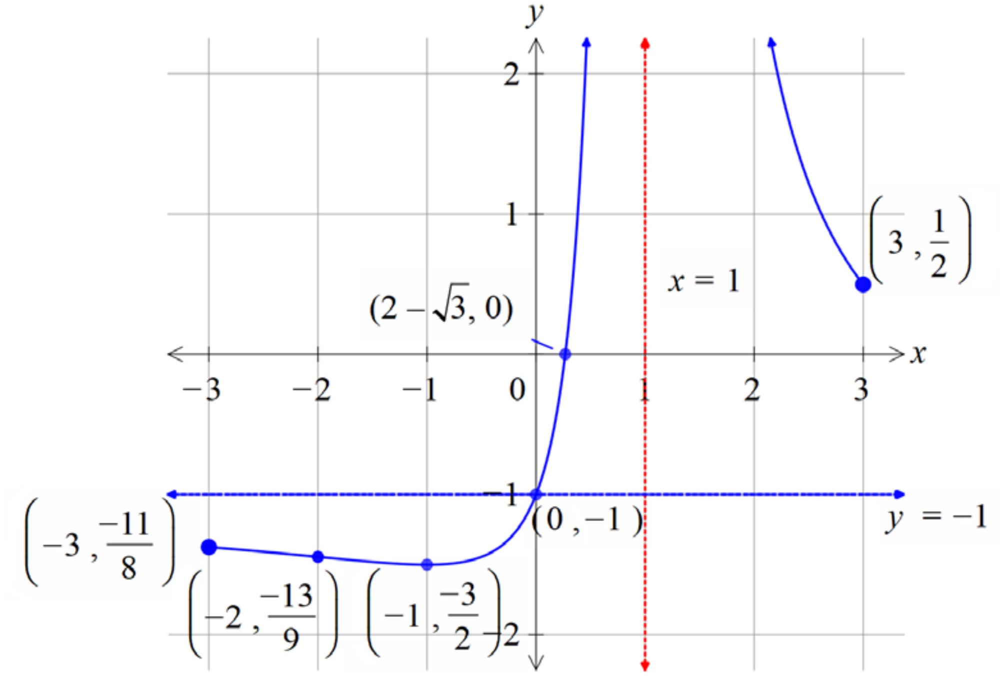
The result $f''(a) = 0$ doesn't always result in an inflection point at $x = a$. An inflection point is when the concavity of a curve changes at a point; that is, the sign of the second derivative changes, so further investigation is sometimes necessary.

For instance, if $f(x) = x^4$ at $x = 0$, we have a second derivative of $f''(x) = 12x^2$. Substituting $x = 0$ into the second derivative would give $f''(0) = 0$. However, when you check the sign of the second derivative either side of $x = 0$, $f''(x)$ will always be greater than zero. Therefore, there is no sign change at $x = 0$ and the concavity of the curve has not changed, so $x = 0$ is not a point of inflection.

For $f''(x) = \dfrac{4(x+2)}{(x-1)^4}$, $f''(x) < 0$ for $x < -2$ and $f''(x) > 0$ for $x > -2$. Therefore, it can be seen that there is a sign change at $x = -2$, so a point of inflection exists at $x = -2$.

Question 1b.

Worked solution



Mark allocation: 3 marks

- 1 mark for a sketch with an accurate shape that shows the asymptotes with their equations and labelled end points
- 1 mark for accurately showing and labelling all stationary points and points of inflection
- 1 mark for accurately showing and labelling all intercepts

Tip

- *Sketch the graph using your calculator. Make the scale and size of the axes on the calculator the same as in the question to improve the accuracy of the sketch.*

Question 2a.i.

Worked solution

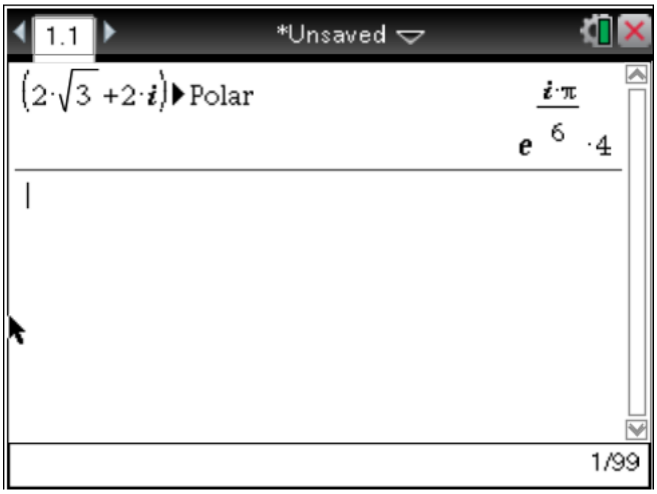
Polar form is given by $r\,\text{cis}(\theta)$.

$$r = \sqrt{(2\sqrt{3})^2 + 2^2} = 4$$

$$\theta = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6}$$

So, in polar form: $4\,\text{cis}\left(\frac{\pi}{6}\right)$.

Alternatively, using technology:



Mark allocation: 1 mark

- 1 mark for correct polar form of $4\,\text{cis}\left(\frac{\pi}{6}\right)$

Question 2a.ii.

Worked solution

Since all coefficients of the equation are real, this means that one of the other solutions is the complex conjugate of z_1.

So $z_2 = 2\sqrt{3} - 2i$.

Then:

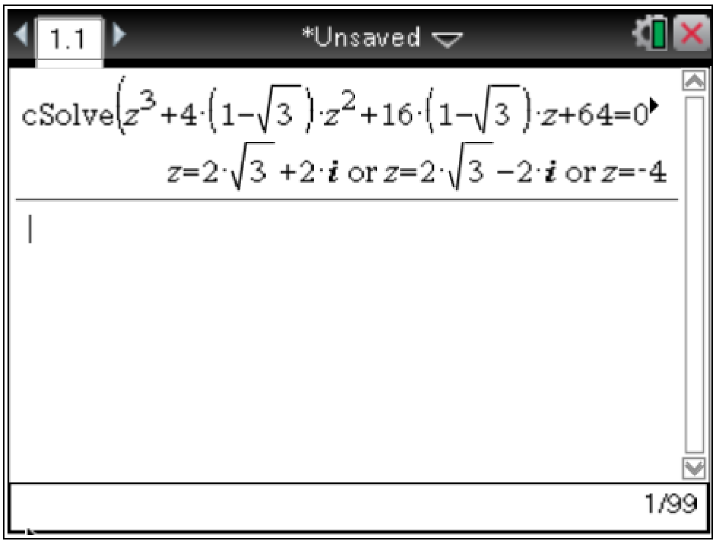
$$\left(z - 2\sqrt{3} - 2i\right)\left(z - 2\sqrt{3} + 2i\right)(z + a) = z^3 + 4\left(1 - \sqrt{3}\right)z^2 + 16\left(\sqrt{3} + 1\right)z + 64$$

$$\left(z^2 - 4\sqrt{3}z + 16\right)(z + a) = z^3 + 4\left(1 - \sqrt{3}\right)z^2 + 16\left(1 - \sqrt{3}\right)z + 64$$

By equating coefficients, it can be seen that $a = 4$.

So $z_3 = -4$.

Alternatively, using technology:

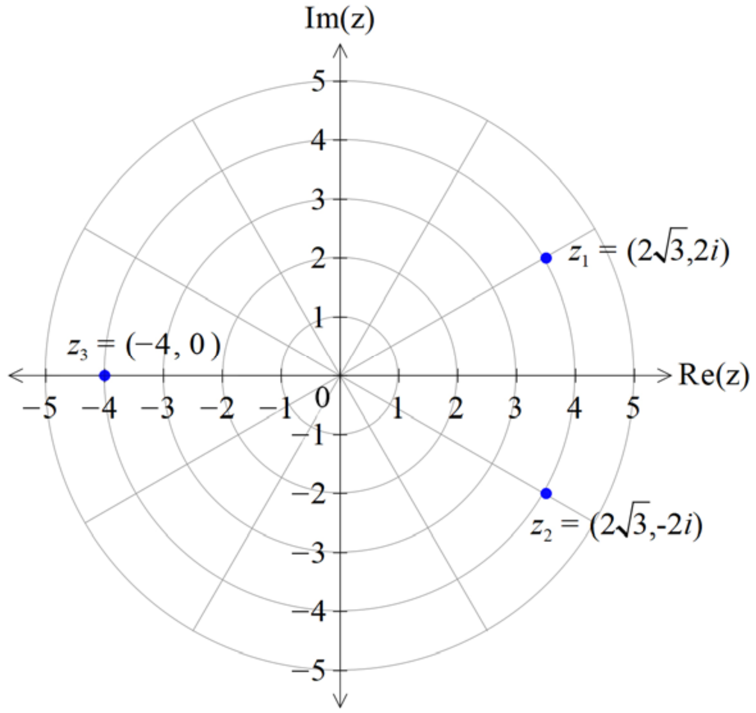


Mark allocation: 2 marks

- 1 mark for a correct root of $z_2 = 2\sqrt{3} - 2i$
- 1 mark for a correct root of $z_3 = -4$

Question 2b.

Worked solution



Mark allocation: 3 marks

- 1 mark for correctly plotted and labelled root $z_1 = 2\sqrt{3} + 2i$
- 1 mark for correctly plotted and labelled root $z_2 = 2\sqrt{3} - 2i$
- 1 mark for correctly plotted and labelled root $z_3 = -4$

Question 2c.

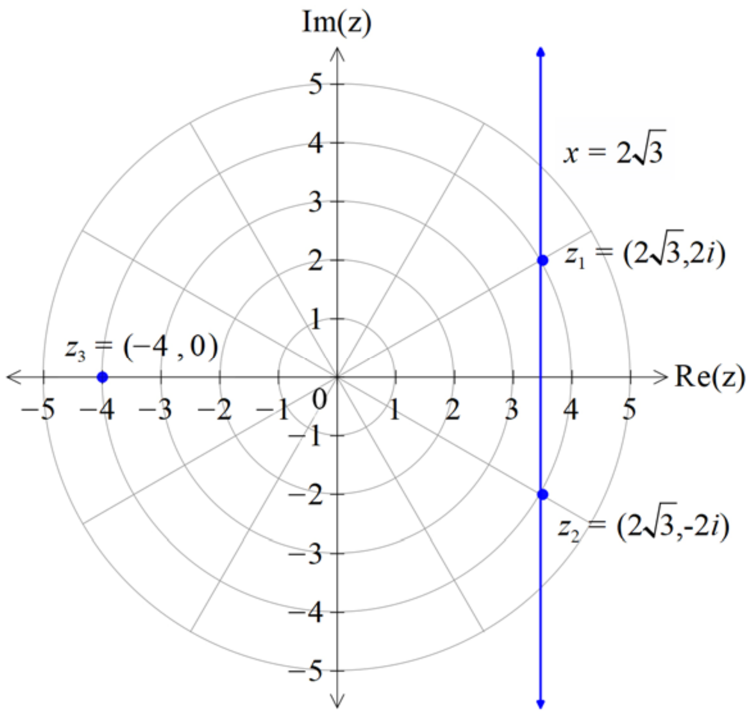
Worked solution

$|z-4\sqrt{3}|=|z|$ is equivalent to:

$$|x+iy-4\sqrt{3}|=|x+iy|$$
$$\sqrt{(x-4\sqrt{3})^2+y^2}=\sqrt{x^2+y^2}$$
$$(x-4\sqrt{3})^2+y^2=x^2+y^2$$

Expanding and collecting like terms leads to:

$$-8\sqrt{3}x+48=0$$
$$\Rightarrow x=\frac{6}{\sqrt{3}}=2\sqrt{3}$$



Mark allocation: 3 marks

- 1 mark for showing $\sqrt{(x-4\sqrt{3})^2+y^2}=\sqrt{x^2+y^2}$
- 1 mark for getting the Cartesian equation $x=2\sqrt{3}$
- 1 mark for correctly sketching the line $|z-4\sqrt{3}|=|z|$ going through the points $z_1=2\sqrt{3}+2i$ and $z_2=2\sqrt{3}-2i$

Question 2d.

Worked solution

The segment described is made up from the area of a sector (which is formed by z_1, z_2 and the origin) with the area of a triangle (which is formed with vertices at z_1, z_2 and the origin) being subtracted from the sector.

$$\begin{aligned}\text{Area} &= \frac{\frac{\pi}{3}}{2\pi}\times\pi\times4^2-\frac{1}{2}\times4\times2\sqrt{3}\\ &= \frac{8\pi}{3}-4\sqrt{3}\\ &= \frac{8\pi-12\sqrt{3}}{3}\end{aligned}$$

Alternatively, use the area of a segment formula $A=\frac{1}{2}r^2\left(\theta-\sin(\theta)\right)$, with $r=4$ and $\theta=\frac{\pi}{3}$.

So, the area is:

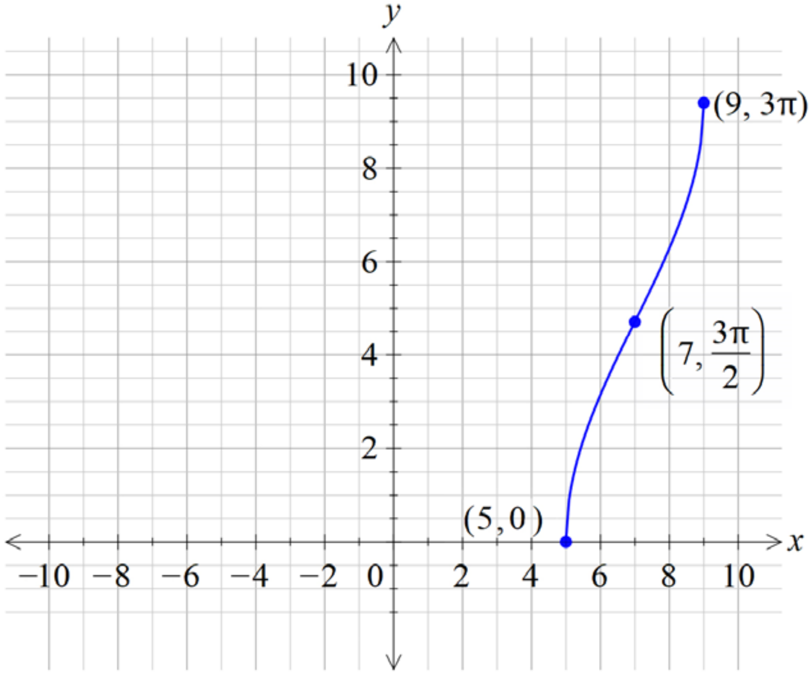
$$\begin{aligned}A &= \frac{1}{2}4^2\left(\frac{\pi}{3}-\sin\left(\frac{\pi}{3}\right)\right)\\ &= 8\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)\\ &= \frac{8\pi-12\sqrt{3}}{3}\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for subtracting the area of a triangle from the area of a sector or using the area of a segment formula
- 1 mark for a correct final answer of $\frac{8\pi-12\sqrt{3}}{3}$

Question 3a.

Worked solution



Mark allocation: 3 marks

- 1 mark for an accurately shaped graph with correctly labelled intercepts
- 1 mark for correctly labelled end points
- 1 mark for correctly labelled point of inflection

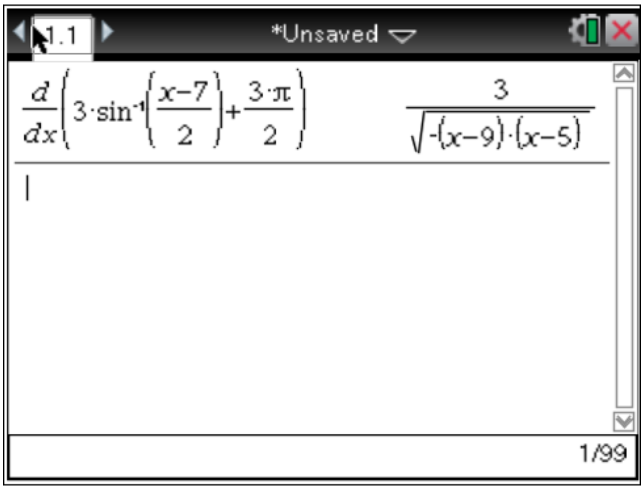
Question 3b.

Worked solution

Arc length is given by $\int_{x_1}^{x_2} \sqrt{1+(f'(x))^2}\, dx$.

The derivative of the curve $f'(x)$ is:

$$f'(x) = \frac{3}{\sqrt{-(x-5)(x-9)}}$$



So, the arc length is:

$$\int_5^9 \sqrt{1+\left(\frac{3}{\sqrt{-(x-5)(x-9)}}\right)^2}\, dx$$

$$= \int_5^9 \sqrt{1+\frac{9}{-(x-5)(x-9)}}\, dx$$

$$= \int_5^9 \sqrt{1-\frac{9}{(x-5)(x-9)}}\, dx$$

So:

$a = 9,\ b = -5,\ c = -9$

or

$a = 9,\ b = -9,\ c = -5$

Mark allocation: 2 marks

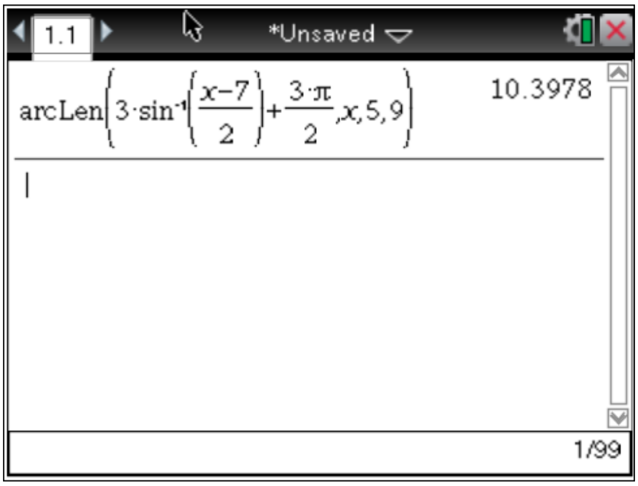
- 1 mark for finding $f'(x) = \dfrac{3}{\sqrt{-(x-5)(x-9)}}$
- 1 mark for correct values of a, b and c

Question 3c.

Worked solution

Evaluating the definite integral in **part b.** gives:

$$\int_5^9 \sqrt{1-\frac{9}{(x+5)(x+9)}}\,dx$$
$$=10.3978...$$
$$=10.398 \text{ cm}$$



Mark allocation: 1 mark

- 1 mark for a correct final answer of 10.398 cm

Question 3d.i.

Worked solution

The volume generated when a curve is rotated about the y-axis is given by:

$$V=\pi\int_a^b x^2 dy$$

Rearranging the equation of the curve in terms of y gives:

$$x=2\sin\left(\frac{1}{3}\left(y-\frac{3\pi}{2}\right)\right)+7$$

Therefore, the definite integral for the volume is:

$$V=\pi\int_0^{3\pi}\left[2\sin\left(\frac{1}{3}\left(y-\frac{3\pi}{2}\right)\right)+7\right]^2 dy$$

Alternatively,

$$V = \pi\int_0^{3\pi}\left[7-2\cos\left(\frac{y}{3}\right)\right]^2 dy$$

Mark allocation: 1 mark

- 1 mark for a correct definite integral of $V = \pi\int_0^{3\pi}\left[2\sin\left(\frac{1}{3}\left(y-\frac{3\pi}{2}\right)\right)+7\right]^2 dy$ or

  $V = \pi\int_0^{3\pi}\left[7-2\cos\left(\frac{y}{3}\right)\right]^2 dy$

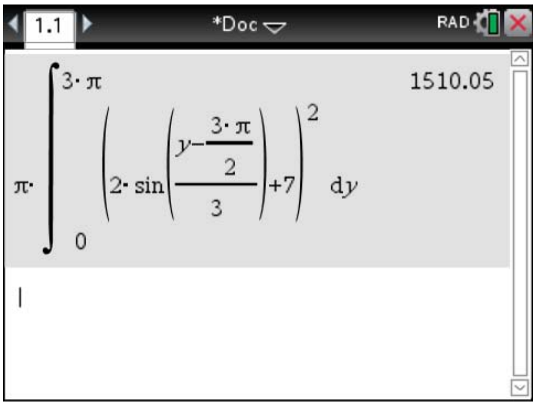
Question 3d.ii.

Worked solution

Evaluating the integral found in **part d.i.** gives:

$$V = \pi\int_0^{3\pi}\left[2\sin\left(\frac{1}{3}\left(y-\frac{3\pi}{2}\right)\right)+7\right]^2 dy$$
$$= 1510.05 \text{ cm}^3$$

Using CAS to evaluate the integral from **part d.i.** gives a volume of 1510.05 cm^3.



Mark allocation: 1 mark

- 1 mark for a correct final answer of 1510.05 cm^3

Question 3e.

Worked solution

Newtons law of cooling is:

$$\frac{dT}{dt} = -k(T - T_{\text{s}})$$

Substituting $T_{\text{s}} = 20$ gives:

$$\frac{dT}{dt} = -k(T - 20)$$

Separating the variables gives:

$$\frac{dT}{(T-20)} = -k.dt$$

Integrating both sides gives:

$$\int \frac{dT}{(T-20)} = \int -k.dt$$

$$\Rightarrow \log_e(T-20) = -kt + c$$

$$\Rightarrow \quad T - 20 = e^{-kt+c}$$

$$\Rightarrow \quad T = 20 + be^{-kt}$$

Where $b = e^c$.

So $a = 20$.

Applying the initial conditions gives:

$$90 = 20 + be^{-k\times 0}$$

$$\Rightarrow b = 70$$

To find k, substitute $T = 60$ and $t = 15$ and solve for k.

$$60 = 20 + 70e^{-15k}$$

$$\Rightarrow k = \frac{-1}{15}\log_e\left(\frac{40}{70}\right)$$

$$= 0.037...$$

$$= 0.04$$

Therefore, $a = 20, b = 70$ and $k = 0.04$, and the temperature of the coffee is given as $T = 20 + 70e^{-0.04t}$.

Alternatively, we have

$$\frac{dT}{(T-20)} = -k.dt$$

Integrating both sides over the given conditions to find k

$$\int_{90}^{60} \frac{dT}{(T-20)} = \int_{0}^{15} -k.dt$$

$$\left[\log_e(T-20)\right]_{90}^{60} = \left[-kt\right]_0^{15}$$

$$\Rightarrow \log_e\left(\frac{40}{70}\right) = -15k$$

$$\Rightarrow k = -\frac{1}{15}\log_e\left(\frac{4}{7}\right) = 0.037...$$

$$= 0.04$$

This then gives

$$\int \frac{dT}{(T-20)} = \int -0.04.dt$$

Integrating both sides gives

$$\log_e(T-20) = -0.04t + c$$

$$\Rightarrow T - 20 = e^{-0.04t+c}$$

$$= T = 20 + be^{-0.04t}$$

Where $b = e^c$

So $a = 20.$

Applying the initial conditions gives

$$90 = 20 + be^{-0.04\times 0}$$

$$\Rightarrow b = 70$$

Therefore $a = 20, b = 70, k = 0.04$ and the temperature of the coffee is given as

$T = 20 + 70e^{-0.04t}$

Mark allocation: 3 marks

- 1 mark for separating the variables and integrating both sides of the equation
- 1 mark for applying the conditions in the question to get $a = 20, b = 70$ and $k = 0.04$, correct to two decimal places
- 1 mark for stating the equation $T = 20 + 70e^{-0.04t}$

Question 4a.i.

Worked solution

The distance can be found using the formula $|\underset{\sim}{\mathrm{r}}_{\mathrm{G}}(t_2)-\underset{\sim}{\mathrm{r}}_{\mathrm{G}}(t_1)|$.

Substituting $t_1=0$ and $t_2=\dfrac{3\pi}{4}$ and then evaluating gives:

$$|\underset{\sim}{\mathrm{r}}_{\mathrm{G}}\left(\frac{3\pi}{4}\right)-\underset{\sim}{\mathrm{r}}_{\mathrm{G}}(0)|$$

$$=\left|\left(1-2\cos\left(2\times\frac{3\pi}{4}\right)-1+2\cos(2\times 0)\right)\underset{\sim}{\mathrm{i}}+\left(2\sin\left(2\times\frac{3\pi}{4}\right)+3-2\sin(0)-3\right)\underset{\sim}{\mathrm{j}}\right|$$

$$=\sqrt{2^2+(-2)^2}$$

$$=\sqrt{8}$$

$$=2\sqrt{2}$$

Mark allocation: 1 mark

- 1 mark for correct final answer of $2\sqrt{2}$

Question 4a.ii.

Worked solution

Speed is given by $\left|\dot{\underset{\sim}{\mathrm{r}}}_{\mathrm{G}}(t)\right|$.

So $\dot{\underset{\sim}{\mathrm{r}}}_{\mathrm{G}}=4\sin(2t)\underset{\sim}{\mathrm{i}}+4\cos(2t)\underset{\sim}{\mathrm{j}}$.

Then:

$$\left|\dot{\underset{\sim}{\mathrm{r}}}_{\mathrm{G}}\left(\frac{3\pi}{4}\right)\right|=\left|4\sin\left(2\times\frac{3\pi}{4}\right)\underset{\sim}{\mathrm{i}}+4\cos\left(2\times\frac{3\pi}{4}\right)\underset{\sim}{\mathrm{j}}\right|$$

$$=\sqrt{\left(4\sin\left(2\times\frac{3\pi}{4}\right)\right)^2+\left(4\cos\left(2\times\frac{3\pi}{4}\right)\right)^2}$$

$$=\sqrt{16}$$

$$=4$$

Mark allocation: 2 marks

- 1 mark for finding $\dot{\underset{\sim}{\mathrm{r}}}_{\mathrm{G}}=4\sin(2t)\underset{\sim}{\mathrm{i}}+4\cos(2t)\underset{\sim}{\mathrm{j}}$
- 1 mark for correct final answer of 4

Question 4b.

Worked solution

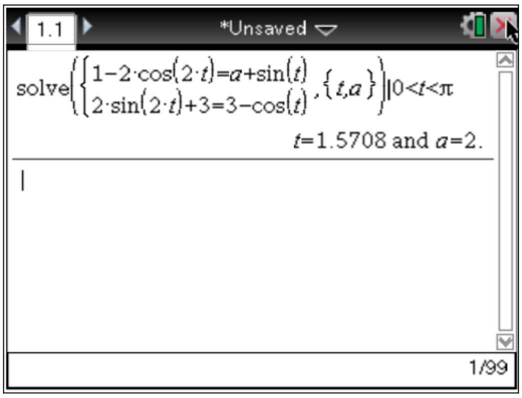
For a collision to occur, both Geoff and Dustin must have the same position at the same time.

Equating the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of the position vectors and solving simultaneously gives:

$1-2\cos(2t)=a+\sin(t)$ [1]

$2\sin(2t)+3=3-\cos(t)$ [2]

$\Rightarrow t=\dfrac{\pi}{2}$ and $a=2$.



Solving equation [2] to get $t=\dfrac{\pi}{2}$ can be done as follows.

$$
\begin{aligned}
& 2\sin(2t)=-\cos(t)\\
\Rightarrow\ & 4\sin(t)\cos(t)+\cos(t)=0\\
\Rightarrow\ & \cos(t)\big(4\sin(t)+1\big)=0
\end{aligned}
$$

So, either $\cos(t)=0$ or $4\sin(t)+1=0$.

For $\cos(t)=0$, $t=\dfrac{\pi}{2}$ is the only solution inside the domain.

For $4\sin(t)+1=0\Rightarrow\sin(t)=-\dfrac{1}{4}$, there is no solution inside the domain.

Thus, $t=\dfrac{\pi}{2}$.

Substituting $t=\dfrac{\pi}{2}$ into either position vector will give the coordinates of the collision, which are:

$$
\begin{aligned}
\underset{\sim}{r}_G\left(\frac{\pi}{2}\right) &= \left(1-2\cos\left(2\times\frac{\pi}{2}\right)\right)\underset{\sim}{i}+\left(2\sin\left(2\times\frac{\pi}{2}\right)+3\right)\underset{\sim}{j}\\
&= 3\underset{\sim}{i}+3\underset{\sim}{j}
\end{aligned}
$$

Mark allocation: 3 marks

- 1 mark for equating the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of the position vectors
- 1 mark for correctly finding $a = 2$
- 1 mark for correct coordinates of the collision $3\underset{\sim}{i} + 3\underset{\sim}{j}$

Tip

- *When solving trigonometric equations using the calculator, it is a good idea to restrict the domain to filter out solutions not in the required domain.*

Question 4c.

Worked solution

The parametric equations that describe the motion of Dustin's path are $x-2=\sin(t)$ and $-(y-3)=\cos(t)$.

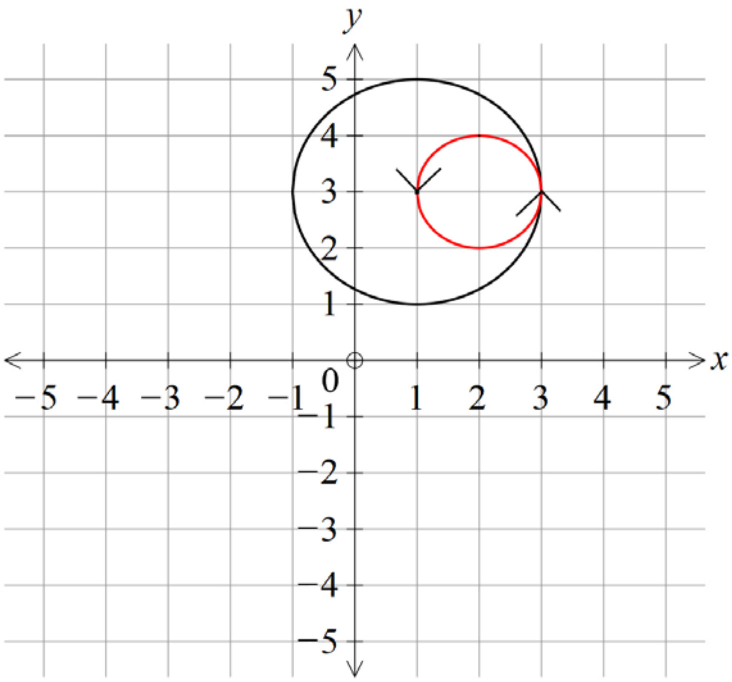
These give the following Cartesian equation for a circle:

$(x-2)^2+(y-3)^2=1$

The initial position of Dustin is when $t=0$, which gives the coordinate (2, 2).

From **part b.**, we know that Dustin and Geoff collide at (3, 3); therefore, Dustin is travelling in an anticlockwise direction.

So, Dustin's path with direction given is:



Mark allocation: 2 marks

- 1 mark for sketching a circle of radius 1 with a centre at (2,3)
- 1 mark for indicating a path that is anticlockwise

Question 4d.

Worked solution

The angle between two moving particles is found by finding the angle between the velocity vectors.

From **part a.ii.**, the velocity vector for Geoff is:

$$\dot{\underset{\sim}{r}}_G = 4\sin(2t)\,\underset{\sim}{i} + 4\cos(2t)\,\underset{\sim}{j}$$

The velocity vector for Dustin is:

$$\dot{\underset{\sim}{r}}_D = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j}$$

So, the angle between the velocity vectors when $t = \frac{\pi}{2}$ is given by:

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right).\dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)}{|\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right)||\dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)|}\right)\\
&= \cos^{-1}\left(\frac{(0\,\underset{\sim}{i} - 4\,\underset{\sim}{j}).(0\,\underset{\sim}{i} + \underset{\sim}{j})}{\sqrt{(0)^2+(-4)^2}\sqrt{(0)^2+(1)^2}}\right)\\
&= \cos^{-1}\left(\frac{-4}{4}\right)\\
&= \pi
\end{aligned}$$

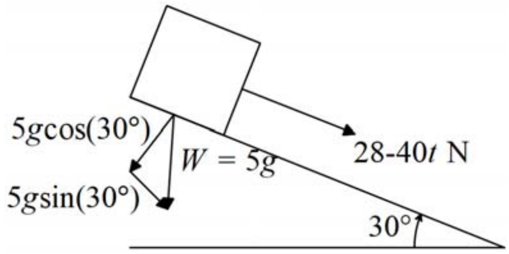
Mark allocation: 3 marks

- 1 mark for finding $\dot{\underset{\sim}{r}}_D = \cos(t)\,\underset{\sim}{i} + \sin(t)\,\underset{\sim}{j}$
- 1 mark for using $\theta = \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right).\dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)}{|\dot{\underset{\sim}{r}}_G\left(\frac{\pi}{2}\right)||\dot{\underset{\sim}{r}}_D\left(\frac{\pi}{2}\right)|}\right)$ or similar
- 1 mark for a correct angle of π

Question 5a.

Worked solution

A free body diagram of all the forces looks like:



Considering the forces acting parallel to the ramp, the following equation of motion can be found.

$$\sum F = ma$$

$$\Rightarrow 28 - 40t + 5g\sin(30°) = 5a$$

Solving for a gives:

$$\begin{aligned} a &= \frac{28 - 40t + 5g\sin(30°)}{5} \\ &= \frac{28 + 2.5g - 40t}{5} \\ &= \frac{52.5 - 40t}{5} \\ &= 10.5 - 8t \end{aligned}$$

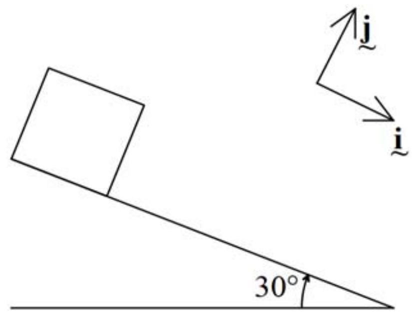
Mark allocation: 2 marks

- 1 mark for the equation of motion $28 - 40t + 5g\sin(30°) = 5a$
- 1 mark for showing appropriate working to get the result $10.5 - 8t$

Question 5b.

Worked solution

We resolve the motion of the block into $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components, where the $\underset{\sim}{i}$ component is parallel to the ramp and the $\underset{\sim}{j}$ component is perpendicular to the ramp, as shown in the diagram below.



The acceleration can then be written as the vector $\underset{\sim}{a}(t) = (10.5 - 8t)\underset{\sim}{i}$.

Integrating to get the velocity vector gives $\underset{\sim}{v}(t) = (10.5t - 4t^2)\underset{\sim}{i} + \underset{\sim}{c}$.

Apply initial conditions to find $\underset{\sim}{c}$:

$$\underset{\sim}{v}(0) = (10.5(0) - 4(0)^2)\underset{\sim}{i} + \underset{\sim}{c} = \underset{\sim}{0}$$
$$\Rightarrow \underset{\sim}{c} = \underset{\sim}{0}$$
$$\Rightarrow \underset{\sim}{v}(t) = (10.5t - 4t^2)\underset{\sim}{i}$$

The speed of the block at 0.7 seconds is then given by:

$$\begin{aligned}\left|\underset{\sim}{v}(0.7)\right| &= \left|10.5(0.7) - 4(0.7)^2\,\underset{\sim}{i}\right| \\ &= \sqrt{(5.39)^2} \\ &= 5.39\end{aligned}$$

Alternatively, use an integral to find the speed of the particle.

Since $a(t) = 10.5 - 8t$, then

$\int_0^{0.7} \left(10.5 - 8t\right) dt = 5.39 \text{ ms}^{-1}$.

Mark allocation: 2 marks

- 1 mark for finding the velocity vector $\underset{\sim}{v}(t) = \left(10.5t - 4t^2\right)\underset{\sim}{i}$
- 1 mark for a correct final answer of 5.39 ms^{-1}

OR

- 1 mark for use of integral with correct integrand and terminals
- 1 mark for correct final answer of 5.39 ms^{-1}

Tip

- *Terminating decimals are exact values when written in full (e.g. 0.25 is just as exact as $\frac{1}{4}$).*

Question 5c.

Worked solution

Since the question is referring to motion parallel to the ramp and the $\underset{\sim}{i}$ component is parallel to the ramp, the magnitude of the $\underset{\sim}{i}$ component of the position vector will represent the distance travelled along the ramp by the block.

From the previous question, the velocity vector is:

$$\underset{\sim}{v}(t) = (10.5t - 4t^2)\,\underset{\sim}{i}$$

Integrating gives:

$$\underset{\sim}{r}(t) = \left(5.25t^2 - \frac{4}{3}t^3\right)\underset{\sim}{i} + \underset{\sim}{b}$$

Assigning the initial position of the block to be $0\,\underset{\sim}{i} + 0\,\underset{\sim}{j}$ and applying this initial condition gives a position vector of:

$$\underset{\sim}{r}(0) = \left(5.25(0)^2 - \frac{4}{3}(0)^3\right)\underset{\sim}{i} + \underset{\sim}{b} = 0\,\underset{\sim}{i} + 0\,\underset{\sim}{j}$$

$$\Rightarrow \underset{\sim}{b} = \underset{\sim}{0}$$

$$\Rightarrow \underset{\sim}{r}(t) = \left(5.25t^2 - \frac{4}{3}t^3\right)\underset{\sim}{i}$$

Substituting the time 0.7 seconds gives:

$$\underset{\sim}{r}(0.7) = \left(5.25(0.7)^2 - \frac{4}{3}(0.7)^3\right)\underset{\sim}{i}$$

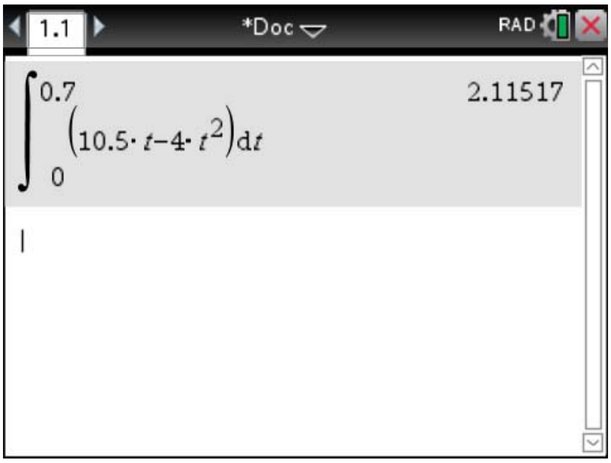
Finding the magnitude of the $\underset{\sim}{i}$ component:

$$\sqrt{\left(5.25(0.7)^2 - \frac{4}{3}(0.7)^3\right)^2} = 2.11517\ldots$$

$$= 2.12$$

So, the distance travelled is 2.12 m.

Alternatively, since $v = 10.5t - 4t^2$, the distance travelled by the particle in the first 0.7 seconds is $\int_0^{0.7} \left(10.5t - 4t^2\right) dt = 2.12$ m.



Mark allocation: 2 marks

- 1 mark for finding the position vector
- 1 mark for a correct final answer of 2.12 m

OR

- 1 mark for use of integral with correct integrand and terminals
- 1 mark for a correct final answer of 2.12 m

Question 5d.

Worked solution

Since gravity is the only force acting on the block, the acceleration vector is now $\underset{\sim}{a} = \left(g\sin(30°)\right)\underset{\sim}{i} = 4.9\,\underset{\sim}{i}$ and is valid only for $t > 0.7$ seconds.

Integrating to get the velocity vector gives $\underset{\sim}{v}(t^*) = \left(4.9(t^*)\right)\underset{\sim}{i} + \underset{\sim}{d}$, where $t^* = t - 0.7$.

Applying the initial condition that $\underset{\sim}{v}(t^*) = 5.3\,\underset{\sim}{i}$ when $t = 0.7$ gives a velocity vector of:

$$\underset{\sim}{v}(t^*) = \left(4.9(t^*) + 5.39\right)\underset{\sim}{i}$$

Integrating the velocity to get the position vector gives:

$$\underset{\sim}{r}(t^*) = \left(\frac{4.9}{2}(t^*)^2 + 5.39(t^*)\right)\underset{\sim}{i} + \underset{\sim}{e}$$

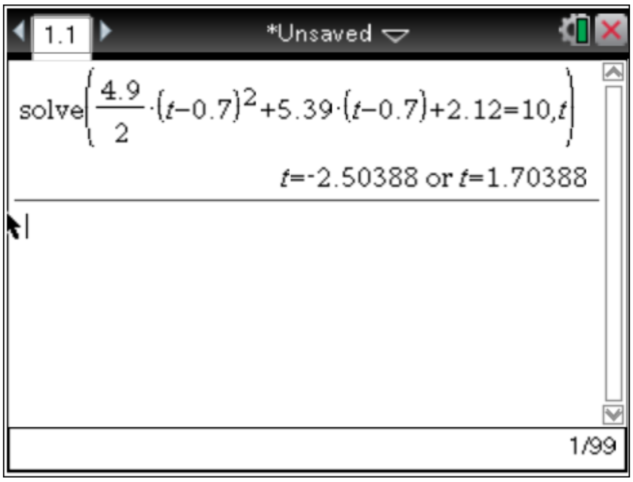
Applying the initial condition that $\underset{\sim}{r}(t^*) = 2.12\,\underset{\sim}{i}$ when $t = 0.7$ gives a position vector of:

$$\underset{\sim}{r}(t^*) = \left(\frac{4.9}{2}(t^*)^2 + 5.39(t^*) + 2.12\right)\underset{\sim}{i}$$

Replacing t^* with $t^* = t - 0.7$ and solving for t when $\underset{\sim}{r}(t - .07) = 10\,\underset{\sim}{i}$ gives:

$$\underset{\sim}{r}(t - 0.7) = \left(\frac{4.9}{2}(t - 0.7)^2 + 5.39(t - 0.7) + 2.12\right)\underset{\sim}{i} = 10\,\underset{\sim}{i}$$

$\Rightarrow \qquad t = -2.50388$ or $t = 1.70388$



Rejecting the negative time gives a total time taken as 1.7 seconds.

Alternatively, the total time can be found as total time $= 0.7 + t$, where *t* is the time taken for the block to reach the bottom of the ramp after the applied force is removed.

For this time period we can say that:

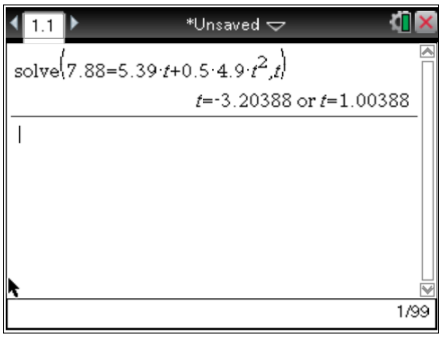
u = initial speed = 5.39 ms^{-1}

x = distance to the bottom of the ramp = 10 – 2.12 = 7.88 m

a = acceleration due to gravity = $g\sin(30°)$ = 4.9 ms^{-2}

Substituting the values into the formula for constant acceleration of $x = ut + \frac{1}{2}at^2$ and solving for *t* gives:

$$7.88 = 5.39t + \frac{1}{2} \times 4.9t^2$$
$$\Rightarrow t = -3.20388 \text{ or } t = 1.00388$$



Rejecting the negative value then gives a total time of:

$$\begin{aligned} \text{Total time} &= 0.7 + 1.00388 \\ &= 1.70388 \\ &= 1.7 \end{aligned}$$

So, the total time taken is 1.7 seconds.

Mark allocation: 3 marks

- 1 mark for using vector calculus techniques to find the position vector
- 1 mark for the equation $\underset{\sim}{r}(t-0.7) = \left(\frac{4.9}{2}(t-0.7)^2 + 5.39(t-0.7) + 2.12\right)\underset{\sim}{i} = 10\underset{\sim}{i}$
- 1 mark for a correct final answer of 1.7 seconds

OR

- 1 mark for using the formula for constant acceleration $x = ut + \frac{1}{2}at^2$
- 1 mark for finding that it takes 1.0 second for the block to reach the bottom of the ramp after the applied force has been removed
- 1 mark for a correct final answer of 1.7 seconds

Question 6a.

Worked solution

Let $\overline{X}$ be the distribution of the sample means.

The standard deviation is then:

$$\begin{aligned}\text{sd}(\overline{X}) &= \frac{\sigma}{\sqrt{n}} \\ &= \frac{3}{\sqrt{100}} \\ &= 0.3\end{aligned}$$

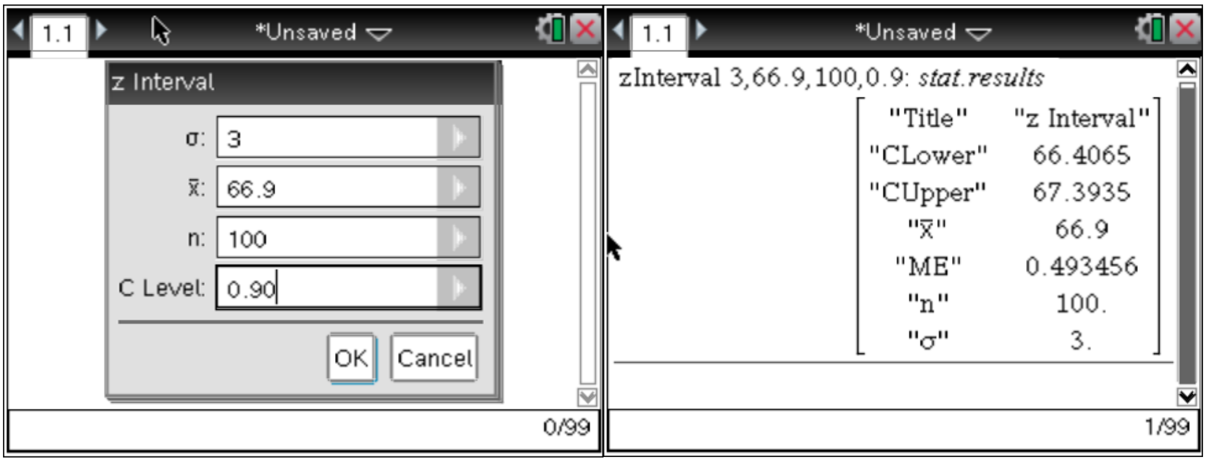
Mark allocation: 1 mark

- 1 mark for showing the calculation $\frac{3}{\sqrt{100}}$

Question 6b.

Worked solution

Begin by finding the 90% confidence interval by using the CAS.



So, the confidence interval is $(66.4065, 67.3935)$.

Note that the formula $\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$, where $z = 1.65$, can be used to find the 90% confidence interval.

Equate the confidence interval found with the confidence interval given in the question and solve for k_1 and k_2, correct to one decimal place.

So $66.9 - k_1 = 66.4065$ and $66.9 + k_2 = 67.3935$.

$\Rightarrow k_1 = 0.5$ and $k_2 = 0.5$.

Mark allocation: 2 marks

- 1 mark for finding the 90% confidence interval $(66.4065, 67.3935)$
- 1 mark for a correct final answer of $k_1 = 0.5$ and $k_2 = 0.5$

Question 6c.

Worked solution

Since the quality control tester claims the diameter exceeds the production company’s mean, this is an upper-tail hypothesis test.

The null and alternative hypotheses are then:

H_0: $\mu = 66.9$

H_1: $\mu > 66.9$

Mark allocation: 1 mark

- 1 mark for H_0: $\mu = 66.9$ and H_1: $\mu > 66.9$

Question 6d.

Worked solution

An expression for the p-value is $\Pr(\overline{X} > 67 \mid \mu = 66.9)$.

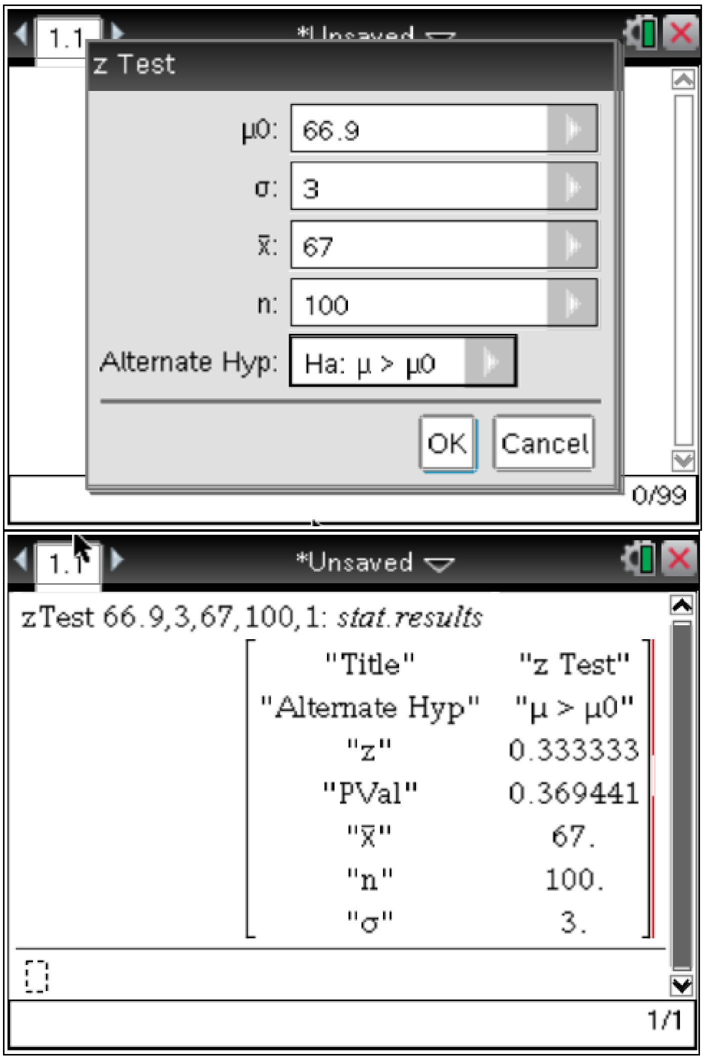
The p-value is then:

$$\Pr(\overline{X} > 67 \mid \mu = 66.9)$$

$$= \Pr\left(Z > \frac{67 - 66.9}{\frac{3}{\sqrt{100}}} \right)$$

$$= 0.3694$$

Alternatively, using the CAS to evaluate the p-value:



Mark allocation: 2 marks

- 1 mark for the expression $\Pr(\overline{X} > 67 \mid \mu = 66.9)$
- 1 mark for a correct p-value of 0.3694

Question 6e.

Worked solution

Since the p-value $= 0.3694 > 0.05$, we cannot reject the null hypothesis at the 5% level of significance. There is insufficient evidence to support the quality control tester's claim.

Mark allocation: 1 mark

- 1 mark for stating that we fail to reject the null hypothesis and there is insufficient evidence to support the quality control tester's claim as $p > 0.05$

Question 6f.

Worked solution

The critical z-value for the test is 2.326.

So, to find the minimum value of the sample mean we solve

$2.326 = \dfrac{\bar{x} - 66.9}{\frac{3}{\sqrt{100}}}$, which gives a minimum value of $67.5978 \approx 67.6$ mm.

Mark allocation: 1 mark

- 1 mark for a correct answer of 67.6 mm

END OF WORKED SOLUTIONS