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SPECIALIST MATHS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2018

Question 1 (2 marks)

$$\int_0^{\frac{\pi}{6}} \cos^2(3x)\,dx = \int_0^{\frac{\pi}{6}} \frac{1}{2}(\cos(6x)+1)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{6}\sin(6x)+x\right]_0^{\frac{\pi}{6}}$$

(1 mark)

$$= \frac{1}{2}\left\{\left(\frac{1}{6}\sin(\pi)+\frac{\pi}{6}\right)-(0+0)\right\}$$

$$= \frac{\pi}{12}$$

(1 mark)

Question 2 (3 marks)

$$5y-2x^2y+x=7$$

$$5\frac{dy}{dx}-2x^2\frac{dy}{dx}-4xy+1=0$$

(1 mark)

Method 1

$$(5-2x^2)\frac{dy}{dx}=4xy-1$$

$$\frac{dy}{dx}=\frac{4xy-1}{5-2x^2}$$

At $(1, 2)$,

$$\frac{dy}{dx}=\frac{4\times1\times2-1}{5-2\times1^2}$$

$$=\frac{7}{3}$$

(1 mark)

$$y-2=\frac{7}{3}(x-1)$$

$$y=\frac{7}{3}x-\frac{1}{3}$$

(1 mark)

Method 2

At $(1, 2)$ we have $5\frac{dy}{dx}-2\frac{dy}{dx}-8+1=0$

$$\frac{dy}{dx}=\frac{7}{3}$$

(1 mark)

$$y-2=\frac{7}{3}(x-1)$$

$$y=\frac{7}{3}x-\frac{1}{3}$$

(1 mark)

Question 3 (3 marks)

Let X be the random variable representing the weight, in grams, of eggs produced at this farm.

$X \sim \text{Normal}\,(\mu_X = 68,\ \sigma_X = 4)$

so $\bar{X} \sim \text{Normal}\left(\mu_{\bar{X}} = 68,\ \sigma_{\bar{X}} = \dfrac{4}{\sqrt{16}}\right)$

i.e. $\bar{X} \sim \text{Normal}\,(\mu_{\bar{X}} = 68,\ \sigma_{\bar{X}} = 1)$ **(1 mark)**

Now $z = \dfrac{\bar{x} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}$

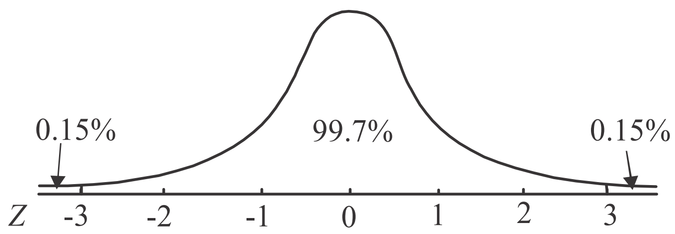
so $z = \dfrac{65 - 68}{1}$

$= -3$ **(1 mark)**

So $\Pr(\bar{X} < 65) = \Pr(Z < -3)$

≈ 0.0015

(1 mark)



Question 4 (3 marks)

Since $z = 2 - i$ is a solution then $z = 2 + i$ is also a solution (conjugate root theorem applies because the coefficients of the terms in the equation are real).

So $(z - 2 + i)(z - 2 - i)$

$= (z - 2)^2 - i^2$ (difference of perfect squares)

$= z^2 - 4z + 5$ (the quadratic factor)

(1 mark)

Let $(z^2 - 4z + 5)(z - b) = z^3 - 7z^2 + (a^2 + 1)\,z - (4a - 1)$ where b is a real constant.
Comparing the coefficients of the z-squared terms, we have

$-4z^2 - bz^2 = -7z^2$

$b = 3$

(1 mark)

Comparing the coefficients of the constant terms, we have

$4a - 1 = 15$

$a = 4$

(1 mark)

Question 5 (5 marks)

a. $|\underset{\sim}{c}| = \sqrt{1+4+4}$

$= 3$

$\hat{\underset{\sim}{c}} = \frac{1}{3}(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$

(1 mark)

b. vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{c}$ is given by

$\underset{\sim}{a} - (\underset{\sim}{a} . \hat{\underset{\sim}{c}})\, \hat{\underset{\sim}{c}}$ **(1 mark)**

$= \underset{\sim}{a} - \left((\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}) . \frac{1}{3}(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}) \right) \hat{\underset{\sim}{c}}$

$= \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} - \frac{1}{3} \times 1 \times \frac{1}{3}(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$

$= \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k} - \frac{1}{9}(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k})$

$= \frac{1}{9}(8\underset{\sim}{i} + 20\underset{\sim}{j} + 16\underset{\sim}{k})$

$= \frac{4}{9}(2\underset{\sim}{i} + 5\underset{\sim}{j} + 4\underset{\sim}{k})$

(1 mark)

c. If $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent then $\alpha \underset{\sim}{a} + \beta \underset{\sim}{c} = \underset{\sim}{b}$ where $\alpha, \beta \in R$.

We require $\alpha(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}) + \beta(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}) = 2\underset{\sim}{i} + 3\underset{\sim}{j} + d\underset{\sim}{k}$

Equating the $\underset{\sim}{i}$ components:

$\alpha + \beta = 2 \quad -(1)$

Equating the $\underset{\sim}{j}$ components:

$2\alpha - 2\beta = 3 \quad -(2)$

Equating the $\underset{\sim}{k}$ components:

$2\alpha + 2\beta = d \quad -(3)$ **(1 mark)**

$(1) \times 2 \quad 2\alpha + 2\beta = 4$

Comparing to (3) gives $d = 4$

(1 mark)

Question 6 (4 marks)

$$-\frac{1}{x}\frac{dy}{dx}=\sqrt{\frac{4-y^2}{4-x^2}}$$
$$=\frac{\sqrt{4-y^2}}{\sqrt{4-x^2}}$$

So $\int\frac{-1}{\sqrt{4-y^2}}\,dy=\int\frac{x}{\sqrt{4-x^2}}\,dx$ (separation of variables) **(1 mark)**

$$\arccos\left(\frac{y}{2}\right)+c_1=\int u^{-\frac{1}{2}}\times-\frac{1}{2}\frac{du}{dx}\,dx$$ where $u=4-x^2$ and $\frac{du}{dx}=-2x$

$$\arccos\left(\frac{y}{2}\right)+c_1=-\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$\arccos\left(\frac{y}{2}\right)+c_1=-\frac{1}{2}u^{\frac{1}{2}}\times 2+c_2$$

(1 mark)

$$\arccos\left(\frac{y}{2}\right)=-\sqrt{4-x^2}+c$$ where $c=c_2-c_1$

Since $y(2)=\sqrt{3}$,

$$\arccos\left(\frac{\sqrt{3}}{2}\right)=-\sqrt{0}+c$$

$$c=\frac{\pi}{6}$$ **(1 mark)**

So $\arccos\left(\frac{y}{2}\right)=-\sqrt{4-x^2}+\frac{\pi}{6}$

$$\cos\left(\frac{\pi}{6}-\sqrt{4-x^2}\right)=\frac{y}{2}$$

The solution is $y=2\cos\left(\frac{\pi}{6}-\sqrt{4-x^2}\right)$.

(1 mark)

Question 7 (3 marks)

$F = ma$ and $a = v\dfrac{dv}{dx}$ (formula sheet)

Now $v = 4\arccos(2x^2 - 1)$

$$\frac{dv}{dx} = 4 \times \frac{-1}{\sqrt{1-(2x^2-1)^2}} \times 4x \qquad \text{(Chain rule)}$$ **(1 mark)**

$$= \frac{-16x}{\sqrt{1-4x^4+4x^2-1}}$$

$$= \frac{-16x}{\sqrt{4x^2(1-x^2)}}$$

$$= \frac{-16x}{2|x|\sqrt{1-x^2}}$$

$$= \frac{-16x}{2x\sqrt{1-x^2}} \quad \text{since } x > 0$$

$$= \frac{-8}{\sqrt{1-x^2}}$$ **(1 mark)**

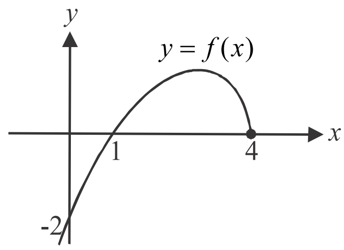
So $a = \dfrac{-32\arccos(2x^2-1)}{\sqrt{1-x^2}}$

and $F = \dfrac{-160\arccos(2x^2-1)}{\sqrt{1-x^2}}$

(1 mark)

Question 8 (5 marks)

a. Do a quick sketch.



$$\text{area of } S=\int_1^4 (x-1)\sqrt{4-x}\,dx \qquad \textbf{(1 mark)}$$

$$=\int_3^0 (3-u)\sqrt{u}\times -1\frac{du}{dx}\,dx$$

$$=\int_0^3 \left(3u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)du \qquad \textbf{(1 mark)}$$

$$=\left[3u^{\frac{3}{2}}\times\frac{2}{3}-u^{\frac{5}{2}}\times\frac{2}{5}\right]_0^3$$

$$=\left\{\left(2\times 3^{\frac{3}{2}}-\frac{2}{5}\times 3^{\frac{5}{2}}\right)-(0-0)\right\}$$

$$=3^{\frac{3}{2}}\left(2-\frac{6}{5}\right)$$

$$=\frac{4}{5}\times 3^{\frac{3}{2}}$$

$$=\frac{12\sqrt{3}}{5}\text{ square units}$$

let $u=4-x$

$\frac{du}{dx}=-1$

Also, $x=4-u$

$x-1=3-u$

Terminals:

$x=1,\ u=3$

$x=4, u=0$

(1 mark)

b.

$$\text{volume}=\pi\int_1^4 y^2\,dx$$

$$=\pi\int_1^4 \left((x-1)\sqrt{4-x}\right)^2 dx \qquad \textbf{(1 mark)}$$

$$=\pi\int_1^4 (x-1)^2(4-x)\,dx$$

$$=\pi\int_1^4 (x^2-2x+1)(4-x)\,dx$$

$$=\pi\int_1^4 (4x^2-8x+4-x^3+2x^2-x)\,dx$$

$$=\pi\int_1^4 (-x^3+6x^2-9x+4)\,dx$$

$$=\pi\left[-\frac{x^4}{4}+2x^3-\frac{9x^2}{2}+4x\right]_1^4$$

$$=\pi\left\{(-64+128-72+16)-\left(-\frac{1}{4}+2-\frac{9}{2}+4\right)\right\}$$

$$=\pi\left(8-\frac{5}{4}\right)$$

$$=\frac{27\pi}{4}\text{ cubic units} \qquad \textbf{(1 mark)}$$

Question 9 (4 marks)

$x = 2\sqrt{4-t}$ $\qquad\qquad$ $y = 2\sqrt{t+4}$

$\frac{dx}{dt} = 2 \times \frac{1}{2}(4-t)^{-\frac{1}{2}} \times -1$ $\qquad\qquad$ $\frac{dy}{dt} = 2 \times \frac{1}{2} \times (t+4)^{-\frac{1}{2}}$

$= \frac{-1}{\sqrt{4-t}}$ $\qquad\qquad$ $= \frac{1}{\sqrt{t+4}}$ **(1 mark)**

$$\text{arc length} = \int_2^4 \sqrt{\frac{1}{4-t} + \frac{1}{t+4}}\, dt$$ (formula sheet) **(1 mark)**

$$= \int_2^4 \sqrt{\frac{t+4+4-t}{(4-t)(t+4)}}\, dt$$

$$= \int_2^4 \sqrt{\frac{8}{16-t^2}}\, dt$$

$$= 2\sqrt{2}\int_2^4 \frac{1}{\sqrt{16-t^2}}\, dt$$

$$= 2\sqrt{2}\left[\sin^{-1}\left(\frac{t}{4}\right)\right]_2^4$$ **(1 mark)**

$$= 2\sqrt{2}\left(\sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$= 2\sqrt{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

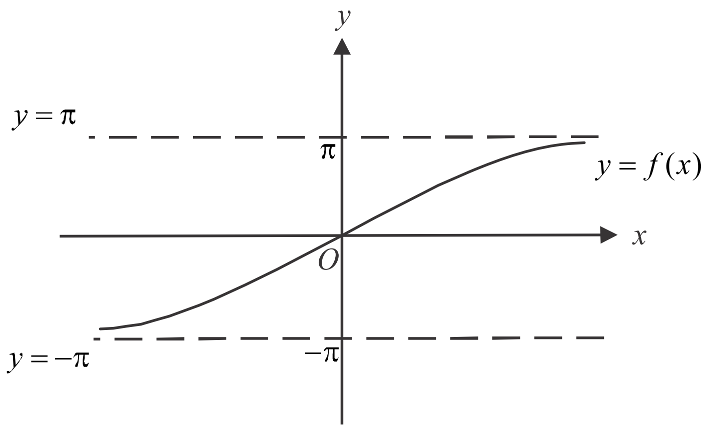
$$= 2\sqrt{2} \times \frac{\pi}{3}$$

$$= \frac{2\sqrt{2}\pi}{3} \text{ units}$$

(1 mark)

Question 10 (8 marks)

a.



(1 mark) – correct asymptotes
(1 mark) – correct shape

b. $f(x) = 2\arctan(3x)$

Let $y = 2\arctan(3x)$

Swap *x* and *y* for inverse.

$$x = 2\arctan(3y)$$

$$\frac{x}{2} = \arctan(3y)$$

$$3y = \tan\left(\frac{x}{2}\right)$$

$$y = \frac{1}{3}\tan\left(\frac{x}{2}\right)$$

So $f^{-1}(x) = \frac{1}{3}\tan\left(\frac{x}{2}\right)$ **(1 mark)**

$$d_{f^{-1}} = r_f$$
$$= (-\pi, \pi)$$

(1 mark)

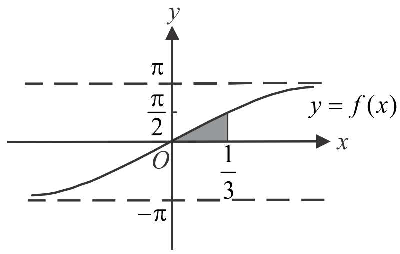
c. $$\int \tan\left(\frac{x}{2}\right)dx = \int \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}dx$$

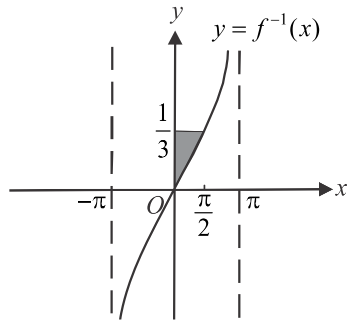
$$= -2\int \frac{-\frac{1}{2}\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}dx$$

$$= -2\log_e\left|\cos\left(\frac{x}{2}\right)\right| + c$$

(1 mark)

d. Do a quick sketch of the graph of $y = f^{-1}(x)$ and compare it to the graph drawn in part **a.**





area required $= \int_0^{\frac{1}{3}} f\,(x)\,dx$ **(1 mark)**

$$= \frac{1}{3} \times \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} f^{-1}(x)\,dx$$ (using the symmetry of the graphs of f and f^{-1})

$$= \frac{\pi}{6} - \int_0^{\frac{\pi}{2}} \frac{1}{3}\tan\left(\frac{x}{2}\right)dx$$

$$= \frac{\pi}{6} - \frac{1}{3}\left[-2\log_e\left|\cos\left(\frac{x}{2}\right)\right|\right]_0^{\frac{\pi}{2}}$$ **(1 mark)**

$$= \frac{\pi}{6} + \frac{2}{3}\left(\log_e\left|\cos\left(\frac{\pi}{4}\right)\right| - \log_e|\cos(0)|\right)$$

$$= \frac{\pi}{6} + \frac{2}{3}\left(\log_e\left(\frac{1}{\sqrt{2}}\right) - \log_e(1)\right)$$

$$= \frac{\pi}{6} + \frac{2}{3}\log_e\left(\frac{1}{\sqrt{2}}\right) \text{ square units}$$

(1 mark)