Neap:

Trial Examination 2017

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

1	A
2	D
3	B
4	D
5	E
6	C
7	C
8	D
9	D
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12	E
13	A
14	A
15	B
16	C
17	A
18	B
19	E
20	C

Question 1 A

The vertical asymptote is $x = 0$.

As $x \to \pm\infty, f(x) \to b$ and so $y \to g(x) + b$.

So $y = g(x) + b$ is a straight-line asymptote.

Question 2 D

$y = -\sec(x+2) - 3$

$y = -\dfrac{1}{\cos(x+2)} - 3$

Considering the graph of $y = -\sec(x + 2) - 3$, vertical asymptotes occur wherecos$(x + 2) = 0$.

$x + 2 = (2k-1)\dfrac{\pi}{2} \Rightarrow x = (2k-1)\dfrac{\pi}{2} - 2$ where k is an integer.

So the implied domain is $R \backslash \left\{(2k-1)\dfrac{\pi}{2} - 2\right\}$.

Question 3 B

Solving $|2x\arccos(x)| = 1$ for x gives $x = -0.271, 0.455, 0.820$.

So $-1 \le x < -0.271$ or $0.455 < x < 0.820$.

Question 4 D

$x + yi = (a + bi)^2$

$\quad = a^2 + 2abi - b^2$

Equating coefficients gives $x = a^2 - b^2$ and y $= 2ab$.

Question 5 E

$$\text{Arg}\left(\frac{z}{\bar{z}}\right) = \frac{\pi}{3}$$

$$\text{Arg}(z) - \text{Arg}(\bar{z}) = \frac{\pi}{3}$$

$$2\text{Arg}(z) = \frac{\pi}{3} \Rightarrow \text{Arg}(z) = \frac{\pi}{6}$$

Question 6 C

$\text{Re}(z) + \text{Im}(z) = 1 \Rightarrow x + y = 1$ and it does not pass through the origin

$z + \bar{z} = 1 \Rightarrow x = \dfrac{1}{2}$ and it does not pass through the origin

$\text{Re}(z) - \text{Im}(z) = 0 \Rightarrow x - y = 0$ and it passes through the origin. So **C** is correct.

For completeness:

$\bar{z}z = 1$ is a circle with centre O and radius 1.

$\text{Re}(z)\text{Im}(z) = 1 \Rightarrow xy = 1$ and it does not pass through the origin.

Question 7 **C**

The most efficient way to perform the implicit differentiation is by use of a CAS.

$x^2 - 4xy + 2y^2 = -2$

$\frac{dy}{dx} = \frac{2y - x}{2(y - x)}$

Alternatively:

$$2x - 4\left(y + x\frac{dy}{dx}\right) + 4y\frac{dy}{dx} = 0$$

$$(4y - 4x)\frac{dy}{dx} = 4y - 2x \Rightarrow \frac{dy}{dx} = \frac{2y - x}{2(y - x)}$$

Tangents parallel to the x-axis will satisfy $2y - x = 0 \Rightarrow y = \frac{x}{2}$.

Question 8 **D**

The two curves intersect at $x = \frac{1}{\sqrt{2}}$.

Let the volume be V.

$$V = \pi\int_0^{\frac{1}{\sqrt{2}}} y^2\,dx - \pi\left(\frac{\sqrt{\pi}}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)$$

$$= \pi\int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}(x)\,dx - \frac{\pi^2}{4\sqrt{2}}$$

Question 9 **D**

Given $\frac{dr}{dt} = -0.2$.

$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

As $C = 2\pi r$, $\frac{dA}{dr} = C$.

Using $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$, we obtain $\frac{dA}{dt} = -0.2C$.

Question 10 **E**

Given $x = \sin^3(t)$ and $y = \cos^3(t)$ for $0 \le t \le \frac{\pi}{6}$.

Let the path length be L.

$$L = \int_0^{\frac{\pi}{6}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{(3\sin^2(t)\cos(t))^2 + (-3\cos^2(t)\sin(t))^2}\,dt$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{9\sin^4(t)\cos^2(t) + 9\cos^4(t)\sin^2(t)}\,dt$$

Question 11 B

Vertical segments will occur when $2x + y = 0 \Rightarrow y = -2x$.

Question 12 E

$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$\frac{1}{3} = \frac{(\underset{\sim}{i} + \underset{\sim}{j} + p\underset{\sim}{k}) \cdot (2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})}{\sqrt{2 + p^2}\sqrt{9}}$$

$$\frac{1}{3} = \frac{4 - p}{3\sqrt{2 + p^2}}$$

Solving for p we obtain $p = \frac{7}{4}$.

Question 13 A

$\int_1^3 \sqrt{(4e^{-t})^2 + (\sin(1 + t))^2}\,dt = 1.642$ (m) (correct to three decimal places) and so option **A** is correct.

Question 14 A

$\overrightarrow{AM} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$ and $\overrightarrow{OM} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})$

$\overrightarrow{AM} \cdot \overrightarrow{AM} = \frac{1}{4}(\underset{\sim}{b} \cdot \underset{\sim}{b} - 2\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a})$ and $\overrightarrow{OM} \cdot \overrightarrow{OM} = \frac{1}{4}(\underset{\sim}{b} \cdot \underset{\sim}{b} + 2\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a})$

$$\begin{aligned} AM^2 + OM^2 &= \frac{1}{2}(\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}) \\ &= \frac{1}{2}(OA^2 + OB^2) \end{aligned}$$

So $OA^2 + OB^2 = 2OM^2 + 2AM^2$.

Question 15 B

The two stones collide when $28t = 35\cos(\alpha)t \Rightarrow \cos(\alpha) = \frac{4}{5}$.
So $\alpha = \cos^{-1}\left(\frac{4}{5}\right)$.

Question 16 C

Use of a differential equation solver to solve $2v\frac{dv}{dx} = 2 - 8v^2$ where $v = 0$ at $x = 0$ gives $v^2 = \frac{1}{4}(1 - e^{-8x})$.

$$\begin{aligned} F &= 2 - 8v^2 \\ &= 2 - 2(1 - e^{-8x}) \\ &= 2e^{-8x} \end{aligned}$$

Question 17 A

Resolving forces vertically:

$$F_y = 30 \times \frac{7}{25} + 22 \times \frac{4}{5} - 26$$
$$= 0 \text{ (N)}$$

Resolving forces horizontally:

$$F_x = 30 \times \frac{24}{25} - 22 \times \frac{3}{5}$$
$$= 15.6 \text{ (N)}$$

$$|\Sigma F| = 15.6 \text{ (N)}$$

Question 18 B

A test to determine whether a new diet promotes weight loss is the only one-sided test presented in the options.

Question 19 E

$\mathrm{E}(\bar{X}) = 21$ and $\mathrm{sd}(\bar{X}) = \frac{2.5}{\sqrt{20}}$

$\bar{X} \sim \mathrm{N}\left(21, \frac{2.5^2}{20}\right)$

$\Pr(\bar{X} > 19.8) = 0.9841$ (correct to four decimal places)

Question 20 C

An approximate $C\%$ confidence interval is given by $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$ where z is such that $\Pr(-z < Z < z) = \frac{C}{100}$.

The width of an appropriate confidence interval is $\bar{x} + z\frac{s}{\sqrt{n}} - \left(\bar{x} - z\frac{s}{\sqrt{n}}\right) = 2z\frac{s}{\sqrt{n}}$.

The width of this confidence interval is 13.3 – 12.4 = 0.9..

Solving $0.9 = 2 \times \frac{0.8}{\sqrt{10}} \times z$ for z gives $z = 1.778...$

$$C = 100\Pr(-1.778\ldots < Z < 1.778\ldots)$$
$$= 92.5$$

The confidence level is 92.5%.

SECTION B

Question 1 (12 marks)

a. The vertical asymptotes are $x = \pm 2$. A1

$f(x) = x + \dfrac{4x - a^3}{x^2 - 4}$ M1

The non-vertical asymptote is $y = x$. A1

b. **i.** $f'(x) = \dfrac{(x^2 - 4)(3x^2) - (x^3 - a^3)(2x)}{(x^2 - 4)^2} \left(= \dfrac{x(x^3 - 12x + 2a^3)}{(x^2 - 4)^2} \right)$ M1

$f'(x) = 0$ when $x = 0$ (and so f has a stationary point at $x = 0$) A1

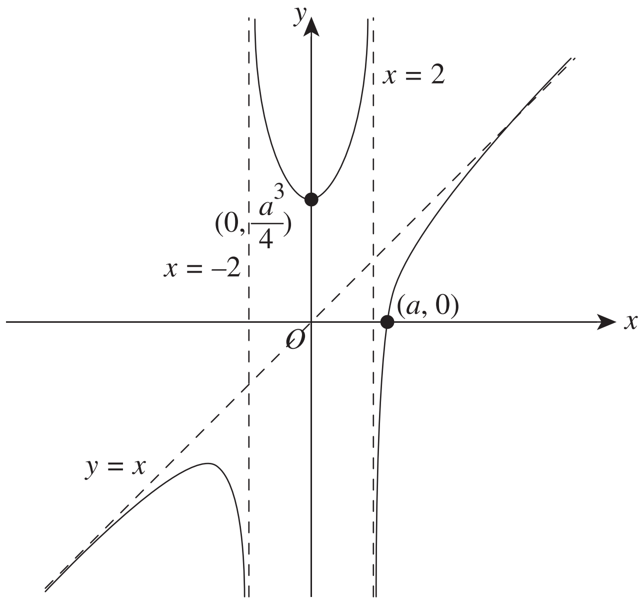
ii. $f''(x) = \dfrac{a^3}{8}$ M1

$\dfrac{a^3}{8} > 0$ for $a > 2$ and so $x = 0$ is a local minimum A1

c. Solving $x^3 - a^3 = x(x^2 - 4)$ for x gives $x = \dfrac{a^3}{4}$. M1 A1

So the graph crosses the non-vertical asymptote.

d.



three correct branches with correctly labelled asymptotes A1

the graph crossing the non-vertical asymptote A1

$\left(0, \dfrac{a^3}{4}\right)$ *and (a, 0)* A1

Question 2 (9 marks)

a. $F = 75 + 90\cos(30°)$ A1

$= 153$ (N) (correct to the nearest newton) A1

b. $R + 90\sin(30°) = 60g$ M1

$R = 543$ (N) A1

c. Using Newton's second law, we obtain $-150 = 60a$.

$a = -2.5$ (m/s^2) A1

$$v\frac{dv}{dx} = -2.5 \Rightarrow x = \int_{1.5}^{0} -\frac{v}{2.5}dv$$ M1

$x = 0.45$ (m) A1

d. Using Newton's second law, we obtain $190 - 145 + P\cos(30°) = 180$. M1

Solving for P, we obtain $P = 156$ (N) (correct to the nearest newton). A1

Question 3 (12 marks)

a. $\text{LHS} = (\cos(\theta) + i\sin(\theta))^2 - 2\cos(\theta)(\cos(\theta) + i\sin(\theta)) + 1$ M1

$= 2\cos^2(\theta) + 2i\sin(\theta)\cos(\theta) - 1 - 2\cos^2(\theta) - 2i\sin(\theta)\cos(\theta) + 1$ A1

$= 0$ (= RHS), and so $z = \cos(\theta) + i\sin(\theta)$ is a solution to the equation A1

b. $z^n + \frac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$ A1

$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

$= 2\cos(n\theta)$ A1

c. **Method 1:**

$$8\cos^3(\theta) + 4\cos^2(\theta) - 4\cos(\theta) - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

$z + \frac{1}{z} = 2\cos(\theta)$ and so $\text{LHS} = \left(z + \frac{1}{z}\right)^3 + \left(z + \frac{1}{z}\right)^2 + \left(z + \frac{1}{z}\right)$ A1

$= \left(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\right) + \left(z^2 + 2 + \frac{1}{z^2}\right) - 2\left(z + \frac{1}{z}\right) - 2$ M1

$= \left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) + 3\left(z + \frac{1}{z}\right) - 2\left(z + \frac{1}{z}\right)$ A1

$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$

= RHS

Method 2:

$8\cos^3(\theta) + 4\cos^2(\theta) - 4\cos(\theta) - 2 = (4\cos^2(\theta) - 2)(2\cos(\theta) + 1)$ A1

$z + \frac{1}{z} = 2\cos(\theta)$ and so $\text{LHS} = \left(z^2 + \frac{1}{z^2}\right)\left(z + \frac{1}{z} + 1\right)$ M1

$= z^3 + z^2 + z + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3}$ A1

$= \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$

= RHS

d. **Method 1:**

$\cos(\theta) + \cos(2\theta) + \cos(3\theta) = 0 \Rightarrow \frac{1}{2}\left(\left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)\right) = 0$

So $\frac{1}{2}(4\cos^2(\theta) - 2)(2\cos(\theta) + 1) = 0$ A1

Attempting to solve $\cos(\theta) = -\frac{1}{2}, \pm\frac{1}{\sqrt{2}}$, ($\cos(\theta) = -\frac{1}{2}$ or $\cos(2\theta) = 0$). M1

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (from $\cos(\theta) = -\frac{1}{2}$) A1

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (from either $\cos(\theta) = \pm\frac{1}{\sqrt{2}}$ or $\cos(2\theta) = 0$) A1

Method 2:

$\cos(\theta) + \cos(2\theta) + \cos(3\theta) = 0 \Rightarrow \cos(2\theta - \theta) + \cos(2\theta) + \cos(2\theta + \theta) = 0$

$\cos(2\theta)\cos(\theta) + \sin(2\theta)\sin(\theta) + \cos(2\theta) + \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) = 0$

$2\cos(2\theta)(2\cos(\theta) + 1) = 0$ A1

Attempting to solve $\cos(\theta) = -\frac{1}{2}$ or $\cos(2\theta) = 0$. M1

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (from $\cos(\theta) = -\frac{1}{2}$) A1

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (from $\cos(2\theta) = 0$) A1

Question 4 (9 marks)

a. $\frac{dN}{dt} = kN(2000 - N)$ A1

Solving $25 = (50)(2000 - 50)k$ for k gives $k = \frac{1}{3900}$. M1

So $\frac{dN}{dt} = \frac{N}{3900}(2000 - N)$.

b. Attempting to solve $\frac{dN}{dt} = \frac{N}{3900}(2000 - N)$, ($t = 0$, $N = 50$) to obtain N in terms of t. M1

$N = \frac{2000e^{\frac{20t}{39}}}{e^{\frac{20t}{39}} + 39}$ (or equivalent) A1

Solving $1500 = \frac{2000e^{\frac{20t}{39}}}{e^{\frac{20t}{39}} + 39}$ for t gives 279 days (correct to the nearest day). M1 A1

c. $\frac{dN}{dt}$ is a maximum when $N = 1000$. A1

Solving $1000 = \frac{2000e^{\frac{20t}{39}}}{e^{\frac{20t}{39}} + 39}$ for t gives 214 days (correct to the nearest day). M1 A1

Question 5 (6 marks)

a. Let $X \sim \text{N}(12, 2.7^2)$ and $Y \sim \text{N}(20, 4.4^2)$.

$$T = \frac{X_1 + X_2 + X_3 + Y_1 + Y_2}{5} \Rightarrow \text{E}(T) = \text{E}\left(\frac{X_1 + X_2 + X_3 + Y_1 + Y_2}{5}\right)$$

$$\begin{aligned}\text{E}(T) &= \frac{1}{5}(3\text{E}(X) + 2\text{E}(Y)) \\ &= \frac{1}{5}(3(12) + 2(20)) \\ &= 15.2\end{aligned}$$ A1

$$\begin{aligned}\text{var}(T) &= \text{var}\left(\frac{X_1 + X_2 + X_3 + Y_1 + Y_2}{5}\right) \\ &= \frac{1}{5^2}(3\text{var}(X) + 2\text{var}(Y)) \\ &= \frac{1}{5^2}(3(2.7^2) + 2(4.4^2)) \\ &= 2.4236\ldots\end{aligned}$$ A1

$\Pr(T < 14) = 0.220$ (correct to three decimal places) A1

b. $\Pr(|X - Y| \le 5) = \Pr(-5 \le X - Y \le 5)$ M1

$X - Y \sim \text{N}(12 - 20,\ 2.7^2 + 4.4^2)$
$X - Y \sim \text{N}(-8,\ 26.65)$ A1

$\Pr(|X - Y| \le 5) = 0.275$ (correct to three decimal places) A1

Question 6 (12 marks)

a. $m = \tan(\alpha)$ and $y = mx$ A1

Hence $y = \tan(\alpha)x$.

b. At P, $\tan(\alpha)x = \tan(\theta)x - \frac{g}{2V^2}(1 + \tan^2(\theta))x^2$

Solving this equation for x gives:

$$x = \frac{2V^2\cos(\theta)(\cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta))}{g\cos(\alpha)}$$ (or equivalent) M1

$x = R\cos(\alpha)$ and $\cos(\alpha)\sin(\theta) - \sin(\alpha)\cos(\theta) = \sin(\theta - \alpha)$ A1

$$\Rightarrow R = \frac{2V^2\sin(\theta - \alpha)\cos(\theta)}{g\cos^2(\alpha)}$$ A1

c. **i.** $\frac{dR}{d\theta} = \frac{2V^2(\cos(\theta)\cos(\theta-\alpha) - \sin(\theta)\sin(\theta-\alpha))}{g\cos^2(\alpha)}$ (or equivalent) M1

$\frac{dR}{d\theta} = 0 \Rightarrow \cos(\theta)\cos(\theta-\alpha) - \sin(\theta)\sin(\theta-\alpha) = 0$ A1

$\cos(2\theta-\alpha) = 0 \Rightarrow \theta = \frac{\alpha}{2} + \frac{\pi}{4}$ (or equivalent) A1

ii. $\frac{d^2R}{d\theta^2} = \frac{-4V^2(\sin(\theta)\cos(\theta-\alpha) + \cos(\theta)\sin(\theta-\alpha))}{g\cos^2(\alpha)}$ (or equivalent) M1

When $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$, $\frac{d^2R}{d\theta^2} = \frac{-4V^2}{g\cos^2(\alpha)}$. A1

As $\frac{4V^2}{g\cos^2(\alpha)} > 0$, then $\frac{d^2R}{d\theta^2} < 0$ and so R is a maximum when $\theta = \frac{\alpha}{2} + \frac{\pi}{4}$ (or equivalent). A1

d. Let m_O be the gradient of the projectile's initial direction at O and so $m_O = \tan(\theta)$.

$\frac{dy}{dx} = -\frac{2\cos^2(\theta)}{2\sin(\theta)\cos(\theta)}$ M1

$$m_O m_P = \frac{\sin(\theta)}{\cos(\theta)} \times \frac{-2\cos^2(\theta)}{2\sin(\theta)\cos(\theta)}$$

$= -1$ A1

So the initial direction of the particle's trajectory is perpendicular to the direction at which it hits the inclined plane.

