The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2017

Trial Written Examination 1 - SOLUTIONS

Question 1

The scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ is given by $\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}}$.

$\left|\underset{\sim}{a}\right| = \sqrt{2^2 + (-1)^2 + (2\sqrt{2})^2} \quad = \sqrt{4+1+8} \quad = \sqrt{13}$.

Therefore $\hat{\underset{\sim}{a}} = \dfrac{2\underset{\sim}{i} - \underset{\sim}{j} + 2\sqrt{2}\underset{\sim}{k}}{\sqrt{13}}$. **[A1]**

Therefore:

$$\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}} = \frac{1}{\sqrt{13}}\left(-\underset{\sim}{i} + m\underset{\sim}{j} - 2\sqrt{3}\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} - \underset{\sim}{j} + 2\sqrt{2}\underset{\sim}{k}\right)$$

$$= \frac{1}{\sqrt{13}}\left(-2 - m - 4\sqrt{6}\right).$$ **[H1]**

Consequential on $\hat{\underset{\sim}{a}}$.

But $\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}} = -\dfrac{\sqrt{26}}{13}$.

Therefore:

$$\frac{1}{\sqrt{13}}\left(-2 - m - 4\sqrt{6}\right) = -\frac{\sqrt{26}}{13}$$

$$\Rightarrow 2 + m + 4\sqrt{6} = \frac{\sqrt{13}\sqrt{26}}{13} \quad = \frac{\sqrt{26}}{\sqrt{13}} \quad = \sqrt{2}$$

$$\Rightarrow m = \sqrt{2} - 2 - 4\sqrt{6}.$$

Answer: $m = \sqrt{2} - 2 - 4\sqrt{6}$. **[H1]**

Consequential on $\underset{\sim}{b} \cdot \hat{\underset{\sim}{a}}$.

©The Mathematical Association of Victoria, 2017

Question 2

a.

Define the random variable $T = Y - X$.

$\Pr(T < 0)$ is required.

- X and Y are normal random variables therefore $T = Y - X$ is a normal random variable.
- $\mu = E(T) = E(Y - X) = E(Y) - E(X) \quad = 76 - 65 \quad = 11$.
- X and Y are independent random variables.

Therefore:

$\sigma^2 = \text{Var}(T) = \text{Var}(Y - X) = \text{Var}(Y) + \text{Var}(X) \quad = 5^2 + 3^2 \quad = 34$.

Therefore:

$\sigma = \text{sd}(T) = \text{sd}(X - Y) = \sqrt{\text{Var}(X - Y)} \quad = \sqrt{34}$.

- $T \sim \text{Normal}\left(\mu = 11,\ \sigma = \sqrt{34}\right)$. **[M1]**

- $\Pr(T < 0) \Leftrightarrow \Pr(Z < a)$ where $Z = \dfrac{T - \mu}{\sigma}$.

Therefore: $a = \dfrac{0 - 11}{\sqrt{34}}$.

Answer: $a = -\dfrac{11}{\sqrt{34}}$. **[H1]**

Consequential on calculated values of mean and standard deviation.

Accept $a = -\dfrac{11\sqrt{34}}{34}$.

b.

Two-tailed test therefore $p = 2\Pr\left(\overline{X} \geq 68 \mid H_0\right)$

where $\overline{X} \sim \text{Norm}\left(\mu_{\overline{X}} = \mu_X = 65,\ \sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n}} = \frac{3}{\sqrt{50}} = \frac{3}{5\sqrt{2}} = \frac{3\sqrt{2}}{10}\right)$ under H_0.

$$Z = \frac{r.v. - mean}{s.d} \qquad = \frac{\overline{X} - \mu_{\overline{X}}}{\sigma_{\overline{X}}} \qquad = \frac{\overline{X} - 65}{\frac{3\sqrt{2}}{10}}$$

. **[M1]**

Substitute $\overline{X} \geq 68$:

$$Z = \frac{68 - 65}{\frac{3\sqrt{2}}{10}} \qquad = \frac{3}{\frac{3\sqrt{2}}{10}} \qquad = \frac{10}{\sqrt{2}} \qquad = 5\sqrt{2}.$$

Therefore:

$p = 2\Pr\left(\overline{X} \geq 68 \mid H_0\right)$

$= 2\Pr(Z \geq 5\sqrt{2})$

$= 2\Pr(Z \leq -5\sqrt{2})$ by symmetry

$= 2\Pr(Z \leq b)$

where $b = -5\sqrt{2}$.

Answer: $b = -5\sqrt{2}$. **[A1]**

Answer must be in simplest surd form.

©The Mathematical Association of Victoria, 2017

Question 3

a.

Vertical asymptotes:

Solve $x-2=0$: $x=2$.

Horizontal and oblique asymptotes:

From polynomial long division:

$$f(x)=x^2+2x+1+\frac{4}{x-2}.$$ **[M1]**

Consider $x\to+\infty$ and $x\to-\infty$:

$$\lim_{x\to\pm\infty}\left(\frac{4}{x-2}\right)=0.$$

Therefore the function $f(x)=\dfrac{x^3-3x+2}{x-2}$ is asymptotic to the parabola $y=x^2+2x+1$.

Therefore $y=x^2+2x+1$ is an oblique asymptote.

Answer: $x=2$, $y=x^2+2x+1$. **[H1]**

Consequential on polynomial long division.

Accept $y=(x+1)^2$ as equation of oblique asymptote.

©The Mathematical Association of Victoria, 2017

b.

• ***y*-intercept:** Solve $y = f(0)$.

$y = -1$.

• ***x*-intercept:** Solve $f(x) = 0$.

$$\frac{x^3 - 3x + 2}{x - 2} = 0 \quad \Rightarrow x^3 - 3x + 2 = 0 \quad \Rightarrow (x-1)(x^2 + x - 2) = 0 \quad \Rightarrow (x-1)(x-1)(x+2) = 0$$

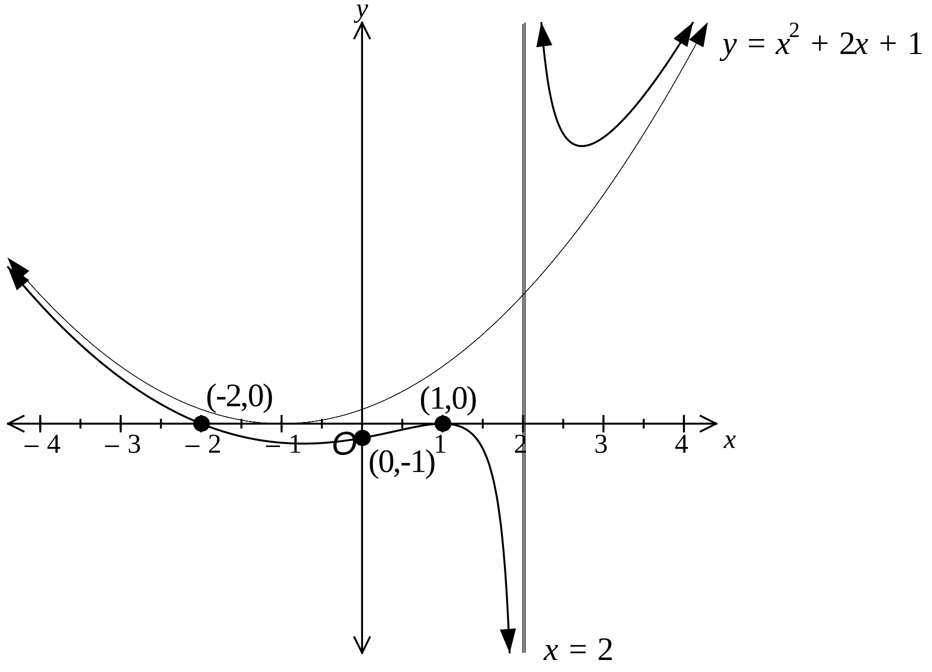
$\Rightarrow x = 1$ or $x = -2$.

• **Shape:**

Option 1: Draw asymptotes, label axes intercepts and then draw a shape that is consistent with these features.

Note: The only way the shape can be consistent with the asymptotes and axes intercepts is for there to also be a maximum turning point at the *x*-intercept $(1, 0)$ since clearly $f(x) < 0$ for $x \in [0, 1)$ and $x \in (1, 2)$ (this is also consistent with the repeated solution $x = 1$).

Option 2: Use addition of ordinates: $f(x) = \underbrace{x^2 + 2x + 1}_{y=g(x)} + \underbrace{\frac{4}{x-2}}_{y=h(x)}$.



Correctly labelled asymptotes (equations found in part a.): [H1]

Consequential on equations found in **part a.** *provided one vertical asymptote and one oblique asymptote were found.*

Coordinates $(0, -1)$, $(1, 0)$ **and** $(-2, 0)$ **of axes intercepts: [A1]**

Correct shape: [A1]

There must be a maximum turning point at $(1, 0)$.

Coordinates of other turning points are NOT required.

Allow the turning point located in the interval $x \in (-2, 1)$ to be either to the left (correct position) or the right of the *y*-axis since it is not possible to know its exact location without further calculation that is beyond the scope of what is required in the question.

©The Mathematical Association of Victoria, 2017

Question 4

Method: 'Hidden quadratic'.

• Re-arrange: $z^4 - \sqrt{3}z^2 + 1 = 0$.

• Substitute $w = z^2$: $w^2 - \sqrt{3}w + 1 = 0$.

Therefore:

$$w = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} \qquad = \frac{\sqrt{3} \pm i}{2}$$

$$\Rightarrow z^2 = \frac{\sqrt{3} \pm i}{2}.$$ **[A1]**

• Find the square roots of $\dfrac{\sqrt{3} \pm i}{2}$ in polar form:

Let $z = r\text{cis}(\theta) \qquad \Rightarrow z^2 = r^2\text{cis}(2\theta)$.

Case 1: $z^2 = \dfrac{\sqrt{3}+i}{2} = \text{cis}\left(\dfrac{\pi}{6} + 2n\pi\right), \; n \in Z$

$$\Rightarrow r^2\text{cis}(2\theta) = \text{cis}\left(\frac{\pi}{6} + 2n\pi\right).$$

Equate moduli: $r^2 = 1 \Rightarrow r = 1$.

Equate arguments: $2\theta = \dfrac{\pi}{6} + 2n\pi \qquad \Rightarrow \theta = \dfrac{\pi}{12} + n\pi$.

$\underline{n = 0}$: $z = \text{cis}\left(\dfrac{\pi}{12}\right)$.

$\underline{n = -1}$: $z = \text{cis}\left(-\dfrac{11\pi}{12}\right)$.

Answer 1: $z = \text{cis}\left(\dfrac{\pi}{12}\right), \; z = \text{cis}\left(-\dfrac{11\pi}{12}\right)$. **[A1]**

Case 2: $z^2 = \frac{\sqrt{3}-i}{2}$.

Note that because all the coefficients of the given quartic are real, the conjugate root theorem is valid and so the conjugate of the solutions found in **Case 1** are solutions.
Note: If $z = r\text{cis}(\theta)$ then $\bar{z} = r\text{cis}(-\theta)$.

Answer 2: $z = \text{cis}\left(-\frac{\pi}{12}\right)$, $z = \text{cis}\left(\frac{11\pi}{12}\right)$. **[H1]**

Consequential on Answer 1.

Discussion: The question does not ask for solutions in cartesian form. However, it is of mathematical interest to consider how one might go about finding such solutions if they were required. The calculations that follow would be very unlikely t be required on a VCAA Examination 1.

There are two methods for finding solutions in cartesian form.

Method 1:

$z^4 - \sqrt{3}z^2 + 1 = 0$ can be factorised as follows:

$$z^4 - \sqrt{3}z^2 + 1 = 0 \quad \Rightarrow z^4 + 1 - \sqrt{3}z^2 = 0$$

$$\Rightarrow \underbrace{(z^2+1)^2}_{z^4+1+2z^2} - 2z^2 - \sqrt{3}z^2 = 0 \qquad \Rightarrow (z^2+1)^2 - \left(2+\sqrt{3}\right)z^2 = 0$$

Factorise as a difference of two squares:

$$\Rightarrow \left(z^2 + 1 - \sqrt{2+\sqrt{3}}z\right)\left(z^2 + 1 + \sqrt{2+\sqrt{3}}z\right) = 0.$$

Apply the Null Factor Law:

Case 1: $z^2 + 1 - \sqrt{2+\sqrt{3}}z = 0 \quad \Rightarrow z^2 - \sqrt{2+\sqrt{3}}z + 1 = 0$

Use the quadratic formula:

$$\Rightarrow z = \frac{\sqrt{2+\sqrt{3}} \pm \sqrt{2+\sqrt{3}-4}}{2} = \frac{\sqrt{2+\sqrt{3}} \pm \sqrt{\sqrt{3}-2}}{2} = \frac{\sqrt{2+\sqrt{3}} \pm i\sqrt{2-\sqrt{3}}}{2}.$$

Case 2: $z^2 + 1 + \sqrt{2+\sqrt{3}}z = 0 \quad \Rightarrow z^2 + \sqrt{2+\sqrt{3}}z + 1 = 0$

Use the quadratic formula:

$$\Rightarrow z = \frac{-\sqrt{2+\sqrt{3}} \pm \sqrt{2+\sqrt{3}-4}}{2} = \frac{-\sqrt{2+\sqrt{3}} \pm \sqrt{\sqrt{3}-2}}{2} = \frac{-\sqrt{2+\sqrt{3}} \pm i\sqrt{2-\sqrt{3}}}{2}.$$

©The Mathematical Association of Victoria, 2017

Method 2:

The square roots of $\dfrac{\sqrt{3}+i}{2}$ can be found in cartesian form as follows:

Let $z = a + ib$, $a, b \in R$

$\Rightarrow z^2 = (a+ib)^2 = a^2 - b^2 + 2abi$.

$z^2 = \dfrac{\sqrt{3}+i}{2} \qquad \Rightarrow a^2 - b^2 + 2abi = \dfrac{\sqrt{3}+i}{2}$.

Equate real parts: $a^2 - b^2 = \dfrac{\sqrt{3}}{2}$. …. (1)

Equate imaginary parts: $2ab = \dfrac{1}{2} \Rightarrow b = \dfrac{1}{4a}$. …. (2)

Substitute (2) into (1):

$a^2 - \dfrac{1}{16a^2} = \dfrac{\sqrt{3}}{2} \qquad \Rightarrow 16a^4 - 1 = 8\sqrt{3}a^2 \qquad \Rightarrow 16a^4 - 8\sqrt{3}a^2 - 1 = 0$.

Use the quadratic formula to solve for a^2:

$$a^2 = \frac{8\sqrt{3} \pm \sqrt{192+64}}{32} = \frac{8\sqrt{3} \pm \sqrt{256}}{32} = \frac{8\sqrt{3} \pm 16}{32} = \frac{\sqrt{3} \pm 2}{4}$$

$\Rightarrow a^2 = \dfrac{\sqrt{3}+2}{4}$ ($a^2 = \dfrac{\sqrt{3}-2}{4}$ is rejected because $a \in R$)

$\Rightarrow a = \pm \dfrac{\sqrt{2+\sqrt{3}}}{2}$.

©The Mathematical Association of Victoria, 2017

Case 1: $a = \dfrac{\sqrt{2+\sqrt{3}}}{2}$.

$$b = \frac{1}{4a} = \frac{1}{2\sqrt{2+\sqrt{3}}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

since

$$\frac{1}{\sqrt{2+\sqrt{3}}} = \frac{\sqrt{2+\sqrt{3}}}{2+\sqrt{3}} = \frac{\sqrt{2+\sqrt{3}} \times (2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \sqrt{2+\sqrt{3}} \times (2-\sqrt{3}) = \sqrt{2+\sqrt{3}} \times \sqrt{(2-\sqrt{3})^2}$$

$$= \sqrt{2+\sqrt{3}} \times \sqrt{7-4\sqrt{3}} = \sqrt{(2+\sqrt{3})(7-4\sqrt{3})} = \sqrt{2-\sqrt{3}}\ .$$

Case 2: $a = -\dfrac{\sqrt{2+\sqrt{3}}}{2}$.

$$b = \frac{1}{4a} = \frac{-1}{2\sqrt{2+\sqrt{3}}} = -\frac{\sqrt{2-\sqrt{3}}}{2}.$$

Therefore $z = \dfrac{\sqrt{2+\sqrt{3}}}{2} + i\dfrac{\sqrt{2-\sqrt{3}}}{2}$ or $z = \dfrac{\sqrt{2+\sqrt{3}}}{2} - i\dfrac{\sqrt{2-\sqrt{3}}}{2}$.

The square roots of $\dfrac{\sqrt{3}-i}{2}$ can be found in cartesian form in a similar way.

©The Mathematical Association of Victoria, 2017

Question 5

a.

The speed of the lift increases while the acceleration is positive.

$a = \frac{1}{3}\sqrt{25 - v^2}$ is positive until $v = 5$.

Therefore the speed of the lift increases until $v = 5$.

Therefore the lift reaches a maximum speed of $v = 5$.

Method 1: Get distance as a function of *v* and substitute the maximum value of *v*.

Since the direction of the lift does not change, the distance it travels is equal to the magnitude of the displacement.

$a = v\frac{dv}{dx} = \frac{\sqrt{25 - v^2}}{3}$ where $v = 0$ when $x = 0$

which is a separable differential equation:

$\int \frac{3v}{\sqrt{25 - v^2}}\, dv = \int dx$ **[M1]**

$\Rightarrow -3\sqrt{25 - v^2} = x + C$.

Substitute $v = 0$ when $x = 0$: $C = -15$.

Therefore:

$-3\sqrt{25 - v^2} = x - 15$. **[A1]**

Substitute $v = 5$ into $-3\sqrt{25 - v^2} = x - 15$: $x = 15$.

Answer: $b = 15$. **[A1]**

©The Mathematical Association of Victoria, 2017

Method 2: Get displacement as a function of *t* and substitute the time at which $v = 5$.

Note: This method includes the solution and answer to **part b.**
If **part b.** is done before **part a.** then the results and answer from **part b.** can be quoted and used in this solution to **part a.**
If **part b.** is not done before **part a.** then the relevant results from this method can be copied as the solution to **part b.**

The acceleration of the lift is upwards. Define the upwards direction as positive.

$$a = \frac{dv}{dt} = \frac{\sqrt{25 - v^2}}{3} \quad \text{where } v = 0 \text{ when } t = 0$$

$$\Rightarrow \frac{dt}{dv} = \frac{3}{\sqrt{25 - v^2}}$$

$$\Rightarrow t = \int \frac{3}{\sqrt{25 - v^2}}\, dv \quad = \underbrace{3\sin^{-1}\left(\frac{v}{5}\right)}_{\text{Get from VCAA formula sheet.}} + C.$$

Substitute $v = 0$ when $t = 0$: $C = 0$.

Therefore:

$$t = 3\sin^{-1}\left(\frac{v}{5}\right)$$

$$\Rightarrow v = 5\sin\left(\frac{t}{3}\right).$$ **[A1]**

Solve $v = 5\sin\left(\frac{t}{3}\right) = 5$:

$$\frac{t}{3} = \frac{\pi}{2} \Rightarrow t = \frac{3\pi}{2}.$$ **[A1]**

$$v = \frac{dx}{dt} = 5\sin\left(\frac{t}{3}\right) \quad \text{where } x = 0 \text{ when } t = 0.$$

©The Mathematical Association of Victoria, 2017

Option 1:

$$x = \int 5\sin\left(\frac{t}{3}\right)dt \qquad = -15\cos\left(\frac{t}{3}\right) + K\,.$$

Substitute $x = 0$ when $t = 0$: $K = 15$.

Therefore

$$x = -15\cos\left(\frac{t}{3}\right) + 15\,.$$

Substitute $t = \frac{3\pi}{2}$ into $x = -15\cos\left(\frac{t}{3}\right) + 15$:

$x = 15$.

Answer: $b = 15$. **[H1]**

Consequential on equation for *v* from **part b.** and answer to **part b.** if the equation for *v* from **part b.** and answer to **part b.** are quoted and used.

Option 2:

$$x = \int_0^t 5\sin\left(\frac{t}{3}\right)dt + 0 \qquad \text{(the integral solution)}$$

$$= -15\cos\left(\frac{t}{3}\right) + 15\,.$$

Substitute $t = \frac{3\pi}{2}$ into $x = -15\cos\left(\frac{t}{3}\right) + 15$:

$x = 15$.

Answer: $b = 15$. **[H1]**

Consequential on equation for *v* from **part b.** and answer to **part b.** if the equation for *v* from **part b.** and answer to **part b.** are quoted and used.

©The Mathematical Association of Victoria, 2017

Option 3:

Distance $= \int_0^{3\pi/2} |v|\, dt \quad = \int_0^{3\pi/2} v\, dt$ since $0 \le v \le 5$.

Substitute $v = 5\sin\left(\frac{t}{3}\right)$:

Distance $= \int_0^{3\pi/2} 5\sin\left(\frac{t}{3}\right) dt = 15$.

Answer: $b = 15$. **[H1]**

Consequential on equation for *v* from **part b.** and answer to **part b.** if the equation for *v* from **part b.** and answer to **part b.** are quoted and used.

Comment: The answers to **part a.** and **part b.** allow a check of consistency:
If the answer $t = t_1$ to **part b.** leads to an answer that is different to the answer found using **Method 1** then one of the answers is wrong.

©The Mathematical Association of Victoria, 2017

b.

The acceleration of the lift is upwards. Define the upwards direction as positive.

$a = \frac{dv}{dt} = \frac{\sqrt{25 - v^2}}{3}$ where $v = 0$ when $t = 0$

$\Rightarrow \frac{dt}{dv} = \frac{3}{\sqrt{25 - v^2}}$

$\Rightarrow t = \int \frac{3}{\sqrt{25 - v^2}}\, dv \quad = 3\sin^{-1}\left(\frac{v}{5}\right) + C$.

Substitute $v = 0$ when $t = 0$: $C = 0$.

Therefore:

$t = 3\sin^{-1}\left(\frac{v}{5}\right)$

$\Rightarrow v = 5\sin\left(\frac{t}{3}\right)$. **[A1]**

Method 1:

Speed is increasing when $a > 0$.

Therefore speed is increasing until $a = \frac{\sqrt{25 - v^2}}{3} = 0$.

$a = 0$ at $v = 5$.

Solve $v = 5\sin\left(\frac{t}{3}\right) = 5$:

$\frac{t}{3} = \frac{\pi}{2} \Rightarrow t = \frac{3\pi}{2}$.

It therefore follows that speed is increasing for $0 \le t < \frac{3\pi}{2}$.

Answer: $t_1 = \frac{3\pi}{2}$. **[H1]**

Consequential on equation for *v*.

Method 2:

Speed is increasing when $a > 0$.

$$a = \frac{dv}{dt} = \frac{5}{3}\cos\left(\frac{t}{3}\right)$$

from which it follows that $a > 0$ for $0 \le t < \frac{3\pi}{2}$.

Answer: $t_1 = \frac{3\pi}{2}$. **[H1]**

Consequential on equation for *v*.

Note: The answers to **part a.** and **part b.** allow a check of consistency:
If the answer $t = t_1$ to **part b.** leads to an answer that is different to the answer found using **Method 1** in **part a.** then one of the answers is wrong.

©The Mathematical Association of Victoria, 2017

c.

The reading of the scales is the size of the force exerted by the parcel on the scales (in units of kg wt).
From Newton's Third Law, the size of this force is equal to the size of the normal reaction force of the scales on the parcel (in units of kg wt).

Let *R* newtons be the size of the normal reaction force of the scales on the parcel.

Define the upwards direction as positive. The acceleration is downwards.

$F_{net} = ma$

$\Rightarrow R - 20g = -20\dfrac{\sqrt{10v - v^2}}{3}$. .... (1) **[M1]**

The value of *R* when $v = 2$ is required.

Substitute $v = 2$ into equation (1) and solve for *R*:

$R = 20g - 20\dfrac{\sqrt{16}}{3} = 20g - \dfrac{80}{3}$ newtons $= 20 - \dfrac{80}{3g}$ kg wt.

Answer: $20 - \dfrac{80}{3g}$. **[A1]**

Accept all equivalent answers (such as $\dfrac{60g - 80}{3g}$).

Accept all answers where the substitution $g = 9.8$ has been made and a correct numerical answer is subsequently given.

Comment: This question can be made more challenging by asking for the reading of the scales π seconds after the lift starts to slow down. Such a question would potentially be worth 4 marks.

Brief solution: The value of *v* when $t = \pi$ is required (so that *R* can be found from equation (1)).

$a = \dfrac{dv}{dt} = -\dfrac{1}{3}\sqrt{10v - v^2}$ $\quad\Rightarrow \dfrac{dt}{dv} = -\dfrac{3}{\sqrt{10v - v^2}} = -\dfrac{3}{\sqrt{25 - (v-5)^2}}$ $\quad\Rightarrow t = -3\sin^{-1}\left(\dfrac{v-5}{5}\right) + C$.

Substitute $v = 5$ when $t = 0$: $C = 0$.

Therefore $t = -3\sin^{-1}\left(\dfrac{v-5}{5}\right)$ $\quad\Rightarrow v = 5 - 5\sin\left(\dfrac{t}{3}\right)$.

Substitute $t = \pi$: $v = 5 - \dfrac{5}{2} = \dfrac{5}{2}$.

Substitute $v = \dfrac{5}{2}$ into equation (1) and solve for *R*: $R = 20 - \dfrac{50\sqrt{3}}{3g}$ kg wt.

©The Mathematical Association of Victoria, 2017

Question 6

Compare $\int_a^2 \frac{\sqrt{x^2-6x+13}}{3-x}\,dx$ with the arc length formula (refer to VCAA detachable formula sheet):

$$\int_a^2 \frac{\sqrt{x^2-6x+13}}{3-x}\,dx = \int_a^2 \sqrt{1+\left(f'(x)\right)^2}\,dx \quad \text{where } a<2$$

$$\Rightarrow \int_a^2 \sqrt{\frac{x^2-6x+13}{(3-x)^2}}\,dx = \int_a^2 \sqrt{1+\left(f'(x)\right)^2}\,dx.$$

Therefore:

$$\sqrt{\frac{x^2-6x+13}{(3-x)^2}} = \sqrt{1+\left(f'(x)\right)^2}$$

$$\Rightarrow \frac{x^2-6x+13}{(3-x)^2} = 1+\left(f'(x)\right)^2$$

$$\Rightarrow \left(f'(x)\right)^2 = \frac{x^2-6x+13}{(3-x)^2}-1 \qquad = \frac{x^2-6x+13-(3-x)^2}{(3-x)^2}$$

$$= \frac{4}{(3-x)^2}$$ **[M1]**

$$\Rightarrow f'(x) = \frac{\pm 2}{|3-x|}.$$

But $|3-x| = 3-x$ since the upper terminal of the arc length integral is $x=2$ and so it follows that $x<3$. This is also consistent with the denominator of the original integrand being $3-x$ prior to being brought inside the square root.

Therefore $f'(x) = \frac{\pm 2}{3-x} = \frac{\mp 2}{x-3}$.

©The Mathematical Association of Victoria, 2017

Integrate with respect to *x*:

$f(x) = \pm 2\ln(3-x) + C$. **[H1]**

Consequential on expression for $\left(f'(x)\right)^2$.

Substitute $(2, -3)$: $C = -3$.

Therefore $f(x) = \pm 2\ln(3-x) - 3$.

Answer:: Any of the following or their equivalent form: **[H1]**

$f(x) = 2\ln(3-x) - 3$, $f(x) = -2\ln(3-x) - 3$, $f(x) = \pm 2\ln(3-x) - 3$.

Consequential on general solution for $f(x)$.

Accept $f(x) = 2\ln|3-x| - 3$, $f(x) = -2\ln|3-x| - 3$, $f(x) = \pm 2\ln|3-x| - 3$ or their equivalent form.

©The Mathematical Association of Victoria, 2017

Question 7

Gradient given by the value of $\frac{dy}{dx}$ at $t=3$.

$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$.

- $x=\arctan\left(2t^2\right)$.

Use the chain rule to get $\frac{dx}{dt}$:

$$\frac{dx}{dt}=\underbrace{\frac{1}{\left(2t^2\right)^2+1}}_{\text{From VCAA formula sheet}}\times 4t$$

$$=\frac{4t}{4t^4+1}.$$ **[M1]**

At $t=3$, $\frac{dx}{dt}=\frac{12}{325}$.

- $y=\arccos\left(\frac{\sqrt{t}}{2}\right)$.

Use the chain rule to get $\frac{dy}{dt}$:

$$\frac{dy}{dt}=\underbrace{\frac{-1}{\sqrt{1-\left(\frac{\sqrt{t}}{2}\right)^2}}}_{\text{From VCAA formula sheet}}\times\frac{1}{4\sqrt{t}}\quad=\frac{-1}{\sqrt{1-\frac{t}{4}}}\times\frac{1}{4\sqrt{t}}\quad=\frac{-2}{\sqrt{4-t}}\times\frac{1}{4\sqrt{t}}$$

$$=\frac{-1}{2\sqrt{t}\sqrt{4-t}}.$$ **[M1]**

At $t=3$, $\frac{dy}{dt}=\frac{-1}{2\sqrt{3}}$.

©The Mathematical Association of Victoria, 2017

Therefore:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \qquad = \left(\frac{325}{12}\right) \times \left(\frac{-1}{2\sqrt{3}}\right) \qquad = \frac{-325}{24\sqrt{3}}.$$

Answer: $\frac{-325}{24\sqrt{3}}$. **[A1]**

Accept $\frac{-325\sqrt{3}}{72}$.

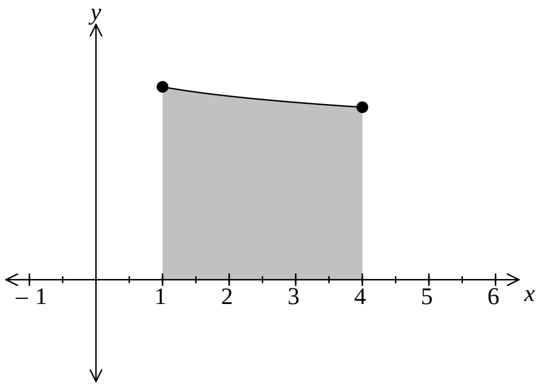
©The Mathematical Association of Victoria, 2017

Question 8

The line $y=0$ is the x-axis.

A rough graph of $y=\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{3+\sqrt{x}}}$ over the relevant interval $1\le x\le 4$ should be drawn and the required area shaded.

Since $y=\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{3+\sqrt{x}}}$ is a strictly decreasing function over its implied domain $x\in[0,+\infty)$, the following graph can be deduced:



a.

Required volume:

$$V=\pi\int_1^4 y^2\,dx$$

$$=4\int_1^4 \frac{1}{3+\sqrt{x}}\,dx\,.$$ **[A1]**

b.

$V = 4\int_1^4 \frac{1}{3+\sqrt{x}}\,dx$ can be evaluated using one of two possible substitutions.

Substitution 1: $u = \sqrt{x}$.

$u = \sqrt{x} \quad \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u} \quad \Rightarrow dx = 2u\,du$.

$x = 1 \quad \Rightarrow u = \sqrt{1} = 1$.

$x = 4 \quad \Rightarrow u = \sqrt{4} = 2$.

Therefore:

$$V = 4\int_1^2 \frac{2u}{3+u}\,du$$ **[M1]**

$$= 8\int_1^2 \frac{u}{u+3}\,du.$$

Substitute $\frac{u}{u+3} = 1 - \frac{3}{u+3}$.

Note: This expression is obtained from either algebraic arrangement or polynomial long division.

Algebraic re-arrangement: $\frac{u}{u+3} = \frac{(u+3)-3}{u+3} = \frac{(u+3)}{u+3} - \frac{3}{u+3} = 1 - \frac{3}{u+3}$.

Therefore

$$V = 8\int_1^2 1 - \frac{3}{u+3}\,du$$ **[M1]**

$= 8\left[u - 3\log_e(u+3)\right]_1^2$ since $u + 3 > 0$

$= 8\left[(2 - 3\log_e(5)) - (1 - 3\log_e(4))\right] \quad = 8(1 + 3\log_e(4) - 3\log_e(5)) \quad = 8\left(1 + 3\log_e\left(\frac{4}{5}\right)\right)$.

Answer: $8\left(1 + 3\log_e\left(\frac{4}{5}\right)\right)$ cubic units. **[A1]**

Unit is not required.

Note that $a = 4$ and $b = 3$.

©The Mathematical Association of Victoria, 2017

Substitution 2: $u = 3 + \sqrt{x}$.

$u = 3 + \sqrt{x} \quad \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2(u-3)} \quad \Rightarrow dx = 2(u-3)\,du$.

$x = 1 \quad \Rightarrow u = 3 + \sqrt{1} = 4$.

$x = 4 \quad \Rightarrow u = 3 + \sqrt{4} = 5$.

Therefore:

$$V = 4\int_4^5 \frac{2(u-3)}{u}\,du$$ **[M1]**

$$= 8\int_4^5 \frac{u-3}{u}\,du.$$

$$= 8\int_4^5 1 - \frac{3}{u}\,du$$ **[M1]**

$= 8\left[u - 3\log_e(u)\right]_4^5$ since $u > 0$

$= 8\left[(5 - 3\log_e(5)) - (4 - 3\log_e(4))\right] \quad = 8\left(1 + 3\log_e(4) - 3\log_e(5)\right)$

$= 8\left(1 + 3\log_e\left(\frac{4}{5}\right)\right)$.

Answer: $8\left(1 + 3\log_e\left(\frac{4}{5}\right)\right)$ cubic units. **[A1]**

Unit is not required.

Note that $a = 4$ and $b = 3$.

©The Mathematical Association of Victoria, 2017

Question 9

$$\frac{dy}{dx}=\frac{\cos^2(2y)}{2x^2+1}$$

which is a separable differential equation

$$\Rightarrow \int \frac{1}{\cos^2(2y)}\,dy = \int \frac{1}{2x^2+1}\,dx \qquad = \frac{1}{2}\int \frac{1}{x^2+\frac{1}{2}}\,dx \qquad = \frac{\sqrt{2}}{2}\int \frac{\frac{1}{\sqrt{2}}}{x^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dx$$

$$\Rightarrow \frac{1}{2}\underbrace{\tan(2y)}_{\substack{\text{From VCAA}\\ \text{formula sheet}}} = \frac{\sqrt{2}}{2}\underbrace{\tan^{-1}\left(\sqrt{2}x\right)}_{\substack{\text{From VCAA}\\ \text{formula sheet:}\\ a=\frac{1}{\sqrt{2}}}}+C$$

Left hand side of equation: **[M1]**

Right hand side of equation: **[M1]**

Accept equation below:

$$\Rightarrow \tan(2y)=\sqrt{2}\tan^{-1}\left(\sqrt{2}x\right)+C\,.$$

Substitute $y\left(-\frac{\sqrt{6}}{6}\right)=0$, that is, $y=0$ when $x=-\frac{\sqrt{6}}{6}$:

$$\tan(0)=\sqrt{2}\tan^{-1}\left(\frac{-\sqrt{2}\sqrt{6}}{6}\right)+C$$

$$\Rightarrow 0=\sqrt{2}\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+C$$

$$\Rightarrow 0=\sqrt{2}\left(-\frac{\pi}{6}\right)+C$$

$$\Rightarrow C=\frac{\pi\sqrt{2}}{6}.$$ **[A1]**

©The Mathematical Association of Victoria, 2017

Therefore:

$$\tan(2y) = \sqrt{2}\tan^{-1}\left(\sqrt{2}x\right) + \frac{\pi\sqrt{2}}{6}$$

$$\Rightarrow 2y = \tan^{-1}\left(\sqrt{2}\tan^{-1}\left(\sqrt{2}x\right) + \frac{\pi\sqrt{2}}{6}\right) \quad = \tan^{-1}\left(\sqrt{2}\left(\tan^{-1}\left(\sqrt{2}x\right) + \frac{\pi}{6}\right)\right)$$

$$\Rightarrow y = \frac{1}{2}\tan^{-1}\left(\sqrt{2}\left(\tan^{-1}\left(\sqrt{2}x\right) + \frac{\pi}{6}\right)\right).$$

Answer: $y = \frac{1}{2}\tan^{-1}\left(\sqrt{2}\left(\tan^{-1}\left(\sqrt{2}x\right) + \frac{\pi}{6}\right)\right)$. **[A1]**

Accept $y = \frac{1}{2}\tan^{-1}\left(\sqrt{2}\tan^{-1}\left(\sqrt{2}x\right) + \frac{\pi\sqrt{2}}{6}\right)$.

©The Mathematical Association of Victoria, 2017

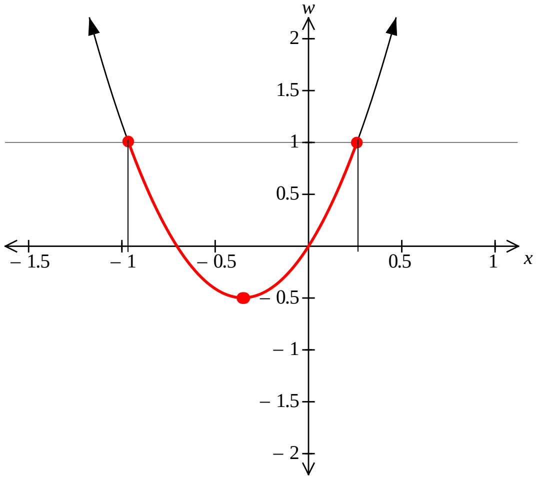
Question 10

• Let $w = 4x^2 + 2\sqrt{2}x$.

• Then $y = -\frac{1}{2}\sin^{-1}(w)$.

Implied domain: Solve $-1 \le w \le 1$.

• Draw a graph of $w = 4x^2 + 2\sqrt{2}x$:



• Solve $w = 1$:

$$4x^2 + 2\sqrt{2}x = 1 \quad \Rightarrow 4x^2 + 2\sqrt{2}x - 1 = 0$$

$$\Rightarrow x = \frac{-2\sqrt{2} \pm \sqrt{\left(2\sqrt{2}\right)^2 - 4(4)(-1)}}{8}$$

$$= \frac{-2\sqrt{2} \pm \sqrt{24}}{8} = \frac{-\sqrt{2} \pm \sqrt{6}}{4}.$$

• Solve $w = -1$: It can be seen from the graph that there is no real solution.

• From the graph of $w = 4x^2 + 2\sqrt{2}x$ it can therefore be seen that the values of x that satisfy $-1 \le w \le 1$ are $x \in \left[\frac{-\sqrt{2} - \sqrt{6}}{4}, \ \frac{-\sqrt{2} + \sqrt{6}}{4}\right]$.

Implied domain: $x \in \left[\frac{-\sqrt{2} - \sqrt{6}}{4}, \ \frac{-\sqrt{2} + \sqrt{6}}{4}\right]$. **[A1]**

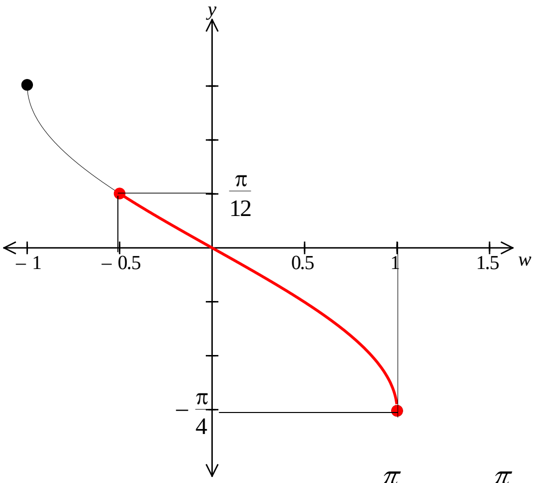
Appropriate method: **[M1]**

Implied range:

• From the graph of $w = 4x^2 + 2\sqrt{2}x$ it can be seen that only the subset

$-\frac{1}{2} \le w \le 1$ of allowed values $-1 \le w \le 1$ is used.

• Draw a graph of $y = -\frac{1}{2}\sin^{-1}(w)$ over the domain $-\frac{1}{2} \le w \le 1$:



• From this graph it is seen that $-\frac{\pi}{4} \le y \le \frac{\pi}{12}$.

Implied range: $\left[-\frac{\pi}{4}, \frac{\pi}{12}\right]$. **[A1]**

Appropriate method: **[M1]**

©The Mathematical Association of Victoria, 2017