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# SPECIALIST MATHS 3 & 4 TRIAL EXAMINATION 2 SOLUTIONS 2017

## Section A – Multiple-choice answers

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1.** | E | **6.** | E | **11.** | C | **16.** | B |
| **2.** | C | **7.** | D | **12.** | E | **17.** | A |
| **3.** | D | **8.** | B | **13.** | B | **18.** | A |
| **4.** | C | **9.** | C | **14.** | E | **19.** | D |
| **5.** | A | **10.** | D | **15.** | B | **20.** | B |

## Section A - Multiple-choice solutions

### Question 1

$x = 2\tan(t) + 1 \qquad y = 5\sec^2(t)$

$\frac{x-1}{2} = \tan(t) \qquad \frac{y}{5} = \sec^2(t)$

$\frac{(x-1)^2}{4} = \tan^2(t)$

Using the identity $1 + \tan^2(x) = \sec^2(x)$ (from the formula sheet) we have,

$$1 + \frac{(x-1)^2}{4} = \frac{y}{5}$$

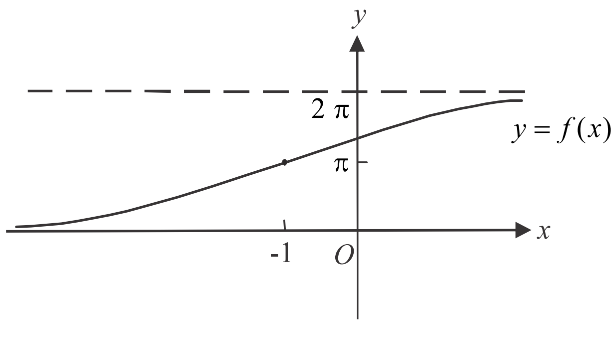
$$y = \frac{5(x-1)^2}{4} + 5$$

The answer is E.

### Question 2

Do a quick sketch.
Note that this graph is the graph of $y = \tan^{-1}(x)$ that has been

- translated 1 unit left
- dilated by a factor of 2 from the $x$-axis
- translated $\pi$ units up




So the domain $= R$.
The range is more interesting.

For $y = \tan^{-1}(x)$, $\quad r = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

For $y = 2\tan^{-1}(x)$, $\quad r = (-\pi, \pi)$

For $y = 2\tan^{-1}(x) + \pi$, $\quad r = (0, 2\pi)$

So range $= (0, 2\pi)$.
The answer is C.

**Question 3**

The period of the graph shown is $\frac{\pi}{2}$.

For $y=\cot(nx)$, period $=\frac{\pi}{n}$.

So $\frac{\pi}{2}=\frac{\pi}{n}$ and so $n=2$.

The rule could be $y=\cot(2x)$ but there is no rule like this on offer. Another possibility is $y=\cot\left(2\left(x-\frac{\pi}{2}\right)\right)$, since this is the graph of $y=\cot(2x)$ that has been translated $\frac{\pi}{2}$ units to the right, thus creating the same graph as $y=\cot(2x)$.

The answer is D.

**Question 4**

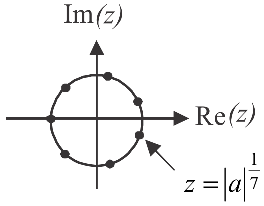
Since $P(z)$ has real coefficients, and $P(z)=0$ has roots $z=3i$ and $z=1-i$, then it must also have the roots $z=-3i$ and $z=1+i$ (Conjugate root theorem). So the minimum number of roots that $P(z)=0$ could have is 5.

The answer is C.

**Question 5**

The roots to the equation $z^7=a$ lie on a circle of radius $|a|^{\frac{1}{7}}$ units and are separated by an argument of $\frac{2\pi}{7}$. The root we are given, ie. $|a|^{\frac{1}{7}}\text{cis}\left(-\frac{\pi}{7}\right)$, is marked on the diagram below.

So the other roots must be




$z=|a|^{\frac{1}{7}}\text{cis}\left(\frac{\pi}{7}\right)$

$z=|a|^{\frac{1}{7}}\text{cis}\left(\frac{3\pi}{7}\right)$

$z=|a|^{\frac{1}{7}}\text{cis}\left(\frac{5\pi}{7}\right)$

$z=|a|^{\frac{1}{7}}\text{cis}\left(-\frac{3\pi}{7}\right)$

$z=|a|^{\frac{1}{7}}\text{cis}\left(-\frac{5\pi}{7}\right)$

$z=|a|^{\frac{1}{7}}\text{cis}\left(\frac{7\pi}{7}\right)$ or $z=|a|^{\frac{1}{7}}\text{cis}\left(-\frac{7\pi}{7}\right)$

$=|a|^{\frac{1}{7}}(\cos(\pi)+i\sin(\pi))$

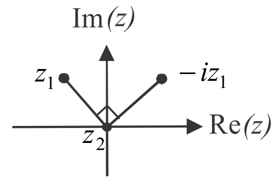
$=|a|^{\frac{1}{7}}(-1+0i)$

$=-|a|^{\frac{1}{7}}$

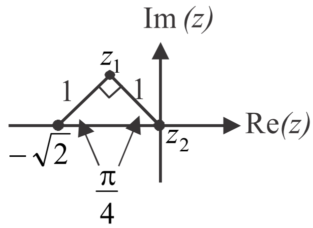
The answer is A.

**Question 6**

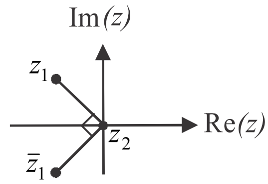
For option A, the complex number $-iz_1$, is obtained by rotating $z_1$ clockwise about the origin by 90°.
A right angled triangle is formed.



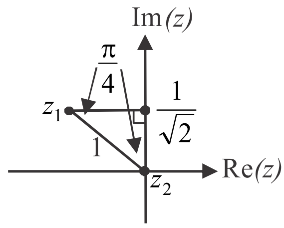

For option B, the complex number $-\sqrt{2}$ lies on the real axis.
Since $\sqrt{1^2+1^2}=\sqrt{2}$ and the length of the sidelength joining $-\sqrt{2}$ and $z_2$ is $\sqrt{2}$, a right-angled triangle is formed.




For option C, $\bar{z}_1=\text{cis}\left(-\frac{3\pi}{4}\right)$.
A right-angled triangle is formed.



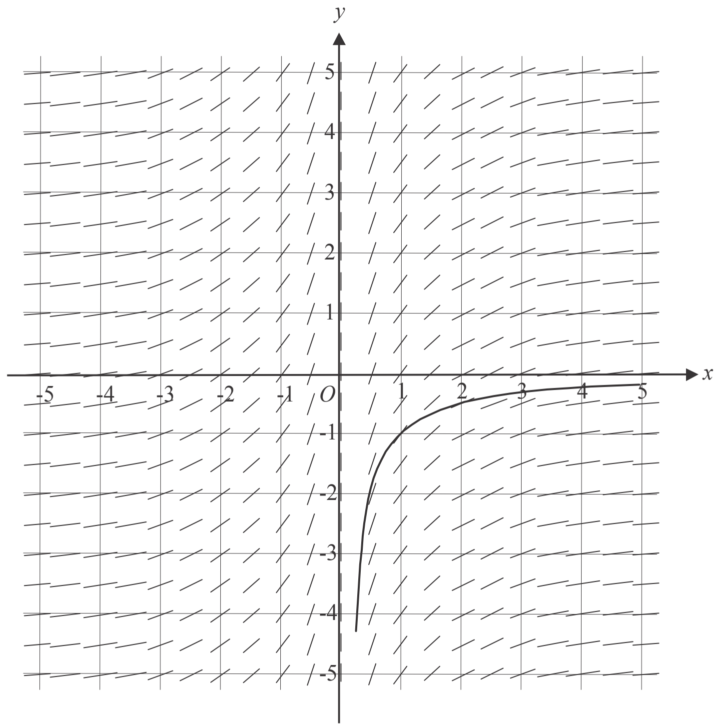

For option D, the complex number $\frac{i}{\sqrt{2}}$ lies on the imaginary axis and with $z_1$ and $z_2$ forms a right-angled isosceles triangle.




By elimination option E must be correct.
Mathematically, if $z_3=i^2z_1$, then the three complex numbers lie along a straight line.
The answer is E.

**Question 7**

Do a quick sketch of this solution.




It appears that this solution could pass through the point $(2,-0.5)$.
The answer is D.

**Question 8**

$$\int_0^{\frac{\pi}{2}} 2\sin(x)\cos^3(x)dx$$

$$= \int_1^0 2 \times -\frac{du}{dx}u^3\,dx$$

$$= 2\int_0^1 u^3\,du$$

$$u = \cos(x)$$

$$\frac{du}{dx} = -\sin(x)$$

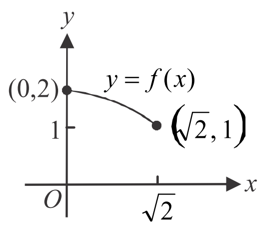
$$x = \frac{\pi}{2}, u = 0$$

$$x = 0, \quad u = 1$$

The answer is B.

**Question 9**

Do a quick sketch to help determine the endpoints of *f*. The terminals of integration are the *y* coordinates of these endpoints because we are rotating about the *y*-axis.




$$\text{volume} = \pi\int_1^2 x^2\,dy$$

$$= \pi\int_1^2 \frac{2(2-y)}{y}dy$$

$$= 2\pi\int_1^2 \frac{2-y}{y}dy$$

The answer is C.

Since $y = \dfrac{4}{2+x^2}$

$$2 + x^2 = \frac{4}{y}$$

$$x^2 = \frac{4}{y} - 2$$

$$= \frac{4-2y}{y}$$

$$= \frac{2(2-y)}{y}$$

**Question 10**

Using the formula from the formula sheet, let $f(x) = \dfrac{dy}{dx} = x(2-x^2)$ .

$f(x) = x(2-x^2), \quad x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n)$

$$\begin{aligned} x_0 &= 1 & y_0 &= 0 \\ x_1 &= 1 + 0.1 & y_1 &= 0 + 0.1 \times f(x_0) \\ &= 1.1 & &= 0.1 \\ x_2 &= 1.1 + 0.1 & y_2 &= 0.1 + 0.1 \times f(x_1) \\ &= 1.2 & &= 0.1869 \\ & & y_3 &= 0.1869 + 0.1 \times f(x_2) \\ & & &= 0.2541 \end{aligned}$$

The answer is D.

**Question 11**

$$\frac{dx}{dt}=\frac{dx_{\text{inflow}}}{dl}\cdot\frac{dl_{\text{inflow}}}{dt}-\frac{dx_{\text{outflow}}}{dl}\cdot\frac{dl_{\text{outflow}}}{dt}$$
$$=0.5\times 6-\frac{x}{900-6t}\times 12$$
$$=3-\frac{12x}{900-6t}$$
$$=3-\frac{2x}{150-t}$$

The answer is C.

Note that the concentration of sugar per litre in the tank after $t$ minutes

$$=\frac{\text{amount of sugar in tank after } t \text{ minutes}}{\text{volume of solution in tank after } t \text{ minutes}}$$
$$=\frac{x}{900-6t}$$

**Question 12**

$$\frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt} \text{ (chain rule)}$$
$$=4\pi r^2\times\frac{-4t}{\sqrt{9-t^2}}$$

When $t=2,\ r=4\sqrt{5}$

So $\frac{dV}{dt}=4\pi\times 80\times\frac{-8}{\sqrt{5}}$
$$=-3596.7051...$$

$V=\frac{4}{3}\pi r^3$ (formula sheet)

$\frac{dV}{dr}=4\pi r^2$

$r=4\sqrt{9-t^2}$ (given)

$\frac{dr}{dt}=\frac{-4t}{\sqrt{9-t^2}}$

At $t=2$, the volume of the sculpture is decreasing at the rate of 3596.7051…$cm^3$/hr.
The closest answer is 3597.
The answer is E.

**Question 13**

Using the cosine rule (formula sheet) we have

$$|\underset{\sim}{w}|^2=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2-2\,|\underset{\sim}{u}|\,|\underset{\sim}{v}|\cos(135^\circ)$$
$$=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2-2\,|\underset{\sim}{u}|\,|\underset{\sim}{v}|\times-\frac{1}{\sqrt{2}}$$
$$=|\underset{\sim}{u}|^2+|\underset{\sim}{v}|^2+\sqrt{2}\,|\underset{\sim}{u}|\,|\underset{\sim}{v}|$$

The answer is B.

**Question 14**

$$\underset{\sim}{a}(t) = e^{\frac{t}{2}}\underset{\sim}{i} + \cos(2t)\underset{\sim}{j} - \sin(t)\underset{\sim}{k}$$

$$\underset{\sim}{v}(t) = 2e^{\frac{t}{2}}\underset{\sim}{i} + \frac{1}{2}\sin(2t)\underset{\sim}{j} + \cos(t)\underset{\sim}{k} + \underset{\sim}{c}$$

The particle starts from rest,
so when $t = 0$, $\underset{\sim}{v}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$

$$\underset{\sim}{v}(0) = 2\underset{\sim}{i} + 0\underset{\sim}{j} + \underset{\sim}{k} + \underset{\sim}{c}$$

So $\underset{\sim}{c} = -2\underset{\sim}{i} - \underset{\sim}{k}$

So $$\underset{\sim}{v}(t) = \left(2e^{\frac{t}{2}} - 2\right)\underset{\sim}{i} + \frac{1}{2}\sin(2t)\underset{\sim}{j} + (\cos(t) - 1)\underset{\sim}{k}$$

$$= 2\left(e^{\frac{t}{2}} - 1\right)\underset{\sim}{i} + \frac{1}{2}\sin(2t)\underset{\sim}{j} + (\cos(t) - 1)\underset{\sim}{k}$$

The answer is E.

**Question 15**

$$v^2 = x^2 - 5x$$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) \quad \text{(formula sheet)}$$

$$= \frac{d}{dx}\left(\frac{1}{2}(x^2 - 5x)\right)$$

$$= x - \frac{5}{2}$$

$F = ma$

$$= 6\left(x - \frac{5}{2}\right)$$

$F = 6x - 15$

The answer is B.

**Question 16**

Initial momentum $= mv$ (in easterly direction)

$$= 6 \times 5$$

$$= 30 \text{ kg ms}^{-1}$$

Final momentum $= 6 \times 1$

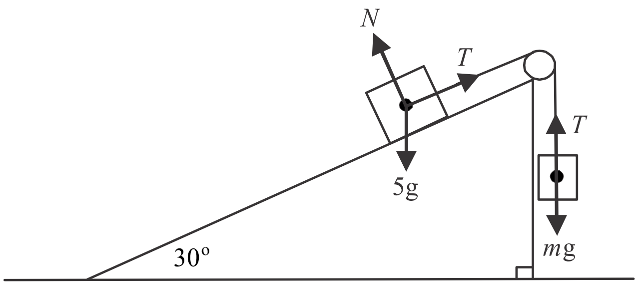
$$= 6 \text{ kg ms}^{-1}$$

Change in momentum is $6 - 30 = -24 \text{ kg ms}^{-1}$ in easterly direction.

The answer is B.

**Question 17**

Draw in the forces.




Note that since the packages are in equilibrium, $a = 0$.

<u>Around the $m$ kg package:</u>

$$T = mg$$

<u>Around the 5 kg package</u> and parallel to the inclined plane:

$$5g\sin(30^\circ) = T$$

So $$\frac{5g}{2} = mg$$

$$m = \frac{5}{2}$$

The answer is A.

**Question 18**

Let the mass of a wrapped chocolate be $W$ and the mass of an unwrapped chocolate be $U$.

$$\begin{aligned}&\mathrm{E}(W_1+W_2+U_1+U_2)\\&=\mathrm{E}(W_1)+\mathrm{E}(W_2)+\mathrm{E}(U_1)+\mathrm{E}(U_2)\\&=32+32+25+25\\&=114\end{aligned}$$

$$\begin{aligned}&\mathrm{var}(W_1+W_2+U_1+U_2)\\&=\mathrm{var}(W_1)+\mathrm{var}(W_2)+\mathrm{var}(U_1)+\mathrm{var}(U_2)\\&=9+9+4+4 \qquad \text{N.B.} \quad \mathrm{var}(W_1+W_2)\neq \mathrm{var}(2W)\\&=26\end{aligned}$$

So the standard deviation is $\sqrt{26}$.
The answer is A.

**Question 19**

For the distribution of the sample mean $\overline{X}$,

$$\mathrm{E}(\overline{X})=\mu \qquad \mathrm{sd}(\overline{X})=\frac{s}{\sqrt{n}}$$

$$=7 \qquad\qquad =\frac{0.5}{\sqrt{64}}$$

$$=0.0625$$

$\Pr(\overline{X}<7.1)=0.9452007...$

The closest answer is 0.9452.
The answer is D.

**Question 20**

A type II error occurs when the null hypothesis $H_0$ is not rejected when it is false.
The answer is B.

## SECTION B

**Question 1** (10 marks)

**a.** Solve $f'(x)=0$ for $x$.

$x=-0.7937005...$

$f(-0.7937005...)=1.88988...$

Stationary point occurs at $(-0.79, 1.89)$ where coordinates are expressed correct to 2 decimal places.

**(1 mark)**

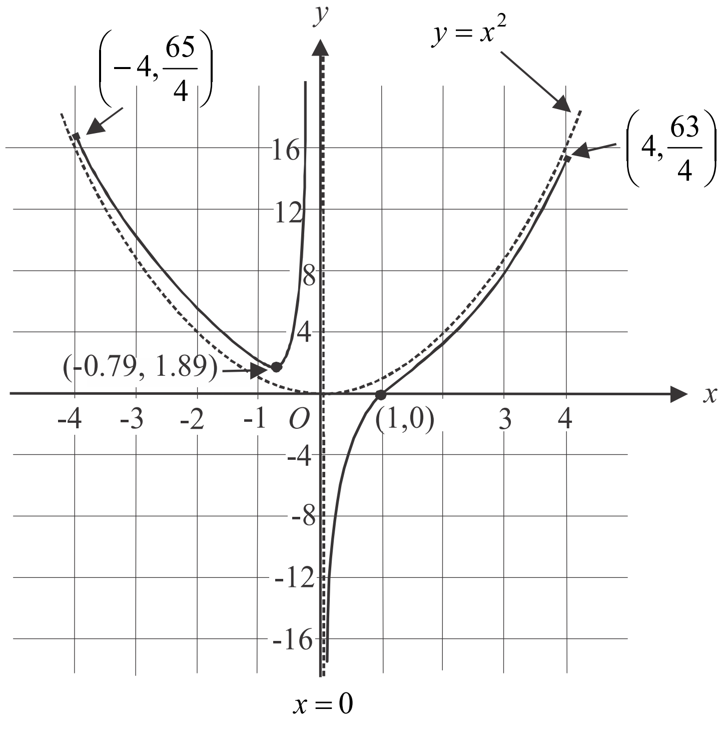
**b.** $f(x)=\dfrac{x^4-x}{x^2}$

$=x^2-\dfrac{1}{x}, \quad x\neq 0$

The vertical asymptote has equation $x=0$. **(1 mark)**

The non-vertical asymptote has equation $y=x^2$. **(1 mark)**

**c.** Note that $f''(x)=0$ when $x=1$ so the point of inflection occurs at (1, 0).




**(1 mark)** – correct shapes including asymptotes
**(1 mark)** – correct domain including endpoints
**(1 mark)** – correct stationary point and point of inflection

**d.** **i.** $$\text{volume} = \int_1^4 \pi y^2\, dx$$

$$= \pi \int_1^4 \left( \frac{x^4 - x}{x^2} \right)^2 dx$$

**(1 mark)**

**ii.** $$\text{volume} = \frac{3807\pi}{20}$$

$$= 598.00216...$$

$$= 598 \text{ cubic units (correct to the nearest whole number)}$$

**(1 mark)**

**e.** Since $x \in (0,4]$ we are looking at the right hand branch of the graph in part **c.**

Solve $f''(x) \geq 0$ for $x$

$$x < 0 \text{ or } x \geq 1$$

So we are looking at the length of the curve from $x = 1$ to $x = 4$. **(1 mark)**

$$\text{length of curve} = \int_1^4 \sqrt{1 + (f'(x))^2}\, dx \qquad \text{(formula sheet)}$$

$$= 16.0591...$$

$$= 16.06 \text{ units correct to two decimal places}$$

**(1 mark)**

**Question 2** (13 marks)

**a.**

$$|z + 2| = 2$$

$$|x + yi + 2| = 2$$

$$\sqrt{(x+2)^2 + y^2} = 2$$

$$(x+2)^2 + y^2 = 4$$

**(1 mark)**

**b.** **i.** $$\tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$$

$$= \frac{\tan\left(\frac{2\pi}{3}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan\left(\frac{2\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} \qquad \text{(compound angle formula - formula sheet)}$$

**(1 mark)**

$$= \frac{-\sqrt{3} + 1}{1 - -\sqrt{3} \times 1}$$

$$= \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$

$$= \frac{1 - 2\sqrt{3} + 3}{1 - 3}$$

$$= \frac{4 - 2\sqrt{3}}{-2}$$

$$= \sqrt{3} - 2 \text{ as required}$$

**(1 mark)**

**ii.** $\text{Arg}(z) = \frac{11\pi}{12}$

Let $z = x + yi$.

$\tan^{-1}\left(\frac{y}{x}\right) = \frac{11\pi}{12}$ **(1 mark)**

$\frac{y}{x} = \tan\left(\frac{11\pi}{12}\right)$

$\frac{y}{x} = \sqrt{3} - 2$ $\left(\text{since } \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11\pi}{12} \text{ and using part } \mathbf{i.}\right)$

$y = (\sqrt{3} - 2)x$

Re-read the question!

So $a = \sqrt{3} - 2$.

**(1 mark)**

**c.** Using your answers to parts **a.** and **b.**, solve $(x+2)^2 + y^2 = 4$ and $y = (\sqrt{3} - 2)x$ for $x$ and $y$ using CAS. **(1 mark)**

$x = -\left(\sqrt{3} + 2\right)$ and $y = 1$.

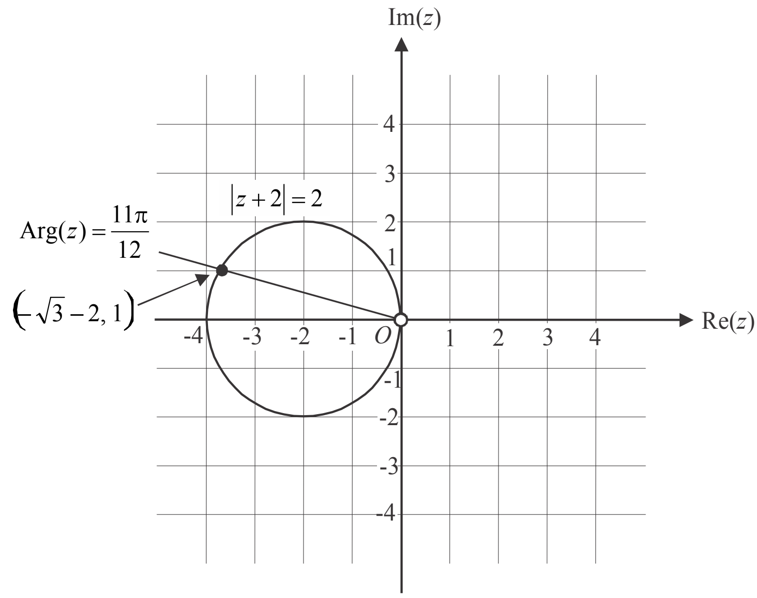
Note that $x < 0$ so reject the $x = 0$ and $y = 0$ solution.

Re-read the question!

The point of intersection is $\left(-\sqrt{3} - 2, 1\right)$.

**(1 mark)**

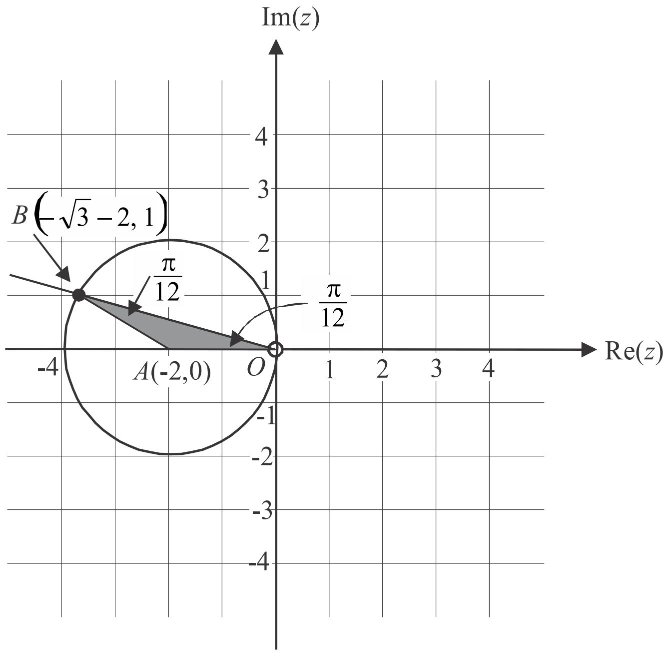
**d.**




**(1 mark)** – correct circle

**(1 mark)** – correct ray including exclusion of origin and point of intersection $\left(-\sqrt{3} - 2, 1\right)$

**e.**




Method 1

$$\text{area of segment} = \frac{1}{2}r^2\left(\theta - \sin(\theta)\right)$$ **(1 mark)**

$$= \frac{1}{2}\times 4\left(\frac{5\pi}{6} - \sin\left(\frac{5\pi}{6}\right)\right)$$

$$= \frac{5\pi}{3} - 1 \text{ square units}$$

**(1 mark)**

Method 2

The triangle with corner points $O(0,0)$, $A(-2,0)$ and $B\left(-\sqrt{3}-2,1\right)$ is isosceles. Since $\angle AOB = \frac{\pi}{12}$, then $\angle ABO = \frac{\pi}{12}$ and so $\angle BAO = \frac{5\pi}{6}$.

$$\text{area of } \Delta ABO = \frac{1}{2}\times 2\times 2\times \sin\left(\frac{5\pi}{6}\right) \qquad \text{(formula sheet)}$$

$$= 1$$

$$\text{area of minor sector } BAO = \left(\frac{5\pi}{6} \div 2\pi\right)\times \pi \times 2^2$$

$$= \frac{5\pi}{3}$$ **(1 mark)**

$$\text{area of minor segment} = \text{area of minor sector } BAO - \text{area of } \Delta ABO$$

$$= \frac{5\pi}{3} - 1 \text{ square units}$$

**(1 mark)**

**f.** Using the diagram from part **d.**, the ray intersects the circle for

$$-\pi < \text{Arg}(z) < \frac{-\pi}{2} \quad \text{and} \quad \frac{\pi}{2} < \text{Arg}(z) \le \pi$$

i.e. $-\pi < \frac{\alpha\pi}{2} < \frac{-\pi}{2}$ and $\frac{\pi}{2} < \frac{\alpha\pi}{2} \le \pi$ **(1 mark)**

So $-2 < \alpha < -1$ and $1 < \alpha \le 2$

Alternatively, we can express this as $\alpha \in (-2,-1) \cup (1,2]$. **(1 mark)**

**Question 3** (10 marks)

**a.** Method 1

At $t=0$, $\underset{\sim}{r}_A = 0\underset{\sim}{i} + \underset{\sim}{j}$

$\underset{\sim}{r}_B = 0\underset{\sim}{i} + 2\underset{\sim}{j}$

The distance between the starting positions is 1 metre.

**(1 mark)**

Method 2

$$|\underset{\sim}{r}_A - \underset{\sim}{r}_B| = \left|\left(\sqrt{3}t - 2\sin(t)\right)\underset{\sim}{i} + (3t+1-2\cos(t))\underset{\sim}{j}\right|$$

$$= \sqrt{(\sqrt{3}t - 2\sin(t))^2 + (3t+1-2\cos(t))^2}$$

At $t=0$, we have $\sqrt{0+(1-2)^2}$

$= \sqrt{1}$

$= 1$

The distance between the starting positions is 1 metre.

**(1 mark)**

**b.** For body $A$, $x = \sqrt{3}t$ and $y = 3t+1$

$t = \dfrac{x}{\sqrt{3}}$ $\qquad t = \dfrac{y-1}{3}$

So $\dfrac{y-1}{3} = \dfrac{x}{\sqrt{3}}$

$y = \sqrt{3}x + 1$ is the Cartesian equation of the path of body $A$, $x \ge 0$ since $t \ge 0$.

The starting point for body $A$ is at (0, 1).

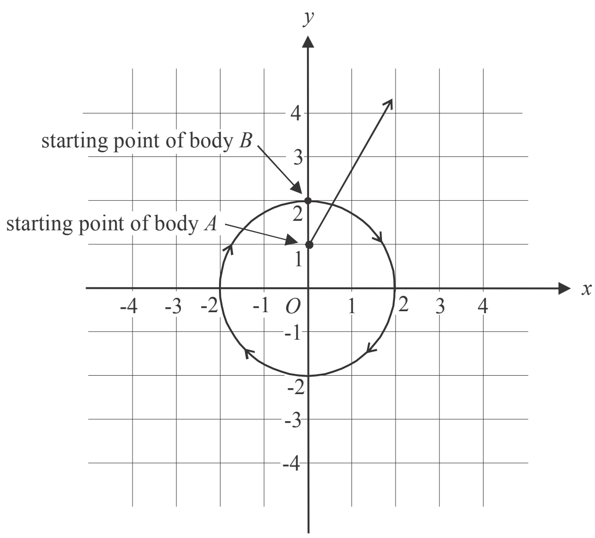
For body $B$, $x = 2\sin(t)$ and $y = 2\cos(t)$

$x^2 = 4\sin^2(t)$ $\qquad y^2 = 4\cos^2(t)$

$x^2 + y^2 = 4$ is the Cartesian equation of the path of body $B$.

The starting point for body $B$ is $(0,2)$.

At $t = \dfrac{\pi}{2}$, $\underset{\sim}{r}_B = 2\underset{\sim}{i} + 0\underset{\sim}{j}$ so body $B$ starts from the point (0, 2) and moves in a clockwise direction.




**(1 mark)** – correct path for body $A$
**(1 mark)** – correct path for body $B$
**(1 mark)** – correct starting points and directions

**c.** The paths of the two bodies are at right angles at $t=\frac{\pi}{6}$ if $\underset{\sim}{v}_A\left(\frac{\pi}{6}\right)\bullet\underset{\sim}{v}_B\left(\frac{\pi}{6}\right)=0$.

Now $\underset{\sim}{v}_A=\sqrt{3}\,\underset{\sim}{i}+3\,\underset{\sim}{j}$

$\underset{\sim}{v}_B=2\cos(t)\,\underset{\sim}{i}-2\sin(t)\,\underset{\sim}{j}$

and $\underset{\sim}{v}_A\left(\frac{\pi}{6}\right)=\sqrt{3}\,\underset{\sim}{i}+3\,\underset{\sim}{j}$ **(1 mark)**

$\underset{\sim}{v}_B\left(\frac{\pi}{6}\right)=\sqrt{3}\,\underset{\sim}{i}-\underset{\sim}{j}$

So $\underset{\sim}{v}_A\left(\frac{\pi}{6}\right)\bullet\underset{\sim}{v}_B\left(\frac{\pi}{6}\right)=3-3$

$=0$ as required

**(1 mark)**

**d.** Using working from part **c.**,

$\underset{\sim}{v}_A=\sqrt{3}\,\underset{\sim}{i}+3\,\underset{\sim}{j}$

$|\underset{\sim}{v}_A|=\sqrt{3+9}$

$=2\sqrt{3}$

$\underset{\sim}{v}_B=2\cos(t)\,\underset{\sim}{i}-2\sin(t)\,\underset{\sim}{j}$

$|\underset{\sim}{v}_B|=\sqrt{4\cos^2(t)+4\sin^2(t)}$

$=2$

Ratio is $2\sqrt{3}:2$

$\sqrt{3}:1$

**(1 mark)** – correct speed of the bodies
**(1 mark)** – correct answer

**e.** Method 1

Body $C$ sits at the point $(0,\sqrt{5})$ ie it is stationary. Using the graph from part **b.**, we see that the minimum distance between body $B$ and body $C$ occurs when body $B$ is located at $(0,2)$. So the minimum distance is $\sqrt{5}-2$ metres. **(1 mark)**

Since the starting point for body $B$ is $(0,2)$, (again from the graph) and since $\underset{\sim}{r}_B=2\sin(t)\underset{\sim}{i}+2\cos(t)\underset{\sim}{j}$, the minimum distance occurs when

$t=0,2\pi,4\pi,...$ i.e. when $t=2k\pi,\ k\in Z^+\cup\{0\}$. **(1 mark)**

Method 2

Distance between body $B$ and body $C$ is given by

$|\underset{\sim}{r}_B-\underset{\sim}{r}_C|=\left|2\sin(t)\,\underset{\sim}{i}+\left(2\cos(t)-\sqrt{5}\right)\underset{\sim}{j}\right|$

$=\sqrt{4\sin^2(t)+\left(2\cos(t)-\sqrt{5}\right)^2}$

$=\sqrt{9-4\sqrt{5}\cos(t)}$

Distance will be a minimum when $\cos(t)=1$ i.e. when $t=2k\pi,\ k\in Z^+\cup\{0\}$.

**(1 mark)**

Minimum distance $=\sqrt{9-4\sqrt{5}}$

$=\sqrt{5}-2$ metres **(1 mark)**

**Question 4** (8 marks)

**a.**

$$\frac{dy}{dx} = e^{a(x-y)}$$
$$= e^{ax-ay}$$
$$= e^{ax} \times e^{-ay}$$
$$= e^{ax} \times \frac{1}{e^{ay}}$$

So $f(x) = e^{ax}$

and $g(y) = \frac{1}{e^{ay}}$

**(1 mark)**

**b.**

$$\frac{dy}{dx} = e^{ax} \times \frac{1}{e^{ay}} \qquad \text{(from part } \textbf{a.}\text{)}$$

$$\int e^{ay}\, dy = \int e^{ax}\, dx \qquad \text{(separation of variables)}$$

**(1 mark)**

$$\frac{1}{a}e^{ay} + c_1 = \frac{1}{a}e^{ax} + c_2$$

$$\frac{1}{a}e^{ay} = \frac{1}{a}e^{ax} + c \quad \text{where } c = c_2 - c_1$$

When $x = 1$, $y = 0$,

$$\frac{1}{a} = \frac{1}{a}e^{a} + c$$

$$c = \frac{1}{a}(1 - e^{a})$$

**(1 mark)**

So $\frac{1}{a}e^{ay} = \frac{1}{a}e^{ax} + \frac{1}{a}(1 - e^{a})$

$$e^{ay} = e^{ax} + 1 - e^{a}$$

$$ay = \log_e(e^{ax} + 1 - e^{a})$$

$$y = \frac{1}{a}\log_e(e^{ax} + 1 - e^{a})$$

**(1 mark)**

**c.** **i.** $y=\log_e\sqrt{f(x)+1-e^2}$

$=\log_e\sqrt{e^{ax}+1-e^2}$ (using our result from part **a.**)

$\frac{dy}{dx}=\frac{ae^{ax}}{2(e^{ax}-e^2+1)}$ (using CAS)

**(1 mark)**

**ii.** $y=\log_e\sqrt{f(x)+1-e^2}$

When $a=2$ and using $f(x)=e^{ax}$ from part **a.**, we have

$y=\log_e\sqrt{e^{2x}+1-e^2}$

Also, $\frac{dy}{dx}=\frac{2e^{2x}}{2(e^{2x}-e^2+1)}$ (from part **c.i.**)

$=\frac{e^{2x}}{e^{2x}-e^2+1}$

For the differential equation $\frac{dy}{dx}=e^{a(x-y)}$, when $a=2$ we have

$\frac{dy}{dx}=e^{2(x-y)}$

$LS=\frac{dy}{dx}$

$=\frac{e^{2x}}{e^{2x}-e^2+1}$

$RS=e^{2x-2y}$

$=e^{2x}\times e^{-2y}$ **(1 mark)**

$=e^{2x}\times e^{-2\log_e\sqrt{e^{2x}+1-e^2}}$

$=e^{2x}\times e^{\log_e\frac{1}{(e^{2x}+1-e^2)}}$

$=\frac{e^{2x}}{e^{2x}+1-e^2}$ **(1 mark)**

$=LS$

Have verified.

Check the initial condition.
i.e. when $x=1, y=0$.

$y=\log_e\sqrt{e^{2x}+1-e^2}$

$RS=\log_e\sqrt{e^2+1-e^2}$

$=\log_e\sqrt{1}$

$=0$

$=LS$

Have verified.

**(1 mark)**

**Question 5** (9 marks)

**a.** We're told $\mu = 6$ and $\sigma = 0.8$.

$$\text{So } \mathrm{E}(\overline{X}) = 6 \text{ and } \mathrm{sd}(\overline{X}) = \frac{0.8}{\sqrt{100}} = 0.08$$

**(1 mark)** – correct mean
**(1 mark)** – correct standard deviation

**b.** **i.** $H_0 : \mu = 6$ **(1 mark)**
$H_1 : \mu > 6$ **(1 mark)**

**ii.** $p = \Pr(\overline{X} > 6.2 | \mu = 6)$ **(1 mark)**

$$= \Pr\left(Z > \frac{6.2 - 6}{0.08}\right)$$
$$= \Pr(Z > 2.5)$$
$$= 0.006209...$$
$$= 0.0062 \text{ (correct to 4 decimal places)}$$

**(1 mark)**

**iii.** Since $p < 0.05$, the sample does provide evidence against $H_0$ and therefore it does provide evidence that the mean amount of sugar is higher than the manufacturer claims.

**(1 mark)**

**c.** For the first test $p = 0.0062$ (from part **b. ii.)**
For the second test $p = 0.0132$ (given).
Since $0.0132 > 0.0062$, this second test provides weaker evidence than the first that the mean amount of sugar is higher than the manufacturer claims.

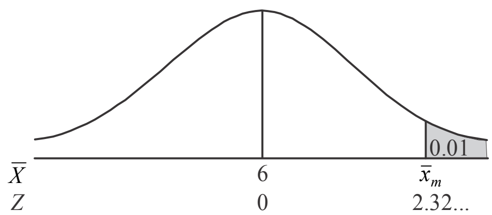
**(1 mark)**

**d.** $\Pr(\overline{X} > \overline{x}_m | \mu = 6) = 0.01$

$$\Pr\left(Z > \frac{\overline{x}_m - 6}{0.08}\right) = 0.01$$

Using invNorm function, we have $z = 2.3263...$

$$\text{So } \frac{\overline{x}_m - 6}{0.08} = 2.3263...$$
$$\overline{x}_m = 6.186107...$$
$$= 6.1861 \text{ (correct to 4 decimal places)}$$




**(1 mark)**

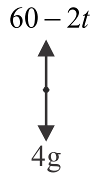
**Question 6** (10 marks)

**a.** Taking upwards as positive, we have,

$(60-2t)-4g=4a$ (equation of motion)

$$a=\frac{60-2t-4g}{4}$$

$$a=\frac{26}{5}-\frac{t}{2} \text{ as required}$$




**(1 mark)**

**b.** $\frac{dv}{dt}=\frac{26}{5}-\frac{t}{2}$ (from part **a.**)

$$v=\int\left(\frac{26}{5}-\frac{t}{2}\right)dt$$

$$v=\frac{26t}{5}-\frac{t^2}{4}+c$$

When $t=0, v=0$ and so $c=0$ .

$$v=\frac{26t}{5}-\frac{t^2}{4}$$

**(1 mark)**

When $t=10$,

$v=27\,\text{ms}^{-1}$

**(1 mark)**

**c.** From $t=10$ onwards, the drone is subject only to gravity and hence acceleration is constant.

Method 1

Taking upwards as positive, we have

$$-4g=4a$$

$$a=-g$$

$$\frac{dv}{dt}=-g$$

$$v=\int -g\,dt$$

$$v=-gt+c$$


When $t=0, v=27$ so $c=27$ (from part **b.**)

$$v=-gt+27$$

**(1 mark)**

At maximum height, $v=0$

$$0=-gt+27$$

So at $t=\frac{27}{g}$ the drone is at its maximum height. **(1 mark)**

$$x=\int_0^{\frac{27}{g}}(-gt+27)\,dt$$

$$=37.193877...$$

The drone reaches 37.19 m higher (correct to two decimal places) before falling to the ground.

**(1 mark)**

Method 2

Because we have constant acceleration we can use the constant acceleration formulae.

At $t = 10$, which is the start of the next phase of motion for the drone, $u = 27\,\text{ms}^{-1}$ from part **a.**

So $u = 27$

$v = 0$ at maximum height **(1 mark)**

$a = -9.8$ (taking upward as positive)

Using $v^2 = u^2 + 2as$

$0 = 27^2 - 2 \times 9.8s$ **(1 mark)**

$s = 37.193877...$

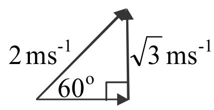
The drone reaches 37.19 m higher (correct to two decimal places) before falling to the ground.

**(1 mark)**

**d.** When the drone's propulsion system fails it is travelling at $2\,\text{ms}^{-1}$ at an angle of 60° with the ground

$$\sin 60^\circ = \frac{\text{opp}}{2}$$

$$\text{opp} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$




So the magnitude of the vertical component of the drone's velocity when its propulsion system fails is $\sqrt{3}\,\text{ms}^{-1}$ and the direction is vertically up.

**(1 mark)**

**e.** Method 1

Considering the vertical component of the drone's motion, and taking the upward direction as positive, we have

$a = -g$ (equation of motion)

$v = \int -g\,dt$

$v = -gt + c$

$t = 0,\ v = \sqrt{3},$ so $c = \sqrt{3}$ (using part **d**.)

$v = -gt + \sqrt{3}$ **(1 mark)**

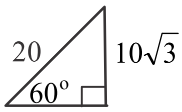
$x = \int \left(-gt + \sqrt{3}\right) dt$

$x = \frac{-gt^2}{2} + \sqrt{3}t + c_1$

The propulsion system fails when the drone is $10\sqrt{3}$ m vertically above the ground.

ie. $\sin(60^\circ) = \frac{\text{opp}}{20}$

$\text{opp} = 10\sqrt{3}$




When $t = 0,\ x = 10\sqrt{3}$ so $c_1 = 10\sqrt{3}$

So $x = \frac{-gt^2}{2} + \sqrt{3}t + 10\sqrt{3}$. **(1 mark)**

When $x = 0$, the drone hits the ground.

Solve $x(t) = 0$ for $t$ using CAS.

$t = -1.711654...$ or $t = 2.065134...$

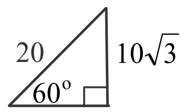
Since $t \geq 0,\ t = 2.065$ (correct to three decimal places) so the drone hits the ground 2.065 seconds after the propulsion system fails. **(1 mark)**

Method 2

Since the drone is subject only to gravity, its acceleration is constant. Take upward direction as positive and consider just the vertical component of the drone's motion.
The propulsion system fails when the drone is $10\sqrt{3}$ m above the ground

i.e. $\sin(60^\circ) = \dfrac{\text{opp}}{20}$

$\text{opp} = 10\sqrt{3}$




When system fails, let $s = 0$, so the drone hits the ground when $s = -10\sqrt{3}$ **(1 mark)**

$u = \sqrt{3}$ (from part **d.**)
$a = -9.8$
$s = -10\sqrt{3}$

$s = ut + \frac{1}{2}at^2$ **(1 mark)**

$-10\sqrt{3} = \sqrt{3}t + \frac{1}{2} \times -9.8t^2$

Solve for $t$, $t = -1.712$ or $t = 2.065$ (to 3 dec. places)

Since $t > 0$, it takes 2.065 seconds. **(1 mark)**

Method 3

Since the drone is subject only to gravity, its acceleration is constant. Take upward direction as positive and consider just the vertical component of the drone's motion.

From when the propulsion system fails until the maximum height is reached,

$u = \sqrt{3}$ (from part **d.**)
$v = 0$ (at max. height)
$a = -9.8$

$v = u + at$

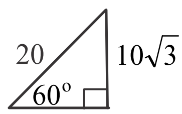
$0 = \sqrt{3} - 9.8t$

$t = \dfrac{\sqrt{3}}{9.8}$ **(1 mark)**

The propulsion system fails when the drone is $10\sqrt{3}$ m above the ground

i.e. $\sin(60^\circ) = \dfrac{\text{opp}}{20}$

$\text{opp} = 10\sqrt{3}$




At this same time the velocity is $\sqrt{3}\,\text{ms}^{-1}$ (from part **d.**).
So between when the propulsion system fails and the maximum height is reached,

$u = \sqrt{3}$
$t = \dfrac{\sqrt{3}}{9.8}$
$a = -9.8$

$s = ut + \frac{1}{2}at^2$

$s = \sqrt{3} \times \dfrac{\sqrt{3}}{9.8} - 4.9 \times \dfrac{3}{9.8^2}$

$= 0.153061...$

The maximum height reached above the ground is

$10\sqrt{3} + 0.153061...$

$= 17.473569...\text{m}$ **(1 mark)**

Between when the maximum height is reached and when the drone hits the ground,

$s = -17.473569...$
$u = 0$
$a = -9.8$

$s = ut + \frac{1}{2}at^2$

$-17.473569... = -4.9t^2$

$t = 1.888394...\text{secs.}$

So the total time between when the propulsion system fails and when the drone hits the ground is $\dfrac{\sqrt{3}}{9.8} + 1.888394... = 2.065134...$

$= 2.065$ seconds (to three decimal places) **(1 mark)**