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SPECIALIST MATHS TRIAL EXAMINATION 1 SOLUTIONS 2017

Question 1 (3 marks)

$$\int_2^3 (x+1)\sqrt{3-x}\,dx$$

$$= \int_1^0 (4-u)\sqrt{u} \times -1\frac{du}{dx}\,dx$$

Let $u = 3 - x$

$\frac{du}{dx} = -1$

So $x = 3 - u$

and $x + 1 = 4 - u$

Also, $x = 3, u = 0$

and $x = 2, u = 1$

$$= \int_0^1 \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$ **(1 mark)**

$$= \left[4u^{\frac{3}{2}} \times \frac{2}{3} - u^{\frac{5}{2}} \times \frac{2}{5}\right]_0^1$$ **(1 mark)**

$$= \left\{\left(4 \times \frac{2}{3} - \frac{2}{5}\right) - (0 - 0)\right\}$$

$$= \frac{8}{3} - \frac{2}{5}$$

$$= \frac{40-6}{15}$$

$$= \frac{34}{15}$$

(1 mark)

Question 2 (3 marks)

$s = \sqrt{4} = 2$

sample size $= 100$

sample mean $\bar{x} = \frac{2\,150}{100} = 21.5$ mL **(1 mark)**

$$95\% \text{ confidence interval} \approx \left(\bar{x} - z\frac{s}{\sqrt{n}},\ \ \bar{x} + z\frac{s}{\sqrt{n}}\right) \quad \text{(formula sheet)}$$

$$= \left(21.5 - 2 \times \frac{2}{10},\ 21.5 + 2 \times \frac{2}{10}\right)$$ **(1 mark)**

$$= (21.5 - 0.4,\ 21.5 + 0.4)$$

$$= (21.1,\ 21.9)$$

(1 mark)

Question 3 (4 marks)

a. $\overrightarrow{OA} = \underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OB} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k} + 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\\ &= \underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}\end{aligned}$$

(1 mark)

b.
$$\begin{aligned}\cos(\theta) &= \frac{\overrightarrow{AO} \bullet \overrightarrow{AB}}{|\overrightarrow{AO}|\,|\overrightarrow{AB}|}\\ &= \frac{15}{3\sqrt{35}}\\ &= \frac{5}{\sqrt{35}}\end{aligned}$$

(1 mark)

c. Vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ are linearly dependent if $\alpha\overrightarrow{OA} + \beta\overrightarrow{OB} = \overrightarrow{OC}$ where α and β are real numbers.
We require
$\alpha(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}) + \beta(2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}) = m\underset{\sim}{j} + 7\underset{\sim}{k}$ **(1 mark)**

So $\alpha + 2\beta = 0$

$\alpha = -2\beta \quad -(1)$

and $-2\alpha + \beta = m \quad -(2)$

and $2\alpha - 3\beta = 7 \quad -(3)$

Put (1) into (3)

$-4\beta - 3\beta = 7$

$\beta = -1$

In (1) $\alpha = 2$

In (2) $-4 - 1 = m$

$m = -5$

(1 mark)

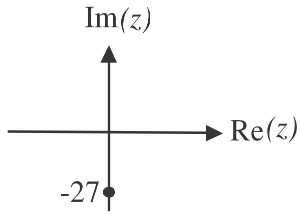
Question 4 (3 marks)

$$z^3 = -27i$$

Let $z = r\text{cis}(\theta)$

So $z^3 = r^3\text{cis}(3\theta)$

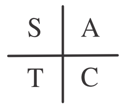
Also $-27i = 27\text{cis}\left(-\frac{\pi}{2}\right)$ **(1 mark)**



So $r^3\text{cis}(3\theta) = 27\text{cis}\left(-\frac{\pi}{2}\right)$

$r^3 = 27, \quad r = 3$

$3\theta = -\frac{\pi}{2} + 2k\pi, \qquad k \in Z$



$\theta = -\frac{\pi}{6} + \frac{2k\pi}{3}$ **(1 mark)**

For $k = 0$, $\theta = -\frac{\pi}{6}$

So $z = 3\text{cis}\left(-\frac{\pi}{6}\right)$

$= 3\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

For $k = 1$, $\theta = -\frac{\pi}{6} + \frac{2\pi}{3}$

$= \frac{\pi}{2}$

So $z = 3\text{cis}\left(\frac{\pi}{2}\right)$

$= 3(0 + i)$

$= 3i$

For $k = 2$, $\theta = -\frac{\pi}{6} + \frac{4\pi}{3}$

$= \frac{7\pi}{6}$

So $z = 3\text{cis}\left(\frac{7\pi}{6}\right)$

$= 3\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$= -3\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

The three solutions are $z = 3\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$, $z = 3i$ and $z = \; = -3\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$. **(1 mark)**

Question 5 (3 marks)

a. $|\underset{\sim}{b}| = \sqrt{4+4+1} = 3$

$\hat{\underset{\sim}{b}} = \frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$ **(1 mark)**

b. scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$

$= \underset{\sim}{a} \bullet \hat{\underset{\sim}{b}}$

$= \frac{1}{3}(5\times 2 + 6\times 2 + 4\times -1)$ (using part **a.**)

$= 6$ (Note, the answer should be a scalar!) **(1 mark)**

c. vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$

$= \underset{\sim}{a} - (\underset{\sim}{a} \bullet \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$

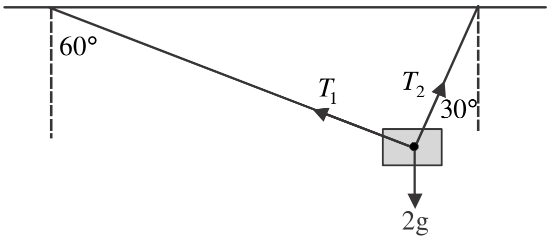
$= 5\underset{\sim}{i} + 6\underset{\sim}{j} + 4\underset{\sim}{k} - 6\times\frac{1}{3}(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$ (from parts **a.** and **b.**)

$= 5\underset{\sim}{i} + 6\underset{\sim}{j} + 4\underset{\sim}{k} - 4\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$= \underset{\sim}{i} + 2\underset{\sim}{j} + 6\underset{\sim}{k}$ (Note, the answer should be a vector!) **(1 mark)**

Question 6 (4 marks)

a.



(1 mark)

b. Resolving horizontally:

$$T_1\sin(60^\circ) = T_2\sin(30^\circ)$$
$$\frac{\sqrt{3}}{2}T_1 = \frac{1}{2}T_2$$
$$T_2 = \sqrt{3}T_1 \qquad -(A)$$

(1 mark)

Resolving vertically:

$$T_1\cos(60^\circ) + T_2\cos(30^\circ) = 2g$$

(1 mark)

$$\frac{1}{2}T_1 + \frac{\sqrt{3}}{2}T_2 = 2g$$
$$\frac{1}{2}T_1 + \frac{3}{2}T_1 = 2g$$
$$T_1 = g$$
$$\text{In } (A) \quad T_2 = \sqrt{3}g$$

(1 mark)

Question 7 (4 marks)

$$\arccos(x) + y\arcsin(x) = \frac{y}{2}$$
$$\frac{-1}{\sqrt{1-x^2}} + \frac{dy}{dx}\arcsin(x) + \frac{y}{\sqrt{1-x^2}} = \frac{1}{2}\frac{dy}{dx}$$

(1 mark)

At the point $(0, \pi)$, we have

$$-1 + \frac{dy}{dx}\times 0 + \pi = \frac{1}{2}\frac{dy}{dx}$$
$$\frac{dy}{dx} = 2(\pi - 1)$$

(1 mark)

Gradient of perpendicular line is therefore $\frac{-1}{2(\pi-1)}$ or $\frac{1}{2(1-\pi)}$. **(1 mark)**

Required equation is

$$y - \pi = \frac{1}{2(1-\pi)}x$$
$$y = \frac{x}{2(1-\pi)} + \pi$$

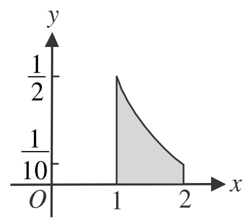
(1 mark)

Question 8 (4 marks)

Do a quick sketch of $y=\dfrac{1}{x(x^2+1)}$
in the vicinity of $x=1$ and $x=2$.

When $x=1$, $y=\dfrac{1}{2}$

When $x=2$, $y=\dfrac{1}{10}$



We know that between $x=1$ and $x=2$, the graph is above the x-axis.

$$\text{area}=\int_1^2 \frac{1}{x(x^2+1)}dx$$ **(1 mark)**

$$\text{Let } \frac{1}{x(x^2+1)} \equiv \frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$=\frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

True iff $\quad 1\equiv A(x^2+1)+(Bx+C)x$

Put $x=0$, $\quad 1=A \quad$ so $\quad A=1$

Put $x=1$, $\quad 1=2+B+C$

$B+C=-1 \qquad -(1)$

Put $x=-1$, $\quad 1=2+B-C$

$B-C=-1 \qquad -(2)$

(1)+(2) $\quad 2B=-2$

$B=-1$

In (1) $\qquad C=0$

$$\text{So area}=\int_1^2\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$$ **(1 mark)**

$$=\left[\log_e|x|-\frac{1}{2}\log_e|x^2+1|\right]_1^2$$ **(1 mark)**

$$=\left(\log_e(2)-\frac{1}{2}\log_e(5)\right)-\left(\log_e(1)-\frac{1}{2}\log_e(2)\right)$$

$$=\log_e(2)-\log_e\left(\sqrt{5}\right)+\log_e\left(\sqrt{2}\right)$$

$$=\log_e\left(\frac{2\sqrt{2}}{\sqrt{5}}\right)$$

$$=\log_e\left(\frac{2\sqrt{10}}{5}\right) \text{ square units.}$$

(1 mark)

Question 9 (5 marks)

$$\frac{\sqrt{x^2-1}}{x}\frac{dy}{dx}=4+y^2$$

$$\int\frac{1}{4+y^2}dy=\int\frac{x}{\sqrt{x^2-1}}dx \qquad \text{(separation of variables)}$$ **(1 mark)**

$$\frac{1}{2}\int\frac{2}{4+y^2}dy=\int u^{-\frac{1}{2}}\times\frac{1}{2}\frac{du}{dx}dx$$

where $u=x^2-1$

$$\frac{du}{dx}=2x$$

$$\frac{1}{2}\arctan\left(\frac{y}{2}\right)+c_1=\frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$\frac{1}{2}\arctan\left(\frac{y}{2}\right)+c_1=\frac{1}{2}u^{\frac{1}{2}}\times 2+c_2$$

$$\frac{1}{2}\arctan\left(\frac{y}{2}\right)=\sqrt{x^2-1}+c \quad \text{where } c=c_2-c_1$$

(1 mark) – left side
(1 mark) – right side

Given $y(1)=2$,

$$\frac{1}{2}\arctan(1)=c$$

$$c=\frac{1}{2}\times\frac{\pi}{4}$$

$$c=\frac{\pi}{8}$$ **(1 mark)**

So $\frac{1}{2}\arctan\left(\frac{y}{2}\right)=\sqrt{x^2-1}+\frac{\pi}{8}$.

$$\arctan\left(\frac{y}{2}\right)=2\sqrt{x^2-1}+\frac{\pi}{4}$$

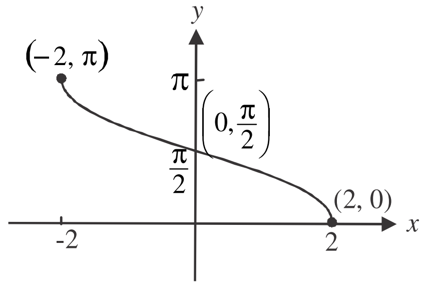
$$\tan\left(2\sqrt{x^2-1}+\frac{\pi}{4}\right)=\frac{y}{2}$$

The solution is $y=2\tan\left(2\sqrt{x^2-1}+\frac{\pi}{4}\right)$.

(1 mark)

Question 10 (7 marks)

a. The graph of $y = \arccos\left(\frac{x}{2}\right)$ is obtained when the graph of $y = \arccos(x)$ is dilated by a factor of 2 from the y-axis.



(1 mark) – correct shape
(1 mark) – correctly labelled intercept and endpoints

b. **i.** $f(x) = \arccos\left(\frac{x}{2}\right)$

Let $y = \arccos\left(\frac{x}{2}\right)$

Swap x and y for inverse

$$x = \arccos\left(\frac{y}{2}\right)$$

$$\cos(x) = \frac{y}{2}$$

$$y = 2\cos(x)$$

So $f^{-1}(x) = 2\cos(x)$ as required.

(1 mark)

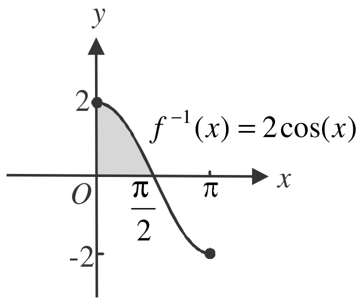
ii. From the graph in part **a.**, $r_f = [0,\pi]$.

Since $d_{f^{-1}} = r_f$

then $d_{f^{-1}} = [0,\pi]$.

(1 mark)

c. Do a quick sketch.
The region to be rotated is shaded.



$$\text{volume} = \int_0^{\frac{\pi}{2}} \pi y^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} 4\cos^2(x)\,dx \qquad \textbf{(1 mark)}$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos(2x) + 1)\,dx$$

$$= 2\pi\left[\frac{1}{2}\sin(2x) + x\right]_0^{\frac{\pi}{2}} \qquad \textbf{(1 mark)}$$

$$= 2\pi\left\{\left(\frac{1}{2}\times 0 + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin(0) + 0\right)\right\}$$

$$= 2\pi \times \frac{\pi}{2}$$

$$= \pi^2 \text{ cubic units} \qquad \textbf{(1 mark)}$$