VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
Year

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER | | | | | | | | | Letter | |

# SPECIALIST MATHEMATICS

## Written examination 2

**Day Date**

**Reading time: *.** to *.** (15 minutes)**
**Writing time: *.** to *.** (2 hours)**

## QUESTION AND ANSWER BOOK

**Structure of book**

| *Section* | *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|---|
| A | 20 | 20 | 20 |
| B | 6 | 6 | 60 |
| | | | Total 80 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

**Materials supplied**
- Question and answer book of 23 pages.
- Formula sheet.
- Answer sheet for multiple-choice questions.

**Instructions**
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

**At the end of the examination**
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.**



Version 2 – April 2016

## SECTION A – Multiple-choice questions

### Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1; an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Take the **acceleration due to gravity** to have magnitude $g$ ms$^{-2}$, where $g = 9.8$

**Question 1**

A circle with centre $(a, -2)$ and radius 5 units has equation $x^2 - 6x + y^2 + 4y = b$, where $a$ and $b$ are real constants.

The values of $a$ and $b$ are respectively

**A.** –3 and 38

**B.** 3 and 12

**C.** –3 and –8

**D.** –3 and 0

**E.** 3 and 18

**Question 2**

The maximal domain and range of the function with rule $f(x) = 3\sin^{-1}(4x-1) + \dfrac{\pi}{2}$ are respectively

**A.** $[-\pi, 2\pi]$ and $\left[0, \dfrac{1}{2}\right]$

**B.** $\left[0, \dfrac{1}{2}\right]$ and $[-\pi, 2\pi]$

**C.** $\left[-\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right]$ and $\left[-\dfrac{1}{2}, 0\right]$

**D.** $\left[0, \dfrac{1}{2}\right]$ and $[0, 3\pi]$

**E.** $\left[-\dfrac{1}{2}, 0\right]$ and $[-\pi, 2\pi]$

**Question 3**

The features of the graph of the function with rule $f(x)=\dfrac{x^2-4x+3}{x^2-x-6}$ include

**A.** asymptotes at $x = 1$ and $x = -2$

**B.** asymptotes at $x = 3$ and $x = -2$

**C.** an asymptote at $x = 1$ and a point of discontinuity at $x = 3$

**D.** an asymptote at $x = -2$ and a point of discontinuity at $x = 3$

**E.** an asymptote at $x = 3$ and a point of discontinuity at $x = -2$

**Question 4**

The algebraic fraction $\dfrac{7x-5}{(x-4)^2(x^2+9)}$ could be expressed in partial fraction form as

**A.** $\dfrac{A}{(x-4)^2}+\dfrac{B}{x^2+9}$

**B.** $\dfrac{A}{x-4}+\dfrac{B}{x-3}+\dfrac{C}{x+3}$

**C.** $\dfrac{A}{(x-4)^2}+\dfrac{Bx+C}{x^2+9}$

**D.** $\dfrac{A}{x-4}+\dfrac{B}{(x-4)^2}+\dfrac{Cx+D}{x^2+9}$

**E.** $\dfrac{A}{x-4}+\dfrac{B}{(x-4)^2}+\dfrac{C}{x^2+9}$

**Question 5**

On an Argand diagram, a set of points that lies on a circle of radius 2 centred at the origin is

**A.** $\{z \in C : z\bar{z} = 2\}$

**B.** $\{z \in C : z^2 = 4\}$

**C.** $\{z \in C : \text{Re}(z^2) + \text{Im}(z^2) = 4\}$

**D.** $\{z \in C : (z+\bar{z})^2 - (z-\bar{z})^2 = 16\}$

**E.** $\{z \in C : (\text{Re}(z))^2 + (\text{Im}(z))^2 = 16\}$

**Question 6**

The polynomial $P(z)$ has real coefficients. Four of the roots of the equation $P(z) = 0$ are $z = 0$, $z = 1 - 2i$, $z = 1 + 2i$ and $z = 3i$.

The **minimum** number of roots that the equation $P(z) = 0$ could have is

**A.** 4

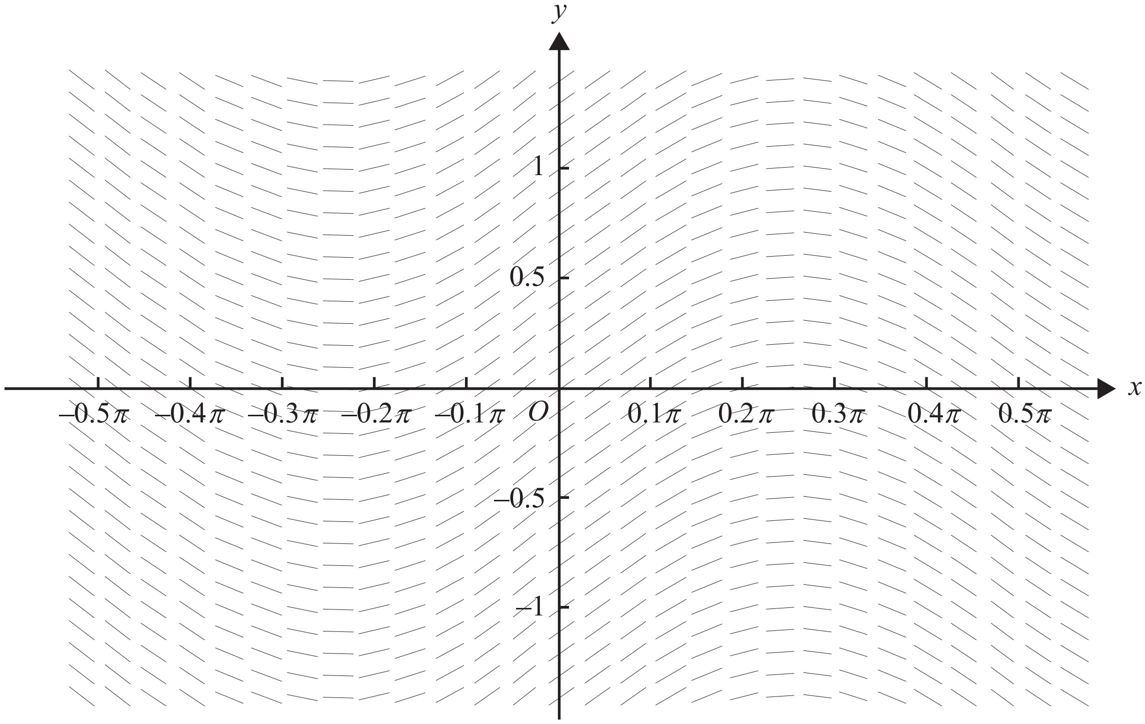
**B.** 5

**C.** 6

**D.** 7

**E.** 8

**Question 7**




The direction (slope) field for a certain first-order differential equation is shown above.
The differential equation could be

**A.** $\frac{dy}{dx} = \sin(2x)$

**B.** $\frac{dy}{dx} = \cos(2x)$

**C.** $\frac{dy}{dx} = \cos\left(\frac{1}{2}y\right)$

**D.** $\frac{dy}{dx} = \sin\left(\frac{1}{2}y\right)$

**E.** $\frac{dy}{dx} = \cos\left(\frac{1}{2}x\right)$

**Question 8**

Let $f:[-\pi, 2\pi] \to R$, where $f(x) = \sin^3(x)$.

Using the substitution $u = \cos(x)$, the area bounded by the graph of $f$ and the $x$-axis could be found by evaluating

**A.** $-\int_{-\pi}^{2\pi} \left(1-u^2\right) du$

**B.** $3\int_{-1}^{1} \left(1-u^2\right) du$

**C.** $-3\int_{0}^{\pi} \left(1-u^2\right) du$

**D.** $3\int_{1}^{-1} \left(1-u^2\right) du$

**E.** $-\int_{-1}^{1} \left(1-u^2\right) du$

**Question 9**

Let $\frac{dy}{dx} = \frac{x+2}{x^2+2x+1}$ and $(x_0, y_0) = (0, 2)$.

Using Euler's method with a step size of 0.1, the value of $y_1$, correct to two decimal places, is

**A.** 0.17

**B.** 0.20

**C.** 1.70

**D.** 2.17

**E.** 2.20

**Question 10**

The curve given by $y = \sin^{-1}(2x)$, where $0 \le x \le \frac{1}{2}$, is rotated about the $y$-axis to form a solid of revolution.

The volume of the solid may be found by evaluating

**A.** $\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\left(1-\cos(2y)\right)dy$

**B.** $\frac{\pi}{8}\int_0^{\frac{1}{2}}\left(1-\cos(2y)\right)dy$

**C.** $\frac{\pi}{8}\int_0^{\frac{\pi}{2}}\left(1-\cos(2y)\right)dy$

**D.** $\frac{1}{8}\int_0^{\frac{\pi}{2}}\left(1-\cos(2y)\right)dy$

**E.** $\frac{\pi}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-\cos(2y)\right)dy$

**Question 11**

The angle between the vectors $3\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}$ and $2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$, correct to the nearest tenth of a degree, is

**A.** 2.0°
**B.** 91.0°
**C.** 112.4°
**D.** 121.3°
**E.** 124.9°

**Question 12**

The scalar resolute of $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$ is

**A.** $\frac{8}{\sqrt{10}}$

**B.** $\frac{8}{9}\left(2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

**C.** 8

**D.** $\frac{4}{5}\left(3\underset{\sim}{i} - \underset{\sim}{k}\right)$

**E.** $\frac{8}{3}$

**Question 13**

The position vector of a particle at time $t$ seconds, $t \geq 0$, is given by $\underset{\sim}{r}(t) = (3-t)\underset{\sim}{i} - 6\sqrt{t}\,\underset{\sim}{j} + 5\underset{\sim}{k}$.
The direction of motion of the particle when $t = 9$ is

**A.** $-6\underset{\sim}{i} - 18\underset{\sim}{j} + 5\underset{\sim}{k}$

**B.** $-\underset{\sim}{i} - \underset{\sim}{j}$

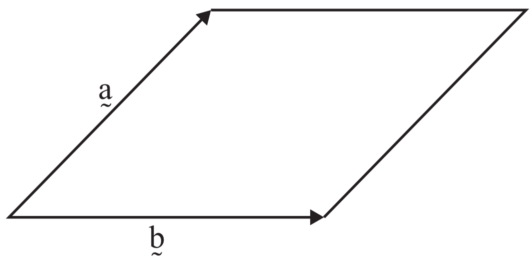
**C.** $-6\underset{\sim}{i} - \underset{\sim}{j}$

**D.** $-\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}$

**E.** $-13.5\underset{\sim}{i} - 108\underset{\sim}{j} + 45\underset{\sim}{k}$

**Question 14**

The diagram below shows a rhombus, spanned by the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.




It follows that

**A.** $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

**B.** $\underset{\sim}{a} = \underset{\sim}{b}$

**C.** $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

**D.** $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$

**E.** $2\underset{\sim}{a} + 2\underset{\sim}{b} = \underset{\sim}{0}$

**Question 15**

A 12 kg mass moves in a straight line under the action of a variable force $F$, so that its velocity $v$ ms$^{-1}$ when it is $x$ metres from the origin is given by $v = \sqrt{3x^2 - x^3 + 16}$.
The force $F$ acting on the mass is given by

**A.** $12\left(3x - \frac{3x^2}{2}\right)$

**B.** $12\left(3x^2 - x^3 + 16\right)$

**C.** $12\left(6x - 3x^2\right)$

**D.** $12\sqrt{3x^2 - x^3 + 16}$

**E.** $12(3 - 3x)$

**Question 16**

The acceleration, $a$ ms$^{-2}$, of a particle moving in a straight line is given by $a = \dfrac{v}{\log_e(v)}$, where $v$ is the velocity of the particle in ms$^{-1}$ at time $t$ seconds. The initial velocity of the particle was 5 ms$^{-1}$.

The velocity of the particle, in terms of $t$, is given by

**A.** $v = e^{2t}$

**B.** $v = e^{2t} + 4$

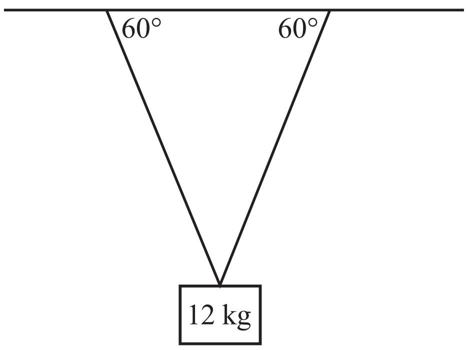
**C.** $v = e^{\sqrt{2t} + \log_e(5)}$

**D.** $v = e^{\sqrt{2t + (\log_e 5)^2}}$

**E.** $v = e^{-\sqrt{2t + (\log_e 5)^2}}$

**Question 17**

A 12 kg mass is suspended in equilibrium from a horizontal ceiling by two identical light strings. Each string makes an angle of 60° with the ceiling, as shown.




The magnitude, in newtons, of the tension in each string is equal to

**A.** $6\,g$

**B.** $12\,g$

**C.** $24\,g$

**D.** $4\sqrt{3}\,g$

**E.** $8\sqrt{3}\,g$

**Question 18**

Given that $X$ is a normal random variable with mean 10 and standard deviation 8, and that $Y$ is a normal random variable with mean 3 and standard deviation 2, the random variable $Z$ defined by $Z = X - 3Y$ will have mean $\mu$ and standard deviation $\sigma$ given by

**A.** $\mu = 1, \sigma = 28$

**B.** $\mu = 19, \sigma = 2$

**C.** $\mu = 1, \sigma = 2\sqrt{7}$

**D.** $\mu = 19, \sigma = 14$

**E.** $\mu = 1, \sigma = 10$

**Question 19**

The mean study score for a large VCE study is 30 with a standard deviation of 7. A class of 20 students may be considered as a random sample drawn from this cohort.

The probability that the class mean for the group of 20 exceeds 32 is

**A.** 0.1007

**B.** 0.3875

**C.** 0.3993

**D.** 0.6125

**E.** 0.8993

**Question 20**

A type I error would occur in a statistical test where

**A.** $H_0$ is accepted when $H_0$ is false.

**B.** $H_1$ is accepted when $H_1$ is true.

**C.** $H_0$ is rejected when $H_0$ is true.

**D.** $H_1$ is rejected when $H_1$ is true.

**E.** $H_0$ is accepted when $H_0$ is true.

**END OF SECTION A**

**TURN OVER**

## SECTION B

**Instructions for Section B**

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude $g$ ms$^{-2}$, where $g = 9.8$

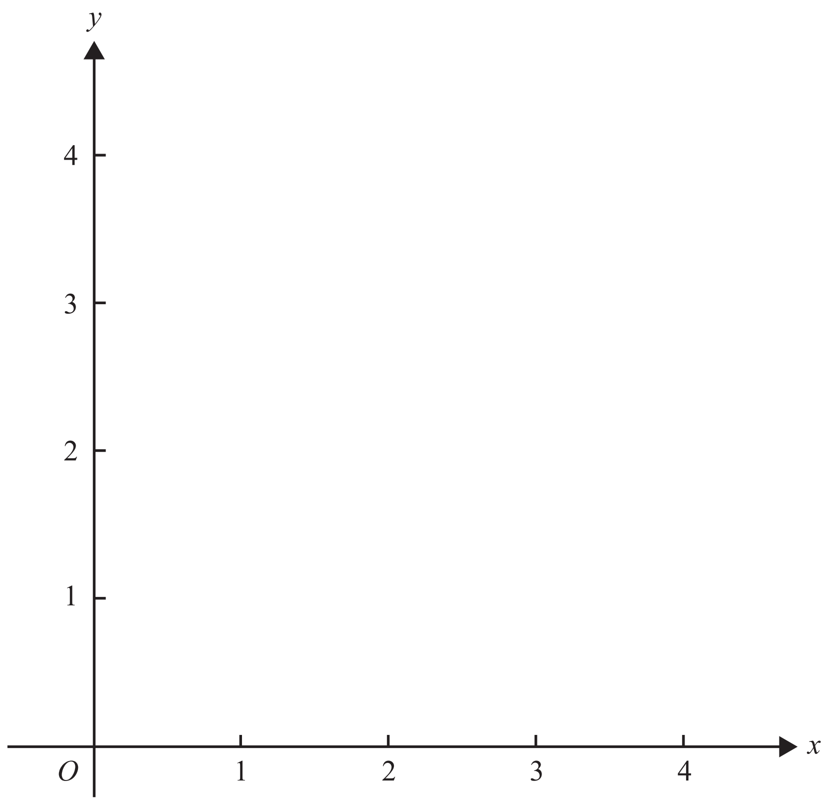
**Question 1** (10 marks)

Consider the function $f: [0, 3) \to R$, where $f(x) = -2 + 2\sec\left(\frac{\pi x}{6}\right)$.

**a.** Evaluate $f(2)$. 1 mark

Let $f^{-1}$ be the inverse function of $f$.

**b.** On the axes below, sketch the graphs of $f$ and $f^{-1}$, showing their points of intersection. 2 marks



**c.** The rule for $f^{-1}$ can be written as $f^{-1}(x) = k\arccos\left(\dfrac{2}{x+2}\right)$.

Find the **exact** value of $k$. 2 marks

Let $A$ be the magnitude of the area enclosed by the graphs of $f$ and $f^{-1}$.

**d.** Write a definite integral expression for $A$ and evaluate it correct to three decimal places. 2 marks

**e.** **i.** Write down a definite integral in terms of $x$ that gives the arc length of the graph of $f$ from $x = 0$ to $x = 2$. 2 marks

**ii.** Evaluate this definite integral correct to three decimal places. 1 mark

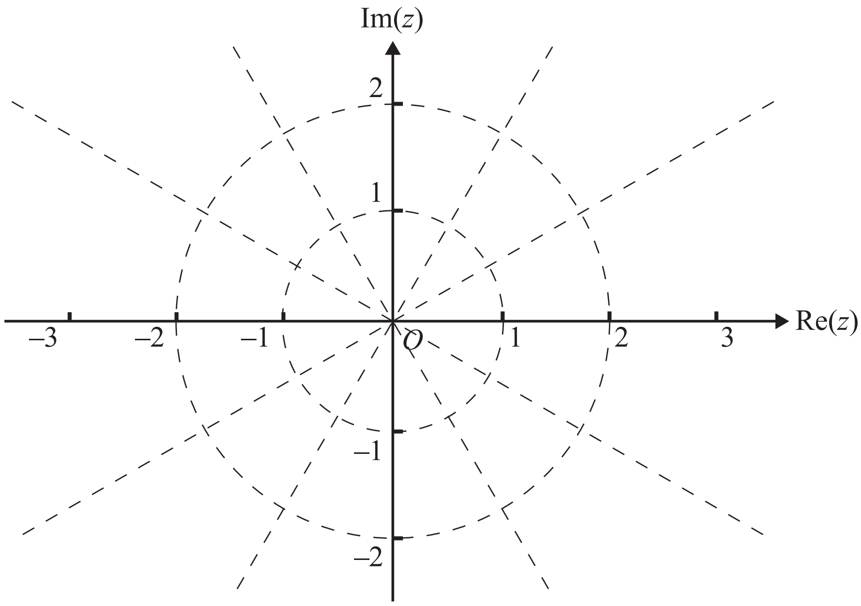
**Question 2** (9 marks)

Let $u = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

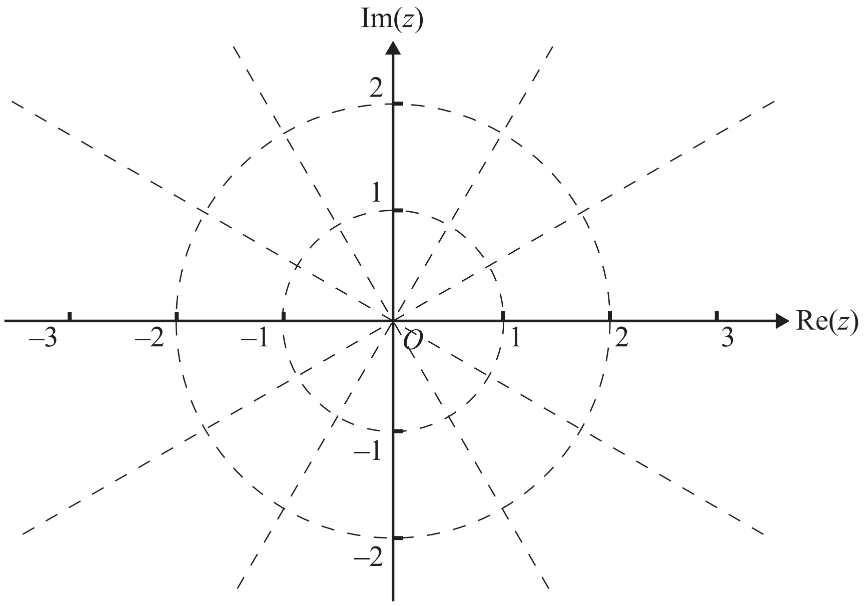
**a.** **i.** Express $u$ in polar form. 2 marks

**ii.** **Hence** show that $u^6 = 1$. 1 mark

**iii.** Plot all roots of $z^6 - 1 = 0$ on the Argand diagram below, labelling $u$ and $w$ where $w = -u$. 2 marks



**b.** **i.** Draw and label the subset of the complex plane given by $S = \{z : |z| = 1\}$ on the Argand diagram below. 1 mark




**ii.** Draw and label the subset of the complex plane given by $T = \{z : |z - u| = |z + u|\}$ on the Argand diagram above. 1 mark

**iii.** Find the coordinates of the points of intersection of $S$ and $T$. 2 marks

**Question 3** (11 marks)

The number of mobile phones, $N$, owned in a certain community after $t$ years may be modelled by $\log_e(N) = 6 - 3e^{-0.4t}$, $t \geq 0$.

**a.** Verify by substitution that $\log_e(N) = 6 - 3e^{-0.4t}$ satisfies the differential equation

$$\frac{1}{N}\frac{dN}{dt} + 0.4\log_e(N) - 2.4 = 0$$

2 marks

**b.** Find the initial number of mobile phones owned in the community. Give your answer correct to the nearest integer. 1 mark

**c.** Using this mathematical model, find the limiting number of mobile phones that would eventually be owned in the community. Give your answer correct to the nearest integer. 2 marks

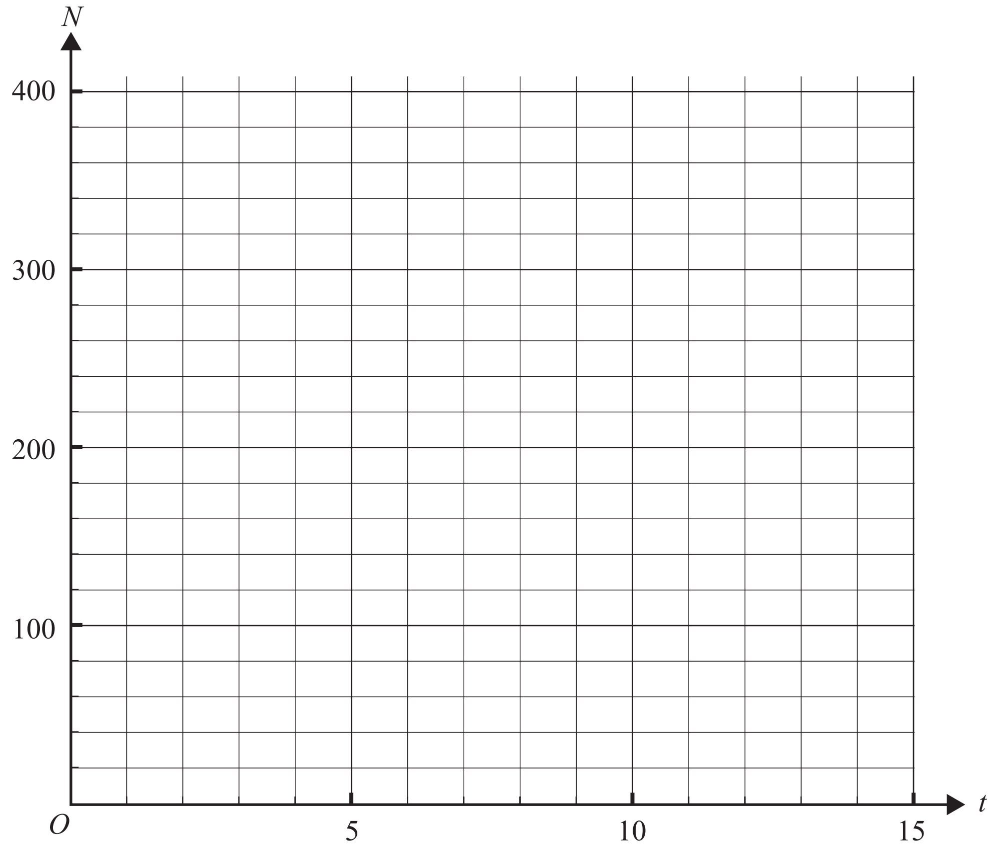
The differential equation in **part a.** can also be written in the form $\frac{dN}{dt} = 0.4N(6 - \log_e(N))$.

**d.** **i.** Find $\frac{d^2N}{dt^2}$ in terms of $N$ and $\log_e(N)$. 2 marks

**ii.** The graph of $N$ as a function of $t$ has a point of inflection.

Find the values of the coordinates of this point. Give the value of $t$ correct to one decimal place and the value of $N$ correct to the nearest integer. 2 marks

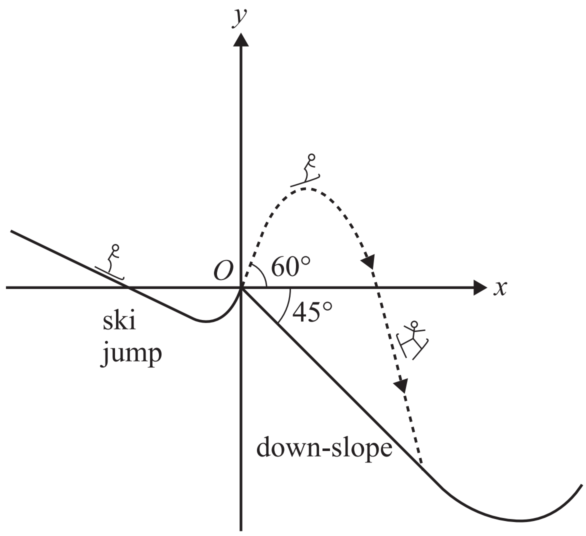
**e.** Sketch the graph of $N$ as a function of $t$ on the axes below for $0 \le t \le 15$. 2 marks



**Question 4** (10 marks)

A skier accelerates down a slope and then skis up a short ski jump, as shown below. The skier leaves the jump at a speed of 12 $\text{ms}^{-1}$ and at an angle of 60° to the horizontal. The skier performs various gymnastic twists and lands on a straight-line section of the 45° down-slope $T$ seconds after leaving the jump.

Let the origin $O$ of a cartesian coordinate system be at the point where the skier leaves the jump, with $\underset{\sim}{i}$ a unit vector in the positive $x$ direction and $\underset{\sim}{j}$ a unit vector in the positive $y$ direction. Displacements are measured in metres and time in seconds.




**a.** Show that the initial velocity of the skier when leaving the jump is $6\underset{\sim}{i} + 6\sqrt{3}\underset{\sim}{j}$. 1 mark

**b.** The acceleration of the skier, $t$ seconds after leaving the ski jump, is given by

$$\ddot{\underset{\sim}{r}}(t) = -0.1t\,\underset{\sim}{i} - (g - 0.1t)\,\underset{\sim}{j},\ 0 \le t \le T$$

Show that the position vector of the skier, $t$ seconds after leaving the jump, is given by

$$\underset{\sim}{r}(t) = \left(6t - \frac{1}{60}t^3\right)\underset{\sim}{i} + \left(6t\sqrt{3} - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right)\underset{\sim}{j},\ 0 \le t \le T$$

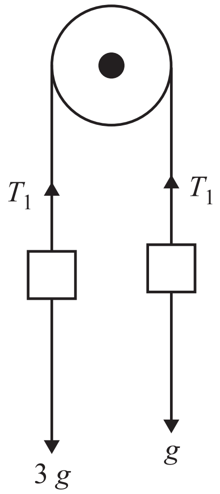
3 marks

**c.** Show that $T = \dfrac{12}{g}\left(\sqrt{3} + 1\right)$. 3 marks

**d.** At what speed, in metres per second, does the skier land on the down-slope? Give your answer correct to one decimal place. 3 marks

**Question 5** (10 marks)

The diagram below shows particles of mass 1 kg and 3 kg connected by a light inextensible string passing over a smooth pulley. The tension in the string is $T_1$ newtons.

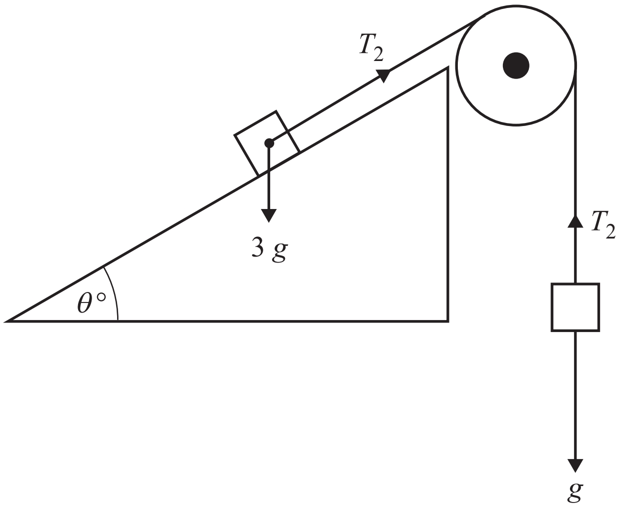



**a.** Let $a$ ms$^{-2}$ be the acceleration of the 3 kg mass downwards.

Find the value of $a$. 2 marks

**b.** Find the value of $T_1$. 1 mark

The 3 kg mass is placed on a smooth plane inclined at an angle of $\theta°$ to the horizontal. The tension in the string is now $T_2$ newtons.




**c.** When $\theta° = 30°$, the acceleration of the 1 kg mass upwards is $b$ ms$^{-2}$.

Find the value of $b$. 3 marks

**d.** For what angle $\theta°$ will the 3 kg mass be at rest on the plane? Give your answer correct to one decimal place. 2 marks

**e.** What angle $\theta°$ will cause the 3 kg mass to accelerate up the plane at $\frac{g}{4}\left(1-\frac{3}{\sqrt{2}}\right)$ ms$^{-2}$? 2 marks

**Question 6** (10 marks)

A certain type of computer, once fully charged, is claimed by the manufacturer to have $\mu = 10$ hours lifetime before a recharge is needed. When checked, a random sample of $n = 25$ such computers is found to have an average lifetime of $\bar{x} = 9.7$ hours and a standard deviation of $s = 1$ hour.

To decide whether the information gained from the sample is consistent with the claim $\mu = 10$, a statistical test is to be carried out.

Assume that the distribution of lifetimes is normal and that $s$ is a sufficiently accurate estimate of the population (of lifetimes) standard deviation $\sigma$.

**a.** Write down suitable hypotheses $H_0$ and $H_1$ to test whether the mean lifetime is less than that claimed by the manufacturer. 2 marks

**b.** Find the $p$ value for this test, correct to three decimal places. 2 marks

**c.** State with a reason whether $H_0$ should be rejected or not rejected at the 5% level of significance. 1 mark

Let the random variable $\overline{X}$ denote the mean lifetime of a random sample of 25 computers, assuming $\mu = 10$.

**d.** Find $C^*$ such that $\Pr\left(\overline{X} < C^* \mid \mu = 10\right) = 0.05$. Give your answer correct to three decimal places. 2 marks

**e.** **i.** If the mean lifetime of all computers is in fact $\mu = 9.5$ hours, find $\Pr\left(\overline{X} > C^* \middle| \mu = 9.5\right)$, giving your answer correct to three decimal places, where $C^*$ is your answer to **part d.** 2 marks

**ii.** Does the result in **part e.i.** indicate a type I or type II error? Explain your answer. 1 mark

**END OF QUESTION AND ANSWER BOOK**