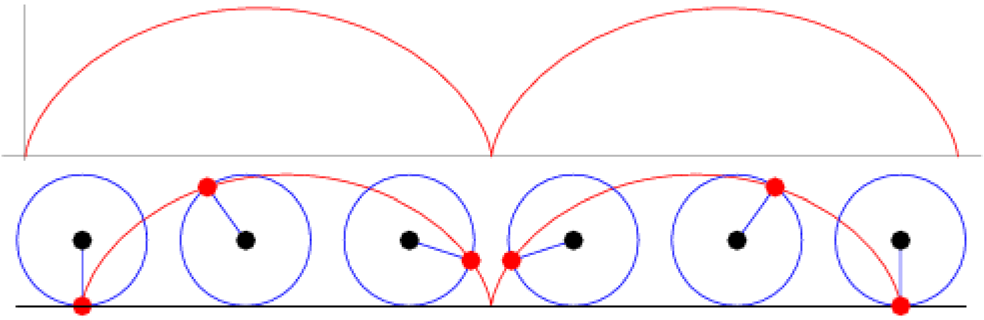
SPECIALIST MATHEMATICS UNIT 4
PRACTICE APPLICATION TASK

Technology Active. **Time Allowed: 75 + 50 Minutes**

NVESTIGATING SOME SPECIAL CURVES: CYCLOID & HYPOCYCLOID

PART A: CYCLOID



A **cycloid** is the curve defined by a fixed point on a wheel as it rolls, or, more precisely, the **locus** of a point on the rim of a circle rolling along a straight line.

A moving point on the cycloid with a cusp at the origin, created by a circle of radius ***a*** centimetres, has the vector equation:

$\underset{\sim}{r}(t) = a(t - \sin t)\underset{\sim}{i} + a(1 - \cos t)\underset{\sim}{j}$, where t is a real parameter.
Distances x and y are in centimetres and time t is in seconds.

Question 1
Assume $a = 2$ cm, $t \geq 0$. Sketch the graph of a moving point using your graphics calculator. Show two humps. Mark all axes intercepts and turning points with exact values. For each point marked state the value of t.

Question 2

a) Write equations for $\underset{\sim}{v}(t)$ and $\underset{\sim}{a}(t)$, the velocity and acceleration vectors.
b) Find $\underset{\sim}{v}(\pi)$ and $\underset{\sim}{a}(\pi)$.
c) Draw $\underset{\sim}{v}(\pi)$ and $\underset{\sim}{a}(\pi)$ on your diagram.
d) Find all points for which velocity and acceleration vectors are perpendicular.

Question 3

a) Find $\underset{\sim}{v}(\frac{\pi}{2})$ and $\underset{\sim}{a}(\frac{\pi}{2})$.
b) Mark the above vectors on the diagram.
c) Find the angle between $\underset{\sim}{v}(\frac{\pi}{2})$ and $\underset{\sim}{a}(\frac{\pi}{2})$.
d) Is the point speeding up or slowing down when $t = \frac{\pi}{2}$? Justify your answer.

e) Repeat a) to d) for $t = \frac{\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

Question 4

a) Find the expression for speed in terms of *t*.
b) Determine the maximum and minimum speeds of the moving point. Show your working.
c) List the values of *t* for which maximum and minimum speeds occur for one cycle of motion.
d) Find the magnitude of acceleration.
e) Hence describe the motion of the moving point for one complete cycle.

Question 5

The following two statements are true for the cycloid under investigation:

Statement One: When the angle between velocity and acceleration vectors is acute, the point is speeding up.

Statement Two: When the angle between velocity ad acceleration vectors is obtuse, the point is slowing down.

Use vector resolutes and diagrams rather than calculations to show that the above two statements are true in general.

Question 6

a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

b) Use related rates to find $\frac{dy}{dx}$ in terms of *t*.

c) Hence find the values of *t* for which $\frac{dy}{dx}$ is undefined.

d) Refer to your diagram. Describe what happens at the points where $\frac{dy}{dx}$ is undefined. What are the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at those points?

Question 7

a) Find the equation of the tangent to the cycloid at $t = \frac{\pi}{2}$.

b) Show that $\underset{\sim}{v}(\frac{\pi}{2})$ lies on the tangent line.

Question 8

a) Find the Cartesian equation of the path of the point moving along the cycloid.
b) Determine the values of *y* for which the Cartesian equation is valid.
c) Draw the curve obtained in Q8a on your graphics calculator. Comment on what it shows.
d) Use the properties of inverse functions and your graphics calculator to show that the area under the original cycloid curve for a single hump is equal to 12π.

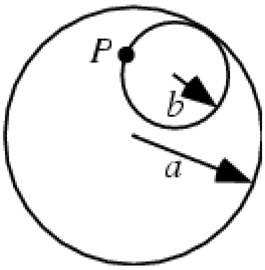
Question 9

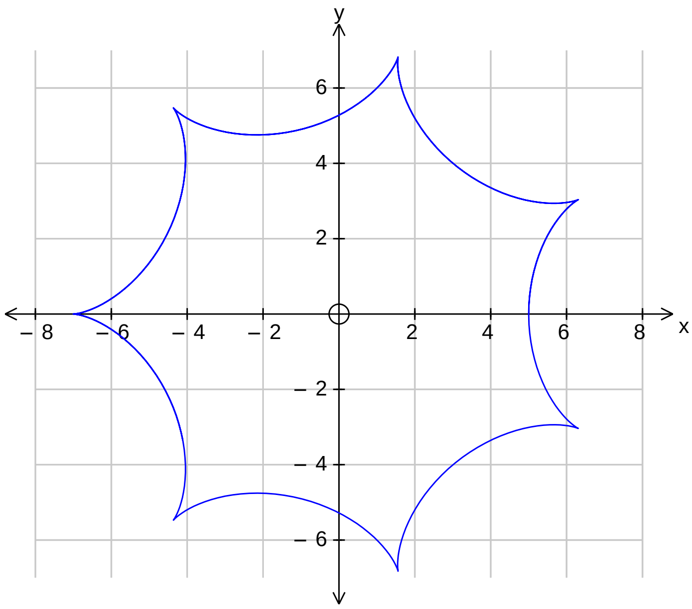
a) Differentiate your Cartesian equation.

b) Compare the results for $\frac{dy}{dx}$ obtained in Questions 6b and 9a.

c) Which method of finding $\frac{dy}{dx}$ is easier?

PART B: HYPOCYCLOID





$x = 6\cos 0.5t - \cos 3t; y = 6\sin 0.5t + \sin 3t$

The graph above shows a hypocycloid of seven cusps. The graph is the path traced by a point on a circle that rolls on the inside of another circle that has 7 times the radius. The position vector $\underset{\sim}{r}(t)$ is given by:

$$\underset{\sim}{r}(t) = (6\cos(0.5t) - \cos(3t))\underset{\sim}{i} + (6\sin(0.5t) + \sin(3t))\underset{\sim}{j}$$

Distances *x* and *y* are in metres and time *t* is in seconds.

Question 10
Plot the path of the object on your graphics calculator. Does the graph agree with the one shown above? Hint: use F2Zoom 5: ZoomSqr. Set up the time in such a way that all graph is shown.

Question 11
Find the exact period of the motion. Justify your answer.

Question 12
Write equations for $\underset{\sim}{v}(t)$ and $\underset{\sim}{a}(t)$, the velocity and acceleration vectors.

Question 13
Find $\underset{\sim}{v}(2)$ and $\underset{\sim}{a}(2)$ correct to two decimal places.

Question 14
Locate the point $\underset{\sim}{r}(2)$ on the graph above and construct $\underset{\sim}{v}(2)$ and $\underset{\sim}{a}(2)$ on your diagram, starting at the head of $\underset{\sim}{r}(2)$. Try to draw to scale using the grid provided.

Question 15
Calculate the speed of the object at $t = 2$.

Question 16
By looking at your diagram, justify whether the object seems to be speeding up or slowing down when $t = 2$.

Question 17
Find the angle between $\underset{\sim}{v}(2)$ and $\underset{\sim}{a}(2)$.

Question 18
Hence find the scalar projection of $\underset{\sim}{a}(2)$ onto $\underset{\sim}{v}(2)$.

Question 19
Calculate the tangential component of acceleration at $t = 2$.

Question 20
Calculate the normal component of acceleration at $t = 2$.

Question 21
Construct the tangential and normal components of acceleration on the graph shown.

Question 22
Describe the physical effects of the tangential and normal components of acceleration at $t = 2$.

Question 23
There is a cusp at the point with approximate coordinates $(6, 3)$. What must be true of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at this point?

Question 24

Use related rates to find $\frac{dy}{dx}$ in terms of *t*.

Question 25

a) Use your graphics calculator to find the value of *t* at the cusp near $(6,3)$.
b) Give the coordinates of the cusp point for this *t* value correct to 2 decimal places.

Question 26

a) What is the value of *t* at the fifth cusp? Explain how you deduced this value?
b) Give the coordinates of the point at the fifth cusp correct to 2 decimal places.