SPECIALIST MATHEMATICS Units 3 & 4

Written examination 1 Solutions

Question 1 (4 marks)

a.

$$|x^2 - 3x - 2| = \begin{cases} x^2 - 3x + 2, & x \in [0,1] \cup [2,3] \\ -(x^2 - 3x + 2), & x \in (1,2) \end{cases}$$

2A

b.

$$\int_0^3 |x^2 - 3x + 2|\,dx = \int_0^1 (x^2 - 3x + 2)\,dx -$$

1M

$$- \int_1^2 (x^2 - 3x + 2)\,dx + \int_2^3 (x^2 - 3x + 2)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1 - \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2$$

$$+ \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_2^3$$

$$= \frac{5}{6} + \frac{1}{6} + \frac{5}{6}$$

$$= \frac{11}{6}$$

1A

Question 2 (4 marks)

a.

$$\cos^4(x) = \left[\cos^2(x)\right]^2$$

$$= \left[\frac{1 - \cos(2x)}{2}\right]^2$$

1M

$$= \frac{1 - 2\cos(2x) + \cos^2(2x)}{4}$$

$$= \frac{1}{4}\cos^2(2x) - \frac{1}{2}\cos(2x) + \frac{1}{4}$$

$a = \frac{1}{4}$, $b = -\frac{1}{2}$ and $c = \frac{1}{4}$ **1A**

b.

$$\int \cos^4 x\,dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\,dx$$

$$= \frac{1}{4}\int dx + \frac{1}{2}\int \cos(2x)\,dx + \frac{1}{4}\int \cos^2(2x)\,dx$$

1M

$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{1}{8}x + \frac{1}{8} \times \frac{1}{4}\sin(4x)$$

$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x)$$

1A

Question 3 (7 marks)

a.

$$\int f'(x)\,dx = \int \frac{1}{2x - 1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x - \frac{1}{2}}\,dx$$

$$= \frac{1}{2}\log_e\left|x - \frac{1}{2}\right| + c$$

1A

$$f(1) = \frac{1}{2}\log_e\left(1 - \frac{1}{2}\right) + c = 0$$

$$c = -\frac{1}{2}\log_e\left(\frac{1}{2}\right)$$

$$c = \frac{1}{2}\log_e(2)$$

1A

$$f(x) = \frac{1}{2}\log_e\left|x - \frac{1}{2}\right| + \frac{1}{2}\log_e(2)$$

$$= \frac{1}{2}\log_e\left|2\left(x - \frac{1}{2}\right)\right|$$

$f(x) = \frac{1}{2}\log_e(2x - 1)$, where the maximal domain is $\left(\frac{1}{2}, \infty\right)$ **1A**

b.

$$\int 5x\sqrt{x}\,dx = 5\int x^{\frac{3}{2}}\,dx$$

$$= 5 \times \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$$ **1A**

$$= 5 \times \frac{2}{5}x^{\frac{5}{2}} + c$$

$$= 2x^{\frac{5}{2}} + c$$

$$g(1) = 2 \times 1^{\frac{5}{2}} + c = 2$$

$$\therefore \; c = 0$$

$$g(x) = 2x^2\sqrt{x},\ x \geq 0$$ **1A**

c.

$$(f \times g)'(x) = f'(x)g(x) + f(x)g'(x)$$ **1M**

$$= \frac{2}{2(2x-1)}2x^{\frac{5}{2}} + \frac{\log_e(2x-1)}{2} \times 5x^{\frac{3}{2}}$$ **1A**

$$= \frac{2x^{\frac{5}{2}}}{2x-1} + \frac{5x^{\frac{3}{2}}\log_e(2x-1)}{2}$$

Question 4 (8 marks)

a.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= -\boldsymbol{i} + \boldsymbol{j} - (\boldsymbol{i} + 2\boldsymbol{j})$$ **1A**

$$= -2\boldsymbol{i} - \boldsymbol{j}$$

$$\overrightarrow{CD} = \overrightarrow{OC} - \overrightarrow{OD}$$

$$= m\boldsymbol{j} - (n\boldsymbol{i} + 4\boldsymbol{j})$$ **1A**

$$= -n\boldsymbol{i} + (m-4)\boldsymbol{j}$$

b.

ABCD is a rhombus $\Rightarrow \overrightarrow{AB} = \overrightarrow{DC}$

$$-2\boldsymbol{i} - \boldsymbol{j} = -n\boldsymbol{i} + (m-4)\boldsymbol{j}$$ **1M**

Equating corresponding coefficients,
$n = 2$ and $-1 = m - 4$
$\therefore\ m = 3$ and $n = 2$ **1A**

c.

$$\left|\overrightarrow{AB}\right| = \sqrt{(-2)^2 + 1^2}$$ **1A**

$$= \sqrt{5}$$

ABCD is a rhombus $\Rightarrow \left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| = \sqrt{5}$ **1A**

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|\cos B$$

$$-4 = \sqrt{5} \cdot \sqrt{5} \cdot \cos B$$

$$\cos B = -\frac{4}{5}$$

$$\sin B = \frac{3}{5}$$ **1A**

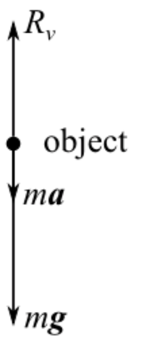
$$\text{Area}_{ABCD} = \left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|\sin B$$

$$= \sqrt{5} \cdot \sqrt{5} \cdot \frac{3}{5}$$ **1A**

$$= 3\ \text{units}^2$$

Question 5 (3 marks)

Let the downwards direction be the positive direction.
The object moves with constant acceleration, *a*.



$$m\boldsymbol{a} = m\boldsymbol{g} - R_v$$ **1M**

R_v is the vertical air resistance

Initial conditions: $u = 0$, $s = 40$ m, $t = 2$ seconds

$$s = ut + \frac{1}{2}at^2 \Rightarrow 40 = \frac{1}{2} \times 2^2 a$$

$a = 20$ m/s^2 **1A**

$$0.4 \times 20 = 0.4 \times 10 - R_v$$ **1A**

$$R_v = -4N$$

Question 6 (3 marks)

$$\frac{\tan(\alpha)+i}{\tan(\alpha)-i}=\frac{\dfrac{\sin(\alpha)}{\cos(\alpha)}+i}{\dfrac{\sin(\alpha)}{\cos(\alpha)}-i}$$ **1M**

$$=\frac{\sin(\alpha)+i\cos(\alpha)}{\sin(\alpha)-i\cos(\alpha)}$$

Multiply both the numerator and the denominator by the conjugate of the denominator.

$$=\frac{\left[\sin(\alpha)+i\cos(\alpha)\right]^2}{\sin^2(\alpha)+\cos^2(\alpha)}$$

$$=\frac{\sin^2(\alpha)-\cos^2(\alpha)+2i\sin(\alpha)\cos(\alpha)}{1}$$

Use double angle formulas.

$$=-\cos(2\alpha)+i\sin(2\alpha)$$ **1M**

$$=\cos(\pi-2\alpha)+i\sin(\pi-2\alpha)$$

$$=\text{cis}(\pi-\alpha),\text{ where }(\pi-\alpha)\in(0,2\pi)$$ **1A**

Question 7 (5 marks)

a.

$n = 64$, $\bar{x}=120$, $s = 10$

The 95% confidence interval is

$$\left(\bar{x}-z\times\frac{s}{\sqrt{n}},\ \bar{x}+z\times\frac{s}{\sqrt{n}}\right)$$ **1M**

The corresponding z score for the 95% confidence interval is 1.96.

$$\left(120-1.96\times\frac{10}{\sqrt{64}},\ 120+1.96\times\frac{10}{\sqrt{64}}\right)$$

Therefore a 95% confidence interval for the mean weight of Igor's snatch lifts is
(117.55 kg, 122.45 kg) **1A**

b.

If repeated samples were taken and the 95% confidence interval was computed for each sample, it can be assumed that 95% of these confidence intervals would contain the population mean. **1A**

c.

Igor is not very likely to beat the world record as the world record is higher than the upper limit of the 95% confidence interval. **2A**

Question 8 (6 marks)

a.

$$z^4=-i$$

$$z^4=\text{cis}\left(\frac{3\pi}{2}\right)$$ **1A**

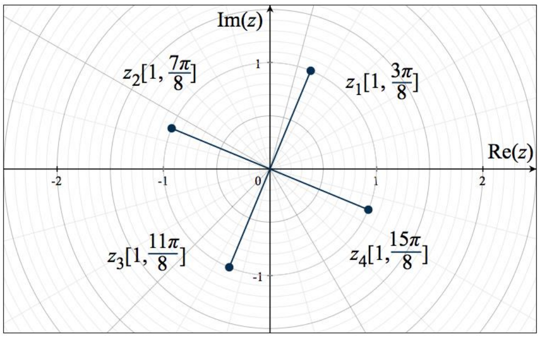
$$z_k=\text{cis}\left(2k\pi+\frac{3\pi}{2}\right),\text{ where }k\in\{0,1,2,3\}$$

The four solutions are:

$$z_0=\text{cis}\left(\frac{3\pi}{8}\right),\ z_1=\text{cis}\left(\frac{7\pi}{8}\right),$$

$$z_2=\text{cis}\left(\frac{11\pi}{8}\right),\ z_3=\text{cis}\left(\frac{15\pi}{8}\right)$$ **1A**

b.



2M

c.

$$(a+ib-1)^4=-i$$

$$(z-1)^4=-i$$ **1M**

$$z-1=z_k,\text{ where }k\in\{0,1,2,3\}\ \ldots[1]$$

The solutions to equation [1] are:

When $k = 0$, $z_0=\text{cis}\left(\dfrac{\pi}{4}\right)$

$$z=\text{cis}\left(\frac{\pi}{4}\right)+1$$

$$=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i+1$$

$$=1+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$$

When k = 1, $z_1 = \text{cis}\left(\frac{3\pi}{4}\right)$

$$z_1 + 1 = \text{cis}\left(\frac{3\pi}{4}\right) + 1$$
$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i + 1$$
$$= 1 + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

When k = 2, $z_2 = \text{cis}\left(\frac{5\pi}{4}\right)$

$$z_2 + 1 = \text{cis}\left(\frac{5\pi}{4}\right) + 1$$
$$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i + 1$$
$$= 1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$$

When k = 3, $z_3 = \text{cis}\left(\frac{7\pi}{4}\right)$

$$z_3 + 1 = \text{cis}\left(\frac{7\pi}{4}\right) + 1$$
$$= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i + 1$$
$$= 1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$$

One possible set of values for a and b is

$a_1 = 1 + \frac{\sqrt{2}}{2}, b_1 = \frac{\sqrt{2}}{2}$ or $a_2 = 1 + \frac{\sqrt{2}}{2}, b_2 = -\frac{\sqrt{2}}{2}$ or $a_2 = 1 - \frac{\sqrt{2}}{2}, b_2 = -\frac{\sqrt{2}}{2}$ or $a_2 = 1 - \frac{\sqrt{2}}{2}, b_2 = \frac{\sqrt{2}}{2}$.

1A