# The Mathematical Association of Victoria

# SPECIALIST MATHEMATICS 2015

# Trial Written Examination 1 - SOLUTIONS

**Question 1**

**a.** **Method 1:**

If $2z+i$ is a factor of $2z^3+9iz^2+10z+7i$ then $z=-\frac{i}{2}$ is a root of $2z^3+9iz^2+10z+7i$:

$2\left(-\frac{i}{2}\right)^3+9i\left(-\frac{i}{2}\right)^2+10\left(-\frac{i}{2}\right)+7i$ **[M1]**

$=2\left(\frac{i}{8}\right)-9i\left(\frac{1}{4}\right)+10\left(-\frac{i}{2}\right)+7i$ **[M1]**

$=\frac{i}{4}-\frac{9i}{4}-5i+7i \qquad =-\frac{8i}{4}-5i+7i \qquad =-2i-5i+7i$

$=0$.

**There must be sufficient correct evidence that** $2\left(-\frac{i}{2}\right)^3+9i\left(-\frac{i}{2}\right)^2+10\left(-\frac{i}{2}\right)+7i=0$**.**

**Method 2:**

IF $2z+i$ is a factor of $2z^3+9iz^2+10z+7i$ then the accompanying quadratic factor must have the form $z^2+az+7$ so that $2z^3+9iz^2+10z+7i=(2z+i)(z^2+az+7)$.

It follows that IF a value of $a$ can be found then the quadratic factor $z^2+az+7$ exists and therefore $2z+i$ is a factor.

When $(2z+i)(z^2+az+7)$ is expanded, the coefficient of $z$ must be equal to 10 by comparison with $2z^3+9iz^2+10z+7i$.

**Clear explanation of the method:** **[M1]**

Therefore:

$$14+ai=10 \qquad \Rightarrow a=\frac{-4}{i}=4i.$$

Expand to check: $(2z+i)(z^2+4iz+7)=2z^3+9iz^2+10z+7i$.

Therefore $2z^3+9iz^2+10z+7i=(2z+i)(z^2+4iz+7)$ **[A1]**

which shows that $2z+i$ is a factor.

The benefit of this method is that the conclusion $2z^3+9iz^2+10z+7i=(2z+i)(z^2+4iz+7)$ can be recruited to answer **part b.**, which saves time. The benefit of using this method as a time investment for answering **part b.** could be recognised during reading time.

**Method 3:** Use polynomial long division.

$$\begin{array}{r l}
 & z^2+4iz+7 \\
2z+i \,\big) & 2z^3+9iz^2+10z+7i \\
 & \underline{2z^3+iz^2} \\
 & \quad 8iz^2+10z+7i \\
 & \quad \underline{8iz^2-4z} \\
 & \qquad 14z+7i \\
 & \qquad \underline{14z+7i} \\
 & \qquad\quad 0
\end{array}$$

**Correct polynomial long division:** **[M1]**

The remainder is zero therefore $2z+i$ is a factor. **[A1]**

The benefit of this method is that $2z^3+9iz^2+10z+7i=(2z+i)(z^2+4iz+7)$ readily follows. This result can be recruited to answer **part b.**, which saves time. The benefit of using this method as a time investment for answering **part b.**, which could be recognised during reading time.

**b.** **Method 1:** Construct the quadratic factor.

Clearly $2z^3 + 9iz^2 + 10z + 7i = (2z + i)(z^2 + az + 7)$.

If the value of $a$ can be found then the quadratic factor is completely specified and its linear factors can be determined.

When $(2z + i)(z^2 + az + 7)$ is expanded, the coefficient of $z$ must be equal to 10 by comparison with $2z^3 + 9iz^2 + 10z + 7i$.

Therefore:

$14 + ai = 10 \qquad \Rightarrow a = \dfrac{-4}{i} = 4i$.

Therefore the quadratic factor is $z^2 + 4iz + 7$. **[M1]**

$z^2 + 4iz + 7 = (z + 2i)^2 + 4 + 7$

$= (z + 2i)^2 + 11$ **[M1]**

$= \left(z + 2i + i\sqrt{11}\right)\left(z + 2i - i\sqrt{11}\right)$.

**Answer:** $z + \left(2 + \sqrt{11}\right)i, \quad z + \left(2 - \sqrt{11}\right)i$. **[A1]**

**Method 2:** Use polynomial long division to find the quadratic factor.

$$
\begin{array}{r}
z^2 + 4iz + 7 \\
2z + i\,\overline{\big)\,2z^3 + 9iz^2 + 10z + 7i} \\
\underline{2z^3 + iz^2 \qquad\qquad\quad} \\
8iz^2 + 10z + 7i \\
\underline{8iz^2 - 4z \qquad} \\
14z + 7i \\
\underline{14z + 7i} \\
0
\end{array}
$$

Therefore the quadratic factor is $z^2 + 4iz + 7$. **[M1]**

$z^2 + 4iz + 7 = (z + 2i)^2 + 4 + 7$

$= (z + 2i)^2 + 11$ **[M1]**

$= \left(z + 2i + i\sqrt{11}\right)\left(z + 2i - i\sqrt{11}\right)$.

**Answer:** $z + \left(2 + \sqrt{11}\right)i, \quad z + \left(2 - \sqrt{11}\right)i$. **[A1]**

**Question 2**

**a.** **Method 1:** Use the formula $y-k=\pm\frac{b}{a}(x-h)$.

The asymptotes of the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ are $y-k=\pm\frac{b}{a}(x-h)$.

Therefore complete the square to get the standard form and hence identify the values of *a*, *b*, *h* and *k*:

$$2x^2-y^2+4y-8=0$$

$$\Rightarrow 2x^2-\left(y^2-4y\right)-8=0$$

$$\Rightarrow 2x^2-\left([y-2]^2-4\right)-8=0$$

$$\Rightarrow 2x^2-[y-2]^2+4-8=0$$

$$\Rightarrow 2x^2-(y-2)^2=4$$

$$\Rightarrow \frac{x^2}{2}-\frac{(y-2)^2}{4}=1.$$ **[A1]**

Substitute $a=\sqrt{2}$, $b=2$, $h=0$ and $k=2$ into $y-k=\pm\frac{b}{a}(x-h)$:

$$y-2=\pm\frac{2}{\sqrt{2}}x.$$

**Answer:** $y=\sqrt{2}x+2$ and $y=-\sqrt{2}x+2$. **[A1]**

**Also accept** $y=\pm\sqrt{2}x+2$.

**Method 2:** Solve for $y$ as a function of $x$ and then take the limit $x \to \infty$.

$2x^2 - y^2 + 4y - 8 = 0$

$\Rightarrow y^2 - 4y + 8 - 2x^2 = 0$

$\Rightarrow y = \dfrac{4 \pm \sqrt{16 - 4(8 - 2x^2)}}{2}$

$= \dfrac{4 \pm \sqrt{8x^2 - 16}}{2}$ **[M1]**

$= 2 \pm \sqrt{2}\sqrt{x^2 - 2}$.

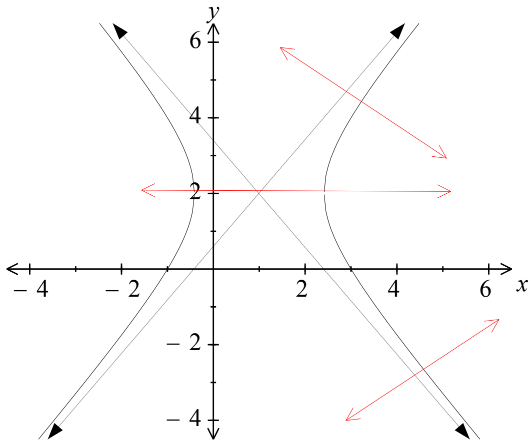
Take the limit $x \to \infty$:

$y \to 2 \pm \sqrt{2}\,|x|$.

**Answer:** $y = \sqrt{2}x + 2$ and $y = -\sqrt{2}x + 2$. **[A1]**

**Also accept** $y = \pm\sqrt{2}x + 2$.

**b.** Sketching a rough graph of $\frac{x^2}{2} - \frac{(y-2)^2}{4} = 1$ (found in **part a. Method 1**) and drawing in a couple of normal at different points helps to see the answer:




Maximum value of $m$ approaches the gradient of the line that is perpendicular to the diagonal asymptote $y = -\sqrt{2}x + 2$.

Minimum value of $m$ approaches the gradient of the line that is perpendicular to the diagonal asymptote $y = \sqrt{2}x + 2$.

$m$ is continuous.

It follows that $\frac{-1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}$.

**Answer:** $\frac{-1}{\sqrt{2}} < m < \frac{1}{\sqrt{2}}$. **[A1]**

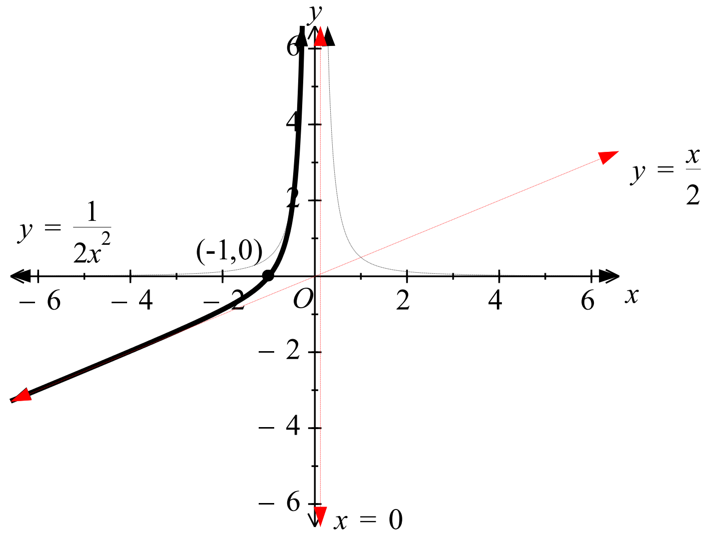
**Also accept** $|m| < \frac{1}{\sqrt{2}}$, $|m| < \frac{\sqrt{2}}{2}$, $\frac{-\sqrt{2}}{2} < m < \frac{\sqrt{2}}{2}$.

**Consequential on answer to part a.:** $\frac{-a}{b} < m < \frac{a}{b}$.

**Question 3**

• Use addition of ordinates to sketch a graph of $y = h(x)$:

$y = f(x) + g(x)$ where $f(x) = \frac{x}{2}$ and $g(x) = \frac{1}{2x^2}$.



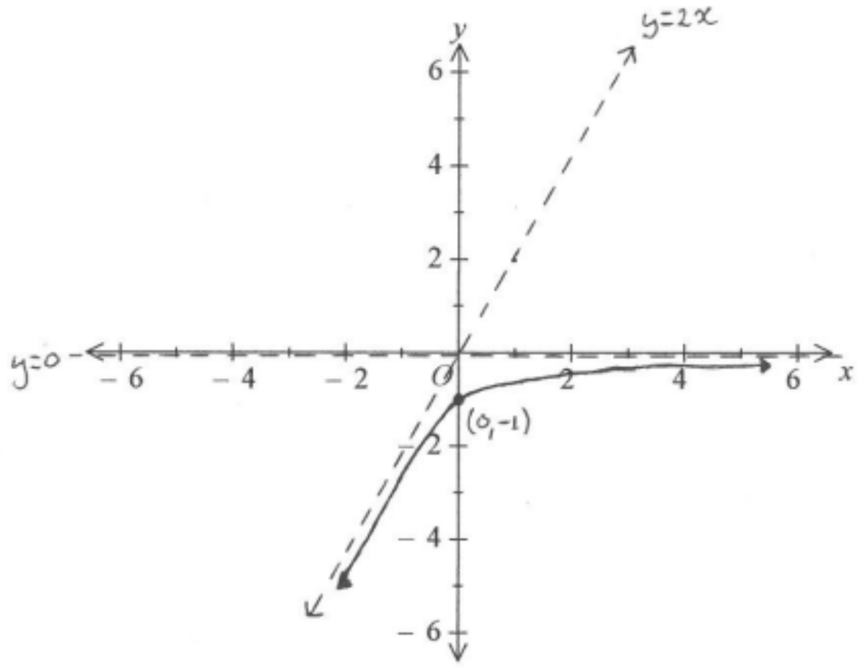

**Correct shape:** [M$\frac{1}{2}$]

**Coordinates $(-1, 0)$ of *x*-intercept:** [M$\frac{1}{2}$]

**Vertical asymptote $x = 0$:** [M$\frac{1}{2}$]

**Diagonal asymptote $y = \frac{x}{2}$:** [M$\frac{1}{2}$]

• Reflect the graph of $y = h(x)$ in the line $y = x$:




**Correct shape:** **[A $\frac{1}{2}$]**

**Coordinates $(0, -1)$ of *y*-intercept:** **[A $\frac{1}{2}$]**

**Horizontal asymptote $y = 0$:** **[A $\frac{1}{2}$]**

**Diagonal asymptote $y = 2x$:** **[A $\frac{1}{2}$]**

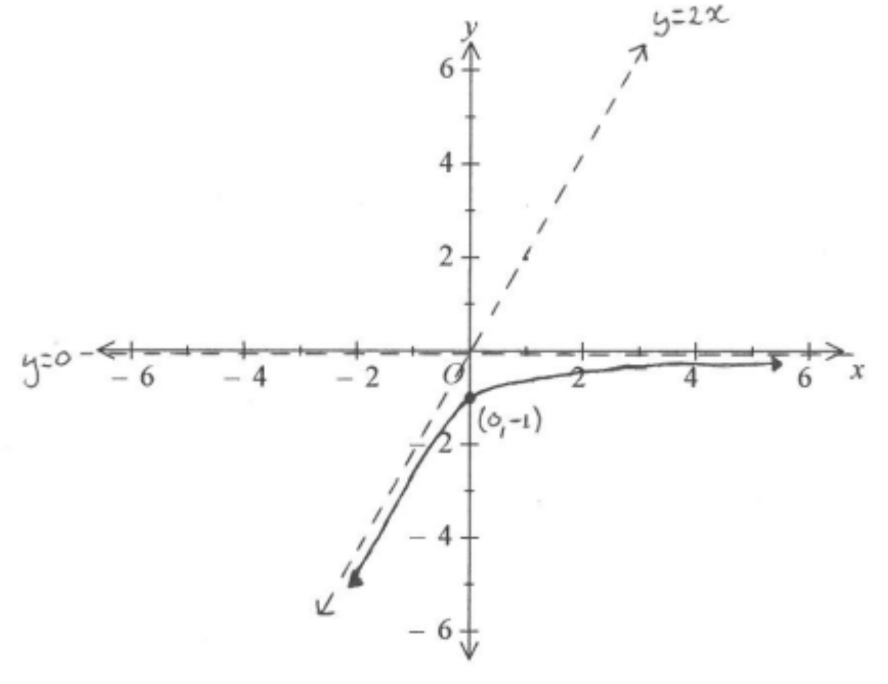
**Add up all half marks and round down**

To find the equations of the asymptotes of $y = h^{-1}(x)$, swap *x* and *y* in the equations of the asymptotes of $y = h(x)$:

• Vertical asymptote $x = 0$ of $y = h(x)$ becomes horizontal asymptote $y = 0$ of $y = h^{-1}(x)$.

• Diagonal asymptote $y = \frac{x}{2}$ of $y = h(x)$ becomes diagonal asymptote $x = \frac{y}{2} \Rightarrow y = 2x$ of $y = h^{-1}(x)$.

**Alternative marking scheme:**




**Correct shape:** **[A1]**

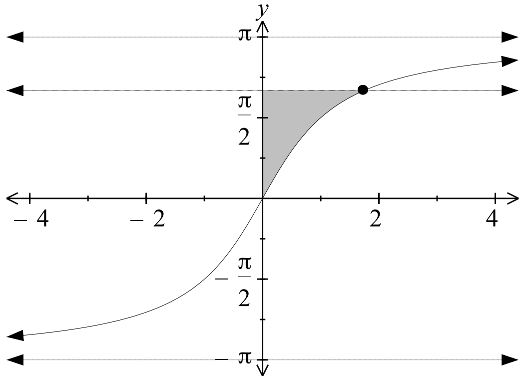
**Coordinates** $(0, -1)$ **of** ***x*-intercept:** **[A1]**

**Horizontal asymptote** $y = 0$**:** **[A1]**

**Diagonal asymptote** $y = 2x$**:** **[A1]**

**Question 4**

**a.** A rough graph of $y = 2\arctan(x)$ should be drawn and the required area shaded:




The equation of the horizontal line is found by substituting $x = \sqrt{3}$ into $y = 2\arctan(x)$:

$$y = 2\arctan(\sqrt{3}) = 2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}.$$

Therefore the required area is $A = \int_0^{\sqrt{3}} \frac{2\pi}{3} - 2\arctan(x)\, dx$.

The integral of $\arctan(x)$ can only be done by hand within the scope of the course by using integration by recognition. However, this technique has not been flagged by a prior question such as "Find the derivative of …..". Therefore the technique of 'definite integration using the inverse function' is required. A rough sketch graph is essential for seeing this.

$$A = \int_0^{2\pi/3} \tan\left(\frac{y}{2}\right) dy.$$

**Correct integrand** $\tan\left(\frac{y}{2}\right)$**:** **[M1]**

**Correct lower and upper integral terminals** $y = 0$ **and** $y = \frac{2\pi}{3}$**:** **[M1]**

Integrate by recognition:

$$A = \left[-2\log_e\left(\cos\left(\frac{y}{2}\right)\right)\right]_0^{2\pi/3}$$ **[M1]**

$$= -2\left(\log_e\left(\cos\left(\frac{\pi}{3}\right)\right) - \log_e(\cos(0))\right) \qquad = -2\log_e\left(\frac{1}{2}\right) + 2\log_e(1).$$

**Answer:** $-2\log_e\left(\frac{1}{2}\right)$ or $2\log_e(2)$ or $\log_e(4)$. **[A1]**

**Units are not required.**

**b.** Required volume:

$$V = \pi \int_0^{2\pi/3} x^2 \, dy$$

$$= \pi \int_0^{2\pi/3} \tan^2\left(\frac{y}{2}\right) dy$$ **[H1]**

**Consequential on their upper terminal found in part a.**

**Consequential on their integrand found in part a.**

$$= \pi \int_0^{2\pi/3} \sec^2\left(\frac{y}{2}\right) - 1 \, dy$$ **[M1]**

Integrate by using the formula on the VCAA detachable formula sheet:

$$= \pi \left[2\tan\left(\frac{y}{2}\right) - y\right]_0^{2\pi/3}$$ **[H1]**

$$= \pi\left(\left(2\tan\left(\frac{\pi}{3}\right) - \frac{2\pi}{3}\right) - \left(\tan(0) - 0\right)\right)$$

$$= \pi\left(2\sqrt{3} - \frac{2\pi}{3}\right).$$

Expand to get required form.

**Answer:** $2\sqrt{3}\pi - \frac{2}{3}\pi^2$. **[A1]**

**Units are not required.**

**Question 5**

**a. i.** Require $-1 \le 2x \le 1$ AND $\frac{\pi}{4} - \arccos(2x) \ne 0$:

- $-1 \le 2x \le -1 \quad \Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$.
- $\frac{\pi}{4} - \arccos(2x) \ne 0 \quad \Rightarrow \arccos(2x) \ne \frac{\pi}{4}$

$\Rightarrow 2x \ne \frac{1}{\sqrt{2}} \quad \Rightarrow x \ne \frac{1}{2\sqrt{2}}$. **[M1]**

**Answer:** $-\frac{1}{2} \le x < \frac{1}{2\sqrt{2}} \cup \frac{1}{2\sqrt{2}} < x \le \frac{1}{2}$. **[A1]**

**Also accept** $-\frac{1}{2} \le x < \frac{\sqrt{2}}{4} \cup \frac{\sqrt{2}}{4} < x \le \frac{1}{2}$ **and answers expressed using bracket notation.**

**ii.** **Method 1:** Use the "Hence …", that is, the answer to **part a.**

- $y = \frac{\pi}{4} - \arccos(2x)$ is a strictly increasing function.

It follows that $y = f(x)$ is a strictly decreasing function.

- The domain of $y = f(x)$ is $-\frac{1}{2} \le x < \frac{1}{2\sqrt{2}} \cup \frac{1}{2\sqrt{2}} < x \le \frac{1}{2}$.

- $f\left(-\frac{1}{2}\right) = -\frac{4}{3\pi}$ and $f\left(\frac{1}{2}\right) = \frac{4}{\pi}$. **[A1]**

- $y = f(x)$ is undefined for $x = \frac{1}{2\sqrt{2}}$.

(Specifically, $\lim_{x \to \frac{1}{2\sqrt{2}}^-} f(x) = -\infty$ and $\lim_{x \to \frac{1}{2\sqrt{2}}^+} f(x) = +\infty$).

It follows from the above dot points that $y \le -\frac{4}{3\pi} \cup y \ge \frac{4}{\pi}$.

**Answer:** $y \le -\frac{4}{3\pi} \cup y \ge \frac{4}{\pi}$. **[A1]**

**Also accept** $\left(-\infty, -\frac{4}{3\pi}\right] \cup \left[\frac{4}{3\pi}, +\infty\right)$.

**Method 2:** "… or otherwise, …". Construct a rough sketch graph of $y = f(x)$ from a rough sketch graph of the reciprocal function $y = \dfrac{1}{f(x)} = \dfrac{\pi}{4} - \arccos(2x)$.

- Graph of $y = \dfrac{1}{f(x)} = \dfrac{\pi}{4} - \arccos(2x)$.

**Domain:**

$-1 \le 2x \le -1$

$\Rightarrow -\dfrac{1}{2} \le x \le \dfrac{1}{2}$.

**Range:**

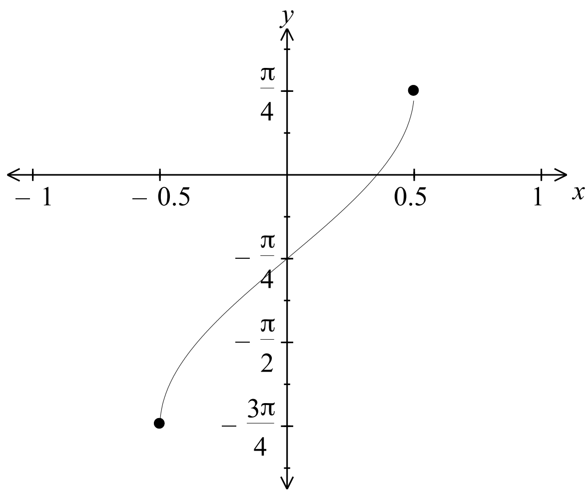
$\dfrac{\pi}{4} - \arccos(-1) \le y \le \dfrac{\pi}{4} - \arccos(1)$

$\Rightarrow -\pi + \dfrac{\pi}{4} \le y \le 0 + \dfrac{\pi}{4}$

$\Rightarrow -\dfrac{3\pi}{4} \le y \le \dfrac{\pi}{4}$.

**Coordinates of endpoints**: Get from the domain and range.

$\left(-\dfrac{3\pi}{4}, -\dfrac{1}{2}\right)$ and $\left(\dfrac{\pi}{4}, \dfrac{1}{2}\right)$.

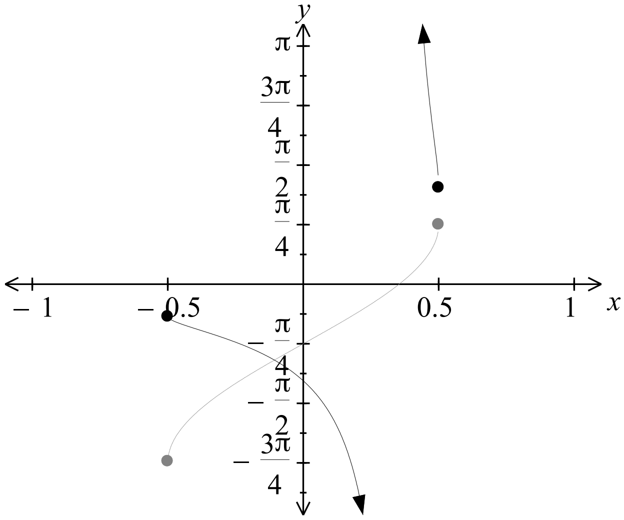



**[M1]**

There is an ***x*-intercept** which means that the graph of $y = f(x)$ will have a vertical asymptote but the location is not relevant.

**Note:** A graph of $y = \dfrac{\pi}{4} - \arccos(2x)$ can be easily sketched from a knowledge of its domain and range.

- Construct a graph of $y = f(x)$:




Coordinates of endpoints: $\left(-\frac{1}{2},\ -\frac{4}{3\pi}\right)$ and $\left(\frac{1}{2},\ \frac{4}{\pi}\right)$.

The range is found from the *y*-coordinates of the endpoints and the shape of this graph.

**Answer:** $y \le -\frac{4}{3\pi} \cup y \ge \frac{4}{\pi}$. **[A1]**

**Also accept** $\left(-\infty, -\frac{4}{3\pi}\right] \cup \left[\frac{4}{3\pi}, +\infty\right)$.

**b.** Use the chain rule.

Let $u = \frac{\pi}{4} - \arccos(2x)$ so that $f(u) = \frac{1}{u}$.

$$f'(x) = f'(u) \times \frac{du}{dx}$$

$$= -\frac{1}{u^2} \times (-1)\frac{-1}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}}$$

$$= -\frac{1}{u^2} \times \frac{1}{\sqrt{\frac{1}{4} - x^2}}$$ **[M1]**

where $u = \frac{\pi}{4} - \arccos(2x)$.

There is no need to simplify (and it is more efficient not to) since only a value $f'(0)$ is required.

Substitute $x = 0$:

$$u(0) = \frac{\pi}{4} - \arccos(0) \qquad = \frac{\pi}{4} - \frac{\pi}{2}$$

$$= \frac{-\pi}{4}$$ **[A1]**

$$\Rightarrow \frac{1}{u^2} = \frac{16}{\pi^2}.$$

Therefore:

$$f'(0) = -\frac{16}{\pi^2} \times \frac{1}{\sqrt{\frac{1}{4}}} \qquad = -\frac{32}{\pi^2}.$$

**Answer:** $-\frac{32}{\pi^2}$. **[A1]**

**Question 6**

Let $\underset{\sim}{u} = (b - a^2)\underset{\sim}{i} + \underset{\sim}{j} - b\underset{\sim}{k}$ and $\underset{\sim}{v} = 2(a - b)\underset{\sim}{i} + 4\underset{\sim}{j} + 3(a - b)\underset{\sim}{k}$.

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel if $\underset{\sim}{u} = \lambda \underset{\sim}{v}$ where $\lambda \in R$.

By considering the ratio of the components on each side of $\underset{\sim}{u} = \lambda \underset{\sim}{v}$ it follows that

$$\underbrace{\frac{b-a^2}{2(a-b)}}_{\text{Ratio of } \underset{\sim}{i}-\text{components}} = \underbrace{\frac{1}{4}}_{\text{Ratio of } \underset{\sim}{j}-\text{components}} = \underbrace{\frac{-b}{3(a-b)}}_{\text{Ratio of } \underset{\sim}{k}-\text{components}}$$

from which it follows that:

- $\dfrac{b - a^2}{2(a - b)} = \dfrac{1}{4}$ **[M1]**

$\Rightarrow 2(b - a^2) = a - b$

$\Rightarrow 3b - a - 2a^2 = 0$. …. (1)

- $\dfrac{-b}{a - b} = \dfrac{1}{4}$ **[M1]**

$\Rightarrow -4b = 3a - 3b$

$\Rightarrow b = -3a$. …. (2)

Substitute equation (2) into equation (1):

$\Rightarrow 3(-3a) - a - 2a^2 = 0$

$\Rightarrow a^2 + 5a = 0$ **[M1]**

$\Rightarrow a(a + 5) = 0$

$\Rightarrow a = 0,\ -5$.

$a = 0$ is rejected since $a, b \in R \setminus \{0\}$.

Substitute $a = -5$ into equation (2): $b = 15$.

**Answer:** $a = -5$, $b = 15$. **[A1]**

**Question 7**

The direction of motion is given by the direction of the velocity vector.

$\underset{\sim}{v} = \dfrac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}} = 60\underset{\sim}{i} - 80\underset{\sim}{j} - 8\underset{\sim}{k}$. **[A1]**

$$\tan(\theta) = \frac{|-8\underset{\sim}{k}|}{\left|60\underset{\sim}{i} - 80\underset{\sim}{j}\right|}$$

$$= \frac{|\underset{\sim}{k} - \text{component}|}{\sqrt{\left(\underset{\sim}{i} - \text{component}\right)^2 + \left(\underset{\sim}{j} - \text{component}\right)^2}}$$

$$= \frac{8}{\sqrt{(60)^2 + (-80)^2}} = \frac{8}{\sqrt{10{,}000}} = \frac{8}{100} = \frac{2}{25}.$$

**Answer:** $\dfrac{2}{25}$. **[A1]**

**Question 8**

$p = mv$

therefore $p = 5v$

therefore the value of $v$ when $x = \frac{5}{2}$ is required

therefore $v = v(x)$ is required.

$a = \sqrt{4 - v^2}$

$\Rightarrow v\frac{dv}{dx} = \sqrt{4 - v^2}$ **[M1]**

$\Rightarrow \frac{dv}{dx} = \frac{\sqrt{4 - v^2}}{v}$

$\Rightarrow \frac{dx}{dv} = \frac{v}{\sqrt{4 - v^2}}$

$\Rightarrow x = -\sqrt{4 - v^2} + C$ **[M1]**

either by recognition or substitution.

Substitute $v = 0$ when $x = 2$ to find $C$:

$2 = -\sqrt{4} + C$

$\Rightarrow C = 4$.

Therefore $x = -\sqrt{4 - v^2} + 4$. **[M1]**

Substitute $x = \frac{5}{2}$:

$\frac{5}{2} = -\sqrt{4 - v^2} + 4$

$\Rightarrow v = \pm\frac{\sqrt{7}}{2}$.

**Answer:** $|p| = \frac{5\sqrt{7}}{2}$ kg m/s. **[A1]**

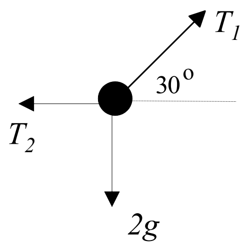
**Units are not required.**

**Question 9**

Let $T_1$ be the tension in the wire attached to the roof at $A$.

Let $T_2$ be the tension in the wire attached to the wall at $B$.

Weight force $= 2g$.




Resolve forces acting on the object in the vertical and horizontal directions.
Take the upwards direction as positive.

**Vertical direction:** $2a = T_1 \sin(30^\circ) - 2g$

$\Rightarrow 2a = \frac{T_1}{2} - 2g$. …. (1) **[A1]**

**Horizontal direction:** $0 = T_2 - T_1 \cos(30^\circ)$

$\Rightarrow T_2 = \frac{\sqrt{3}}{2} T_1$. …. (2) **[A1]**

From equation (2) it follows that $T_2 < T_1$.
It follows that if the wire attached to the roof at $A$ breaks then both wires will break.
It is therefore sufficient to find the maximum acceleration so that the wire attached to the ceiling does not break.
The restriction $T_1 < 9g$ is therefore required.

Substitute $T_1 < 9g$ into equation (1):

$$2a < \frac{9g}{2} - 2g$$

$$\Rightarrow a < \frac{5g}{4}.$$

**Answer:** $\frac{5g}{4}$ m/s$^2$. **[A1]**

**Units are not required.**