# Year 2015

# VCE

# Specialist Mathematics

# Trial Examination 2

# Solutions


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**SECTION 1**

# ANSWERS

| | | | | | |
|---|---|---|---|---|---|
| 1 | A | B | C | **D** | E |
| 2 | **A** | B | C | D | E |
| 3 | A | B | C | D | **E** |
| 4 | A | B | **C** | D | E |
| 5 | A | **B** | C | D | E |
| 6 | A | **B** | C | D | E |
| 7 | **A** | B | C | D | E |
| 8 | A | B | **C** | D | E |
| 9 | A | B | **C** | D | E |
| 10 | A | B | C | **D** | E |
| 11 | A | B | C | **D** | E |
| 12 | A | B | C | D | **E** |
| 13 | A | B | C | **D** | E |
| 14 | A | B | C | D | **E** |
| 15 | **A** | B | C | D | E |
| 16 | A | B | **C** | D | E |
| 17 | A | **B** | C | D | E |
| 18 | **A** | B | C | D | E |
| 19 | A | B | C | **D** | E |
| 20 | A | B | C | D | **E** |
| 21 | A | **B** | C | D | E |
| 22 | A | B | **C** | D | E |

## SECTION 1

**Question 1** **Answer D**

The hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $a>0$ and $b>0$ has asymptotes

$y=\pm\frac{b}{a}(x-h)+k$ One asymptote is $y=-2x+4$, so that $\frac{b}{a}=2$. Taking the negative sign,

$y=-\frac{b}{a}x+k+\frac{b}{a}h$, so that $k+\frac{b}{a}h=k+2h=4$, only $h=1$ and $k=2$ satisfies $k+2h=4$

**Question 2** **Answer A**

$x^2+2px+4y^2-4qy=r^2$, completing the square

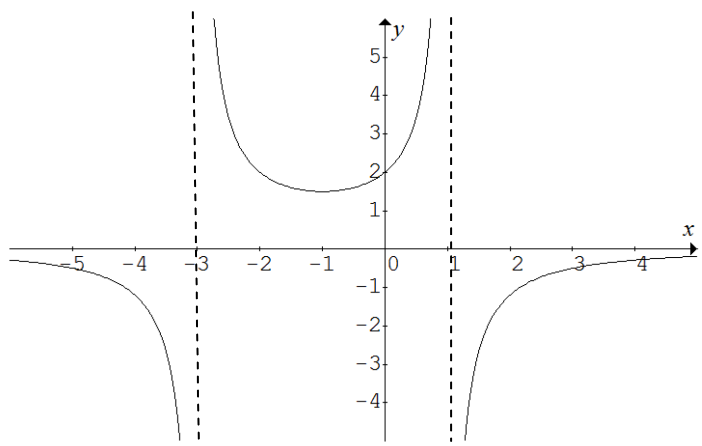
$$\left(x^2+2px+p^2\right)+4\left(y^2-qy+\frac{q^2}{4}\right)=r^2+p^2+q^2$$

$$(x+p)^2+4\left(y-\frac{q}{2}\right)^2=r^2+p^2+q^2 \quad \text{so that} \quad \frac{(x+p)^2}{r^2+p^2+q^2}+\frac{\left(y-\frac{q}{2}\right)^2}{\left(\frac{r^2+p^2+q^2}{4}\right)}=1$$

centre is $\left(-p,\frac{q}{2}\right)$ and $b^2=\frac{r^2+p^2+q^2}{4}$ $\quad b=\frac{\sqrt{r^2+p^2+q^2}}{2}$

**Question 3** **Answer E**

since $x=-3$ is a vertical asymptote, the denominator contains $(x+3)$




$$y=\frac{A}{3+bx-x^2}=\frac{A}{-\left(x^2-bx-3\right)}$$

$$=\frac{A}{-(x+3)(x-1)}=\frac{A}{3-2x-x^2}$$

so $b=-2$. Since it crosses the $y$-axis at

$y=2=\frac{A}{3}$ then $A=6$.

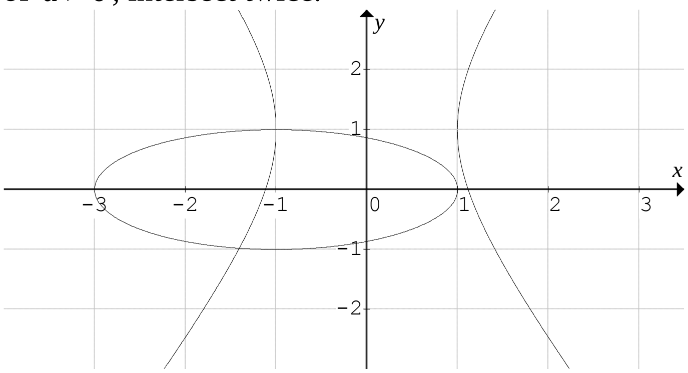
The line $x=1$ is also a vertical asymptote.

The $x$-axis, $y=0$ is a horizontal asymptote.

$y=\frac{6}{(x+3)(1-x)}=\frac{6}{4-(x+1)^2}$, when $x=-1$ $y=\frac{3}{2}$ is a minimum stationary point.

**Question 4** **Answer C**

The ellipse $\frac{(x+a)^2}{4a^2}+\frac{y^2}{a^2}=1$ and the hyperbola $\frac{x^2}{a^2}-\frac{(y-a)^2}{4a^2}=1$, when $a=1$, or any value of $a>0$, intersect twice.




**Question 5** **Answer B**

The range of $y=\sin^{-1}(x)$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

The range of $y=\frac{2a}{\pi}\sin^{-1}\left(\frac{x}{a}\right)$ is $\frac{2a}{\pi}\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]=[-a,a]$

**Question 6** **Answer B**

$\underset{\sim}{r}(t)=4\sin(t)\underset{\sim}{i}+\cos(2t)\underset{\sim}{j}$, for $t\geq 0$.

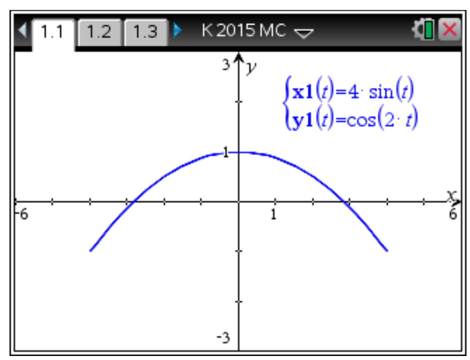
The parametric equations are

$x=4\sin(t)$ and $y=\cos(2t)$.

$y=\cos(2t)=1-2\sin^2(t)=1-2\left(\frac{x}{4}\right)^2$

so that $y=1-\frac{x^2}{8}$, since $t\geq 0$ $x\in[-4,4]$

The particle moves on part of a parabola.




**Question 7** **Answer A**

$z=\pm\sqrt{a}\,i$ and $z=\pm\sqrt{b}$ are roots, so that $(z^2+a)$ and $(z^2-b)$ are both factors,

expanding $P(z)=(z^2+a)(z^2-b)$, gives $P(z)=z^4+(a-b)z^2-ab$

**Question 8** **Answer C**

$z=\sqrt{a}+\sqrt{2}i$, then $|z|=\sqrt{\left(\sqrt{a}\right)^2+\left(\sqrt{2}\right)^2}=\sqrt{a+2}$. Now $|z|^3=27\Rightarrow|z|=\sqrt[3]{27}=3$

$\sqrt{a+2}=3\ \Rightarrow a+2=9$ so that $a=7$.

**Question 9** **Answer C**

$z = a\,\text{cis}\left(\frac{\pi}{b}\right)$ the conjugate is the reflection in the real axis, so that $\bar{z} = a\,\text{cis}\left(-\frac{\pi}{b}\right)$.

The reciprocal of the conjugate, $\frac{1}{\bar{z}} = \frac{1}{a}\text{cis}\left(\frac{\pi}{b}\right)$

**Question 10** **Answer D**

$\frac{dy}{dx} = \cos\left(x^2\right) \quad \Rightarrow \quad y = \int_0^x \cos\left(u^2\right)du + c$

when $y = 1$, $x = 1$

$1 = \int_0^1 \cos\left(u^2\right)du + c \;\Rightarrow\; c = 1 - \int_0^1 \cos\left(u^2\right)du$

$y = \int_0^x \cos\left(u^2\right)du + 1 - \int_0^1 \cos\left(u^2\right)du$

when $x = 2$

$y = \int_0^2 \cos\left(u^2\right)du + \int_1^0 \cos\left(u^2\right)du + 1$ by properties of definite integrals

$y = \int_1^2 \cos\left(u^2\right)du + 1$

**Question 11** **Answer D**

$\underset{\sim}{r}(t) = 15t\sqrt{2}\,\underset{\sim}{i} + \left(15t\sqrt{2} - 4.9t^2\right)\underset{\sim}{k}$ for $t \geq 0$

$\underset{\sim}{r}(t) = Vt\cos(\alpha)\underset{\sim}{i} + \left(Vt\sin(\alpha) - \frac{1}{2}gt^2\right)\underset{\sim}{k}$

$V\cos(\alpha) = 15\sqrt{2}$ and $V\sin(\alpha) = 15\sqrt{2}$

so that $\tan(\alpha) = 1 \;\Rightarrow\; \alpha = 45^0$ and $V = 30\text{ m/s}$

time of flight $T = \frac{2V\sin(\alpha)}{g} = \frac{2 \times 30\sin\left(45^0\right)}{9.8} = 4.33$ seconds

maximum height $H = \frac{V^2\sin^2(\alpha)}{2g} = \frac{30^2\sin^2\left(45^0\right)}{2 \times 9.8} = 22.96$ metres

the range $R = \frac{V^2\sin(2\alpha)}{g} = \frac{30^2\sin\left(90^0\right)}{9.8} = 91.84$ metres

Only Colin and David are correct.

**Question 12** **Answer E**

the gradient of the normal is $M_N = 2\sqrt{m}$ where $m = \dfrac{y-1}{x+1}$

$$-\frac{dx}{dy} = 2\sqrt{\frac{y-1}{x+1}} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}\sqrt{\frac{x+1}{y-1}} \quad \text{or} \quad 2\frac{dy}{dx} + \sqrt{\frac{x+1}{y-1}} = 0$$

**Question 13** **Answer D**

The area is below the $x$-axis, the area is $A = -\int_0^1 \dfrac{x^2-1}{\sqrt{3x+1}}\,dx$

Let $u = 3x+1$ , $\dfrac{du}{dx} = 3 \Rightarrow dx = \dfrac{1}{3}du$ and $x = \dfrac{1}{3}(u-1)$ $x^2 = \dfrac{1}{9}\left(u^2-2u+1\right)$

$$x^2-1 = \frac{1}{9}\left(u^2-2u+1\right)-1 = \frac{1}{9}\left(u^2-2u+1-9\right) = \frac{1}{9}\left(u^2-2u-8\right)$$

$$= \frac{1}{9}(u-4)(u-2)$$

terminals, when $x=1$ $u=4$ and when $x=0$ $u=1$, then

$$A = -\frac{1}{27}\int_1^4 \frac{(u-4)(u-2)}{\sqrt{u}}\,du = \frac{1}{27}\int_1^4 \frac{(4-u)(u+2)}{\sqrt{u}}\,du$$

**Question 14** **Answer E**

along the $y$-axis, when $x=0$ , $m = -\dfrac{1}{2}$,

along the $x$-axis when $y=0$ , $m=1$,

when $x=-y$ , $m=0$,

when $x=2y$ , $(2,1)$ , $(-2,-1)$ , $(4,2)$ , $(-4,-2)$ the gradient $m$ is infinite,

only $m = \dfrac{dy}{dx} = \dfrac{x+y}{x-2y}$ satisfies these conditions.

**Question 15** **Answer A**

$$\underset{\sim}{r}(t) = 4\sin(t)\underset{\sim}{i} + \cos(2t)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}(t) = 4\cos(t)\underset{\sim}{i} - 2\sin(2t)\underset{\sim}{j}$$

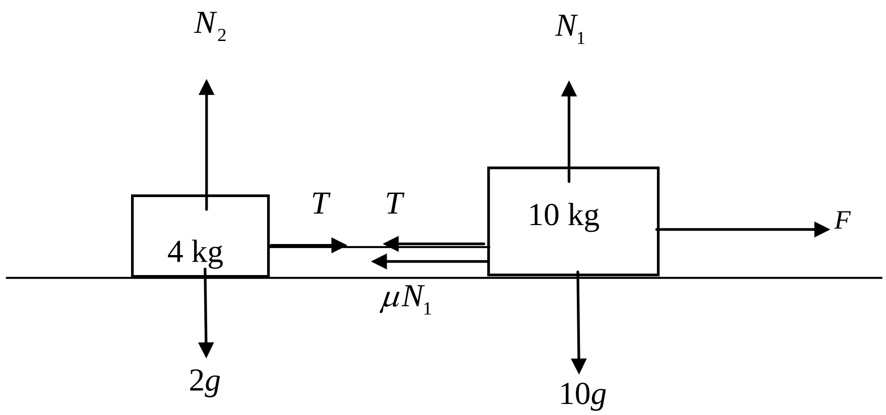
$$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{\left(4\cos(t)\right)^2 + \left(-2\sin(2t)\right)^2} = \sqrt{16\cos^2(t)+4\sin^2(2t)}$$

$$= \sqrt{16\cos^2(t)+4\left(2\sin(t)\cos(t)\right)^2} = \sqrt{16\cos^2(t)+16\sin^2(t)\cos^2(t)}$$

$$= \sqrt{16\cos^2(t)\left(1+\sin^2(t)\right)} = \sqrt{16\cos^2(t)\left(2-\cos^2(t)\right)}$$

when $\cos(t)=1$ $\left|\dot{\underset{\sim}{r}}(t)\right|_{\max} = 4$ , $m=2$ $p_{\max} = mv = 8$ kg m/s

**Question 16** **Answer C**




Resolving horizontally around the 10 kg mass, $(1)\ F - T - \mu N_1 = 10a$

Resolving vertically around the 10 kg mass, $(2)\ N_1 - 10g = 0 \Rightarrow N_1 = 10g$

Resolving horizontally around the 4 kg mass, $(3)\ T = 4a$

substituting $\mu = 0.5$ , $N_1 = 10g$ $T = 4a$

$(1)$ becomes $F - 4a - 5g = 10a$ or $(1)$ becomes $F = 14a + 5g = 14a + 49$

If $F = 50 = 14a + 49 \Rightarrow a = \dfrac{1}{14}$

If $F = 49 \Rightarrow a = 0$ in limiting equilibrium, or the boxes are on the point of moving.

If $F < 49$ the boxes are not on the point of moving. **C** is false.

**Question 17** **Answer B**

Resolving in the east direction $(1)\ 10\cos\left(40^0\right) + 5\cos\left(50^0\right) - F_3\cos(\theta) = 0$

Resolving in the north direction $(2)\ 10\sin\left(40^0\right) - 5\sin\left(50^0\right) - F_3\sin(\theta) > 0$

$(1) \Rightarrow\ F_3\cos(\theta) = 10\cos\left(40^0\right) + 5\cos\left(50^0\right) = 10.874$

$(2) \Rightarrow F_3\sin(\theta) < 10\sin\left(40^0\right) - 5\sin\left(50^0\right) = 2.598$

**Question 18** **Answer A**

To find when the ball hits the ground, use constant acceleration formulae.

$u = 4$ , $s = -1.6$ , $a = -9.8$ , $t = ?$ using $s = ut + \frac{1}{2}at^2$ gives $-1.6 = 4t - 4.9t^2$ ,

solving since $t > 0$ gives $t = 1.11$ seconds.

**Question 19** **Answer D**

$$\dot{\underset{\sim}{r}}(t)=4e^{\frac{t}{2}}\underset{\sim}{i}-2\sin(2t)\underset{\sim}{j}$$

$$\underset{\sim}{r}(t)=\int 4e^{\frac{t}{2}}dt\,\underset{\sim}{i}-\int 2\sin(2t)dt\,\underset{\sim}{j}$$

$$\underset{\sim}{r}(t)=8e^{\frac{t}{2}}\underset{\sim}{i}+\cos(2t)\underset{\sim}{j}+\underset{\sim}{c} \quad \text{now} \quad \underset{\sim}{r}(0)=3\underset{\sim}{i}$$

$$3\underset{\sim}{i}=8\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{c} \;\Rightarrow\; \underset{\sim}{c}=-5\underset{\sim}{i}-\underset{\sim}{j}$$

$$\underset{\sim}{r}(t)=8e^{\frac{t}{2}}\underset{\sim}{i}+\cos(2t)\underset{\sim}{j}+\left(-5\underset{\sim}{i}-\underset{\sim}{j}\right)$$

$$\underset{\sim}{r}(t)=\left(8e^{\frac{t}{2}}-5\right)\underset{\sim}{i}+\left(\cos(2t)-1\right)\underset{\sim}{j}$$

**Question 20** **Answer E**

$$a=\frac{d\left(\frac{1}{2}v^2\right)}{dx}=x\cos(x)$$

$$\frac{1}{2}v^2=\int x\cos(x)dx=\cos(x)+x\sin(x)+c \text{ by CAS.}$$

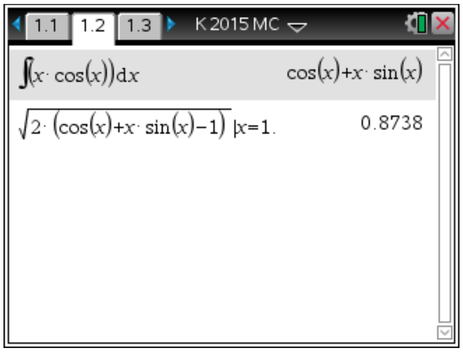
Now initially when $t=0$ , $x=0$ and $v=0$

$$0=1+c \;\Rightarrow\; c=-1$$

$$\frac{1}{2}v^2=\cos(x)+x\sin(x)-1$$

$$v=\sqrt{2\left(\cos(x)+x\sin(x)-1\right)} \quad \text{when} \quad x=1$$

$$v=\sqrt{2\left(\cos(1)+\sin(1)-1\right)}\approx 0.87$$



**Question 21** **Answer B**

$y_1 = \frac{\pi x}{8} \Rightarrow x_1 = \frac{8y}{\pi}$ and

$y_2 = \sin^{-1}\left(\frac{x}{4}\right) \Rightarrow x_2 = 4\sin(y)$,

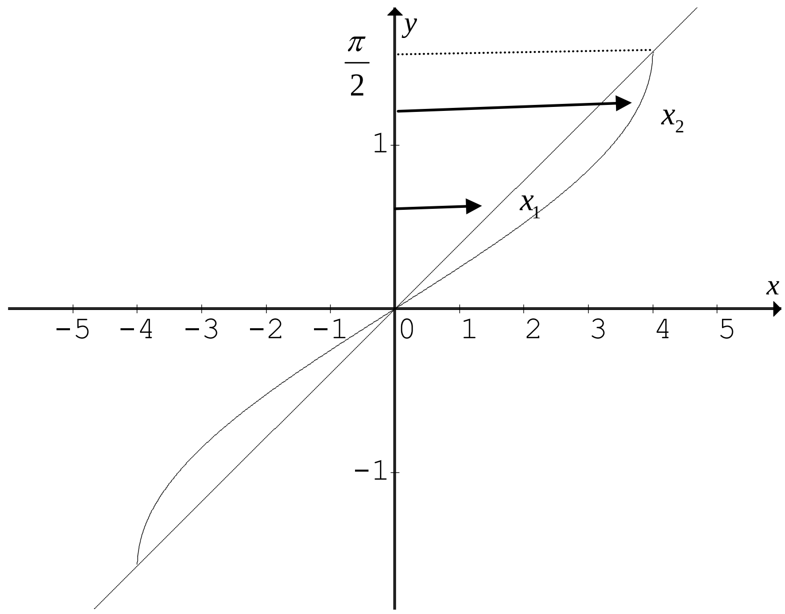
the curves intersect at $x = 4,\ y = \frac{\pi}{2}$

The volume is $V = \pi\int_a^b \left(x_2^2 - x_1^2\right)dy$

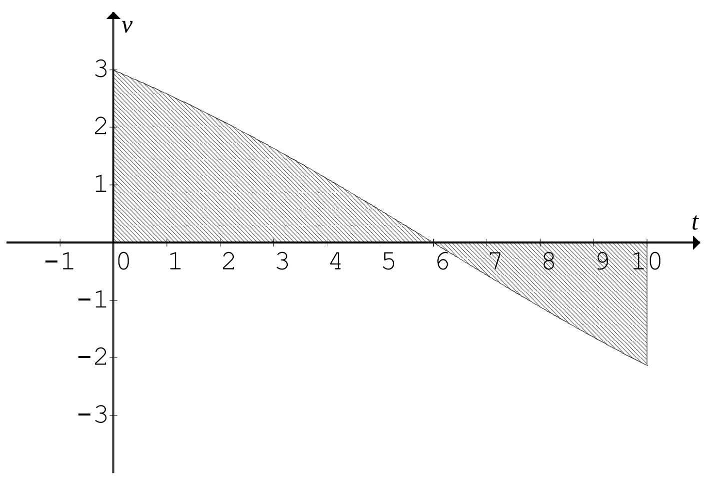
$$V = \pi\int_0^{\frac{\pi}{2}} \left(16\sin^2(y) - \frac{64y^2}{\pi^2}\right)dy$$

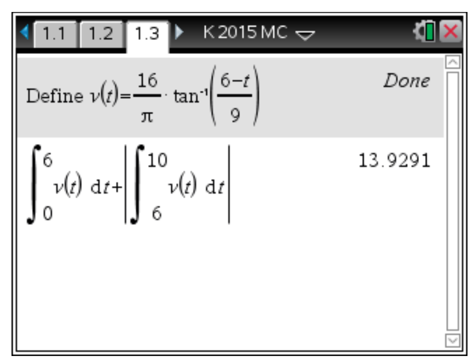
but $y$ is a dummy variable

$$V = \pi\int_0^{\frac{\pi}{2}} \left(16\sin^2(x) - \frac{64x^2}{\pi^2}\right)dx$$




**Question 22** **Answer C**




The distance travelled is $\int_0^6 \frac{16}{\pi}\tan^{-1}\left(\frac{6-t}{9}\right)dt + \left|\int_6^{10} \frac{16}{\pi}\tan^{-1}\left(\frac{6-t}{9}\right)dt\right| \approx 14$

**END OF SECTION 1 SUGGESTED ANSWER**

**SECTION 2**

**Question 1**

**a.i.** $x = 2\left(t - \sin(t)\right)$ $\quad y = 2\left(1 - \cos(t)\right)$

$\dot{x} = \dfrac{dx}{dt} = 2\left(1 - \cos(t)\right)$ $\quad \dot{y} = \dfrac{dy}{dt} = 2\sin(t)$ A1

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{\dot{y}}{\dot{x}}$

$\dfrac{dy}{dx} = \dfrac{2\sin(t)}{2\left(1-\cos(t)\right)} = \dfrac{\sin(t)}{1-\cos(t)}$

$= \dfrac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)} = \dfrac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}$ M1

$= \cot\left(\dfrac{t}{2}\right)$

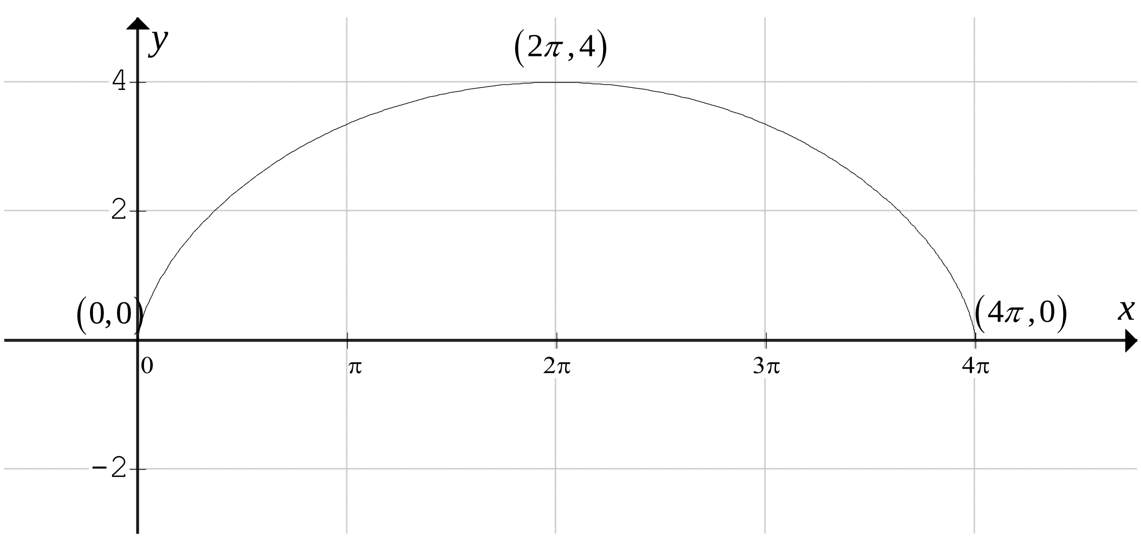
**ii.** gradient is zero, $\dfrac{dy}{dx} = \dfrac{\sin(t)}{1-\cos(t)} = 0 \Rightarrow \sin(t) = 0$ and $\cos(t) \neq 1$ M1

only solution in $t \in [0, 2\pi]$ is $t = \pi$

$x(\pi) = 2\left(\pi - \sin(\pi)\right) = 2\pi$ , $y(\pi) = 2\left(1 - \cos(\pi)\right) = 4$

turning point at $(2\pi, 4)$ A1

**b.** correct graph, shape, restricted domain, endpoints, $(0,0)$, $(4\pi, 0)$ and maximum turning point $(2\pi, 4)$ G2



**c.** $\underset{\sim}{r}(t)=2(t-\sin(t))\underset{\sim}{i}+2(1-\cos(t))\underset{\sim}{j}=x(t)\underset{\sim}{i}+y(t)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t)=\dot{x}(t)\underset{\sim}{i}+\dot{y}(t)\underset{\sim}{j}$ the speed is given by $|\dot{r}(t)|=\sqrt{\dot{x}^2+\dot{y}^2}$

$$|\dot{r}(t)|=\sqrt{(2(1-\cos(t)))^2+(2\sin(t))^2}$$

$$=\sqrt{4(1-2\cos(t)+\cos^2(t))+4\sin^2(t)}$$ M1

$$=\sqrt{4+4(\cos^2(t)+\sin^2(t))-8\cos(t)}$$

$$=\sqrt{8-8\cos(t)}$$

$$=\sqrt{8(1-\cos(t))}$$

$$=\sqrt{8\times 2\sin^2\left(\frac{t}{2}\right)} \quad \text{since } 0\le t\le 2\pi$$ A1

$$=4\sin\left(\frac{t}{2}\right)$$

$u=4$ , $k=\frac{1}{2}$ A1

**d.** $A=\int_0^{4\pi} y\,dx$

$y=2(1-\cos(t))$ , $x=2(t-\sin(t))$ , $dx=2(1-\cos(t))dt$

when $t=0$ $x=0$ , $x=4\pi \Rightarrow t=2\pi$

$$A=\int_0^{2\pi} 4(1-\cos(t))^2\,dt=4\int_0^{2\pi}\left(2\sin^2\left(\frac{t}{2}\right)\right)^2 dt$$

$$=16\int_0^{2\pi}\sin^4\left(\frac{t}{2}\right)dt$$ M1

$c=16$ , $n=4$ and $k=\frac{1}{2}$ A1

**e.** $V=\pi\int_0^{4\pi} y^2\,dx$

$$V=\pi\int_0^{2\pi} 8(1-\cos(t))^3\,dt=8\pi\int_0^{2\pi}\left(2\sin^2\left(\frac{t}{2}\right)\right)^3 dt$$

$$=64\pi\int_0^{2\pi}\sin^6\left(\frac{t}{2}\right)dt$$

$p=64\pi$ and $m=6$ A1

**Question 2**

**a.** Given that $\sin\left(\frac{2\pi}{5}\right)=\frac{1}{4}\left(\sqrt{2\left(5+\sqrt{5}\right)}\right)$

$$\cos^2\left(\frac{2\pi}{5}\right)=1-\sin^2\left(\frac{2\pi}{5}\right)=1-\left(\frac{1}{4}\left(\sqrt{2\left(5+\sqrt{5}\right)}\right)\right)^2$$

M1

$$=1-\frac{1}{16}\left(2\left(5+\sqrt{5}\right)\right)=\frac{1}{16}\left(16-2\left(5+\sqrt{5}\right)\right)$$

$$=\frac{1}{16}\left(6-2\sqrt{5}\right)=\frac{1}{16}\left(5-2\sqrt{5}+1\right)$$

$$=\left(\frac{1}{4}\left(\sqrt{5}-1\right)\right)^2 \text{ since } \cos\left(\frac{2\pi}{5}\right)>0 \text{ and } \sqrt{5}-1>0$$ A1

$$\cos\left(\frac{2\pi}{5}\right)=\frac{1}{4}\left(\sqrt{5}-1\right)$$

**b.i.** $u=\frac{1}{2}\left(\sqrt{5}-1\right)+\frac{1}{2}\left(\sqrt{2\left(5+\sqrt{5}\right)}\right)i=2\cos\left(\frac{2\pi}{5}\right)+2\sin\left(\frac{2\pi}{5}\right)i$

$u=2\operatorname{cis}\left(\frac{2\pi}{5}\right)$ A1

**ii.** $\operatorname{Arg}\left(u^3\right)=3\times\frac{2\pi}{5}-2\pi$

$\operatorname{Arg}\left(u^3\right)=-\frac{4\pi}{5}$ A1

**c.** $\left(\left(\sqrt{5}-1\right)+\left(\sqrt{2\left(5+\sqrt{5}\right)}\right)i\right)^n$

$$=\left(4\cos\left(\frac{2\pi}{5}\right)+4i\sin\left(\frac{2\pi}{5}\right)\right)^n=4^n\left(\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^n=4^n\operatorname{cis}\left(\frac{2n\pi}{5}\right)$$

$=4^n\left(\cos\left(\frac{2n\pi}{5}\right)+i\sin\left(\frac{2n\pi}{5}\right)\right)$ is a real number, M1

so that the imaginary part must be zero $\sin\left(\frac{2n\pi}{5}\right)=0$

$$\frac{2n\pi}{5}=k\pi$$

$n=\frac{5k}{2}$ where $k\in Z$ A1

**d.** $z^5 - 32 = 0$

$z^2 = 32$

$z^5 = 32\text{cis}(2k\pi)$

$z = \sqrt[5]{32}\text{cis}\left(\frac{2k\pi}{5}\right)$

$= 2\,\text{cis}\left(\frac{2k\pi}{5}\right)$

$k = 0 \qquad z_1 = 2\,\text{cis}(0) = 2$ M1

$k = 1 \qquad z_2 = 2\,\text{cis}\left(\frac{2\pi}{5}\right) = \frac{1}{2}\left(\left(\sqrt{5}-1\right) + \left(\sqrt{2\left(5+\sqrt{5}\right)}\right)i\right)$

$k = -1 \qquad z_3 = 2\,\text{cis}\left(-\frac{2\pi}{5}\right) = \frac{1}{2}\left(\left(\sqrt{5}-1\right) - \left(\sqrt{2\left(5+\sqrt{5}\right)}\right)i\right)$ A1

$k = 2 \qquad z_4 = 2\,\text{cis}\left(\frac{4\pi}{5}\right) = \frac{1}{2}\left(-\left(\sqrt{5}+1\right) + \left(\sqrt{2\left(5-\sqrt{5}\right)}\right)i\right)$

$k = -2 \qquad z_5 = 2\,\text{cis}\left(-\frac{4\pi}{5}\right) = -\frac{1}{2}\left(\left(\sqrt{5}+1\right) + \left(\sqrt{2\left(5-\sqrt{5}\right)}\right)i\right)$ A1

there are 5 roots, they form the sides of a regular pentagon ( 5 sided figure )

all the roots are equally spaced by $\frac{2\pi}{5}$ or $72^0$ around a circle of radius 2,

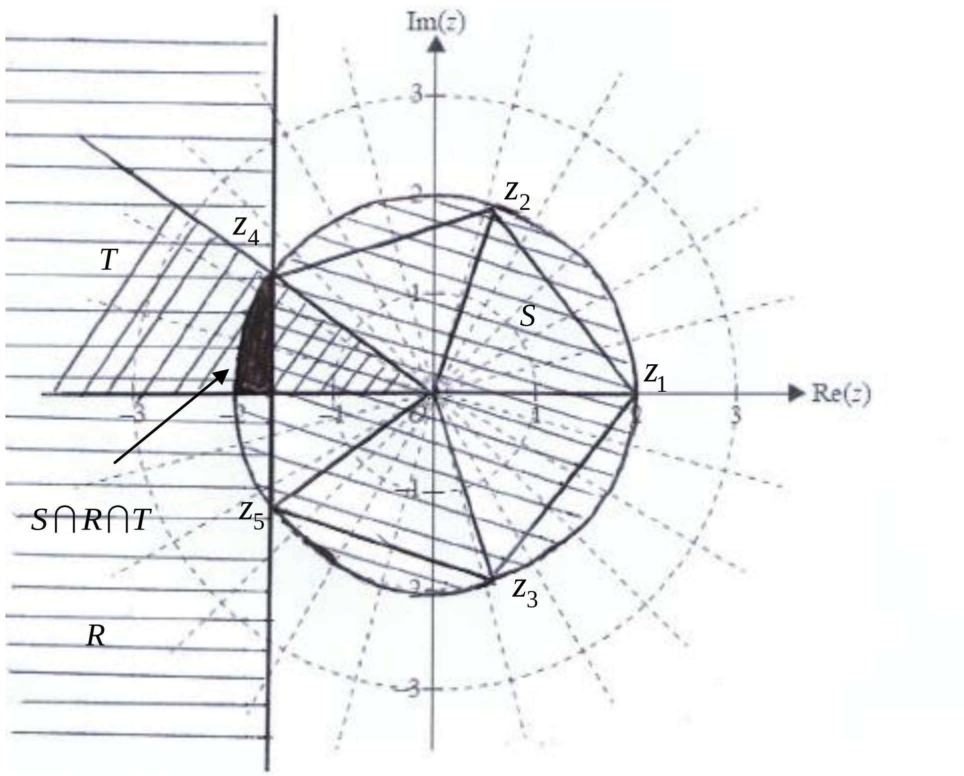
there is one real root and two pairs of complex conjugates, $z_3 = \overline{z}_2$ and $z_5 = \overline{z}_4$

**e.** $S = \{z : |z| \le 2\}$ is the inside of a circle, including the boundary with centre at the origin and radius 2.

$R = \{z : 2\,\text{Re}(z) + \left(\sqrt{5}+1\right) \le 0\ \} \Rightarrow \text{Re}(z) \le -\frac{1}{2}\left(\sqrt{5}+1\right) \approx -1.618$, shaded region to the left of a line parallel to the imaginary axis joining the roots $z_4$ and $z_5$ A1

$T = \{z : \text{Arg}(z) \ge \frac{4\pi}{5}\} \Rightarrow \frac{4\pi}{5} \le \text{Arg}(z) \le \pi$ the wedge from $144^0$ at the point $z_4$ to the real axis.

$S \cap R \cap T$ is the shaded part of the segment. A1




**f.** half the area of a segment, of angle $72^0$ or $\frac{2\pi}{5}$ and radius 2

$$A = \frac{1}{2}\left(\frac{1}{2}r^2\left(\theta - \sin(\theta)\right)\right)$$

$$A = \frac{1}{4} \times 2^2\left(\frac{2\pi}{5} - \sin\left(\frac{2\pi}{5}\right)\right)$$

$$A = \frac{2\pi}{5} - \frac{1}{4}\left(\sqrt{2\left(5+\sqrt{5}\right)}\right)$$ A1

**Question 3**

**a.i.** $\overrightarrow{PG} = \overrightarrow{PO} + \overrightarrow{OG}$

Since $P$ is the midpoint of $OA$,

$$= -\frac{1}{2}\overrightarrow{OA} + \overrightarrow{OG}$$

$$= \underset{\sim}{g} - \frac{1}{2}\underset{\sim}{a}$$

$\overrightarrow{QG} = \overrightarrow{QO} + \overrightarrow{OG}$

Since $Q$ is the midpoint of $OB$

$$= -\frac{1}{2}\overrightarrow{OB} + \overrightarrow{OG}$$

$$= \underset{\sim}{g} - \frac{1}{2}\underset{\sim}{b}$$ A1

Since $\overrightarrow{PG}$ is perpendicular to $\overrightarrow{OA}$, $\overrightarrow{PG}.\overrightarrow{OA} = 0$

$$\left(\underset{\sim}{g} - \frac{1}{2}\underset{\sim}{a}\right).\underset{\sim}{a} = 0$$

$$\underset{\sim}{g}.\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{a}.\underset{\sim}{a} = 0$$ M1

$$\underset{\sim}{g}.\underset{\sim}{a} = \frac{1}{2}|\underset{\sim}{a}|^2$$

Similarly since $\overrightarrow{QG}$ is perpendicular to $\overrightarrow{OB}$, $\overrightarrow{QG}.\overrightarrow{OB} = 0$

$$\left(\underset{\sim}{g} - \frac{1}{2}\underset{\sim}{b}\right).\underset{\sim}{b} = 0$$

$$\underset{\sim}{g}.\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{b}.\underset{\sim}{b} = 0$$ A1

$$\underset{\sim}{g}.\underset{\sim}{b} = \frac{1}{2}|\underset{\sim}{b}|^2$$

**ii.** $\overrightarrow{RG} = \overrightarrow{RA} + \overrightarrow{AP} + \overrightarrow{PG}$

$$= \frac{1}{2}\overrightarrow{BA} + \overrightarrow{AP} + \overrightarrow{PG}$$ Since $R$ is the midpoint of $AB$ M1

$$= \frac{1}{2}\left(\underset{\sim}{a} - \underset{\sim}{b}\right) - \frac{1}{2}\underset{\sim}{a} + \left(\underset{\sim}{g} - \frac{1}{2}\underset{\sim}{a}\right)$$

$$= \underset{\sim}{g} - \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$$ A1

Consider $\overrightarrow{RG}.\overrightarrow{AB} = \left(\underset{\sim}{g} - \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)\right).\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$

$$= \underset{\sim}{g}.\underset{\sim}{b} - \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right).\left(\underset{\sim}{b} - \underset{\sim}{a}\right) - \underset{\sim}{g}.\underset{\sim}{a}$$

$$= \underset{\sim}{g}.\underset{\sim}{b} - \frac{1}{2}\left(\underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{b}.\underset{\sim}{b} - \underset{\sim}{a}.\underset{\sim}{a} - \underset{\sim}{b}.\underset{\sim}{a}\right) - \underset{\sim}{g}.\underset{\sim}{a}$$ from **i.** A1

$$= \frac{1}{2}|\underset{\sim}{b}|^2 - \frac{1}{2}|\underset{\sim}{b}|^2 + \frac{1}{2}|\underset{\sim}{a}|^2 - \frac{1}{2}|\underset{\sim}{a}|^2 = 0$$

so therefore $\overrightarrow{RG}$ is perpendicular to $\overrightarrow{AB}$ A1

**b.** If the vectors $\underset{\sim}{u}$ , $\underset{\sim}{v}$ and $\underset{\sim}{w}$ form a linearly dependant set of vectors.

$\underset{\sim}{w} = \alpha\underset{\sim}{u} + \beta\underset{\sim}{v}$

$9\underset{\sim}{i} - 7\underset{\sim}{j} - 8\underset{\sim}{k} = \alpha\left(3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}\right) + \beta\left(-2\underset{\sim}{i} + \underset{\sim}{j} + t\underset{\sim}{k}\right)$

$\underset{\sim}{i} \Rightarrow (1) \quad 9 = 3\alpha - 2\beta$

$\underset{\sim}{j} \Rightarrow (2) -7 = -2\alpha + \beta$ M1

$\underset{\sim}{k} \Rightarrow (3) -8 = -4\alpha + t\beta$

$(1) + 2(2) \Rightarrow \alpha = 5$

substituting into $(1)$

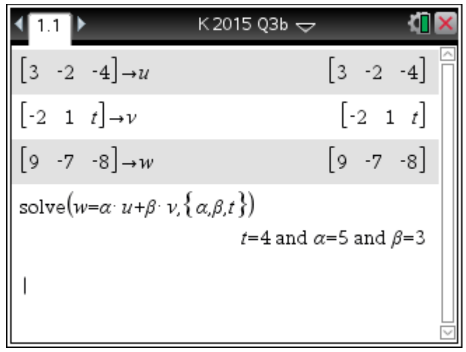
$2\beta = 3\alpha - 9 = 6 \quad \Rightarrow \quad \beta = 3$ M1

so that $\underset{\sim}{w} = 5\underset{\sim}{u} + 3\underset{\sim}{v}$

substituting into $(3)$

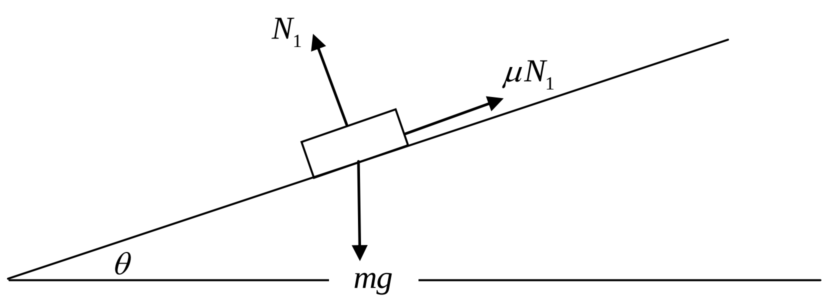
$-8 = -20 + 3t \quad \Rightarrow 3t = 12$

so that $t = 4$ A1




## Question 4

**i.**




resolving parallel and down the plane $(1) \quad mg\sin(\theta) - \mu N_1 = 0$ A1

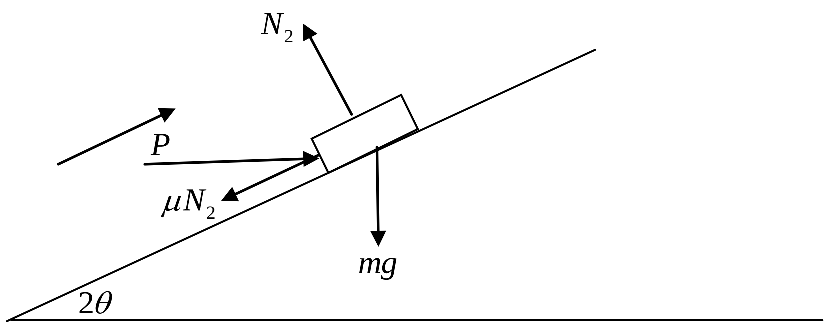
resolving perpendicular to the plane $(2) \quad N_1 - mg\cos(\theta) = 0$

$(2) \Rightarrow \quad N_1 = mg\cos(\theta)$ into $(1) \quad mg\sin(\theta) - \mu mg\cos(\theta) = 0$ A1

$mg\sin(\theta) = \mu mg\cos(\theta)$

$\mu = \tan(\theta)$

**ii.**




$mg$ is the weight force
$N_2$ is the normal reaction
$\mu N_2$ the frictional force A1
$P$ the pushing force

**iii.** resolving parallel and up the plane $(3)\ P\cos(2\theta)-\mu N_2-mg\sin(2\theta)=0$ A1

resolving perpendicular to the plane $(4)\ N_2-P\sin(2\theta)-mg\cos(2\theta)=0$ A1

$(4) \Rightarrow N_2=P\sin(2\theta)+mg\cos(2\theta)$ into $(3)$

$P\cos(2\theta)-\mu(P\sin(2\theta)+mg\cos(2\theta))-mg\sin(2\theta)=0$ M1

$P(\cos(2\theta)-\mu\sin(2\theta))=mg(\sin(2\theta)+\mu\cos(2\theta))$

substitute $\mu=\tan(\theta)$ M1

$P(\cos(2\theta)-\tan(\theta)\sin(2\theta))=mg(\sin(2\theta)+\tan(\theta)\cos(2\theta))$

$$P\left(\cos(2\theta)-\frac{\sin(\theta)\sin(2\theta)}{\cos(\theta)}\right)=mg\left(\sin(2\theta)+\frac{\sin(\theta)\cos(2\theta)}{\cos(\theta)}\right)$$ A1

$$P\left(\frac{\cos(2\theta)\cos(\theta)-\sin(\theta)\sin(2\theta)}{\cos(\theta)}\right)=mg\left(\frac{\sin(2\theta)\cos(\theta)+\sin(\theta)\cos(2\theta)}{\cos(\theta)}\right)$$ M1

$P\cos(3\theta)=mg\sin(3\theta)$

$P=mg\tan(3\theta)$

**iv.** $2mg=mg\tan(3\theta)$

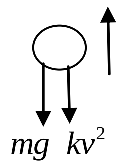
$\tan(3\theta)=2$

$\theta=\frac{1}{3}\tan^{-1}(2)$

$=21^0\,9'$ A1

## Question 5

**a.i.** $m = 58.8 \text{ gm} = 0.0588 \text{ kg} \quad R = kv^2 \quad k = 0.00294$




$$m\ddot{x} = -\left(mg + kv^2\right)$$

$$0.0588\ddot{x} = -\left(0.0588 \times 9.8 + 0.00294v^2\right)$$ M1

$$\ddot{x} = -\left(9.8 + \frac{0.00294v^2}{0.0588}\right) = -\left(9.8 + \frac{v^2}{20}\right) = -\left(\frac{9.8 \times 20 + v^2}{20}\right)$$

$$\ddot{x} = -\frac{\left(196 + v^2\right)}{20}$$

**ii.** Use $\ddot{x} = \dfrac{dv}{dt} = -\dfrac{\left(196 + v^2\right)}{20}$ , $v(0) = 4$

inverting $\dfrac{dt}{dv} = -\dfrac{20}{\left(196 + v^2\right)}$ M1

$$t = \int \frac{-20}{196 + v^2}\, dv$$

$$t = -\frac{20}{\sqrt{196}} \tan^{-1}\left(\frac{v}{\sqrt{196}}\right) + c$$

A1

$$t = -\frac{10}{7} \tan^{-1}\left(\frac{v}{14}\right) + c$$

to find $c$ use $v = 4$ when $t = 0$

$$0 = -\frac{10}{7} \tan^{-1}\left(\frac{4}{14}\right) + c \quad \Rightarrow \quad c = \frac{10}{7} \tan^{-1}\left(\frac{2}{7}\right)$$ A1

$$t = \frac{10}{7}\left(\tan^{-1}\left(\frac{2}{7}\right) - \tan^{-1}\left(\frac{v}{14}\right)\right)$$

$$\frac{7t}{10} = \tan^{-1}\left(\frac{2}{7}\right) - \tan^{-1}\left(\frac{v}{14}\right)$$

$$\tan^{-1}\left(\frac{v}{14}\right) = \tan^{-1}\left(\frac{2}{7}\right) - \frac{7t}{10}$$

$$v = 14\tan\left(\tan^{-1}\left(\frac{2}{7}\right) - \frac{7t}{10}\right)$$ A1

**iii.** When $v = 0$ $t = \dfrac{10}{7} \tan^{-1}\left(\dfrac{2}{7}\right) \approx 0.398$ seconds A1

**iv.** use $\ddot{x} = v\dfrac{dv}{dx} = -\dfrac{\left(196+v^2\right)}{20}$

$\dfrac{dv}{dx} = -\dfrac{\left(196+v^2\right)}{20v}$

inverting $\dfrac{dx}{dv} = \dfrac{-20v}{196+v^2}$ M1

$H = \displaystyle\int_4^0 \frac{-20v}{196+v^2}\,dv + 1.6$ A1

alternatively $v = \dfrac{dx}{dt} = 14\tan\left(\tan^{-1}\left(\dfrac{2}{7}\right) - \dfrac{7t}{10}\right)$ M1

$H = 14\displaystyle\int_0^{0.398} \tan\left(\tan^{-1}\left(\frac{2}{7}\right) - \frac{7t}{10}\right)dt + 1.6$ A1

**v.** $H = 0.785 + 1.6$

$H = 2.385$ metres A1

**b.i.** $\underset{\sim}{r}(t) = 36t\underset{\sim}{i} + 4\sin(\pi t)\underset{\sim}{j} + h\cos(\pi t)\underset{\sim}{k}$

when it hits the ground $h\cos(\pi t) = 0 \Rightarrow \pi t = \dfrac{\pi}{2}$ $T = \dfrac{1}{2} = 0.5$ seconds A1

**ii.** $\underset{\sim}{r}\left(\dfrac{1}{2}\right) = 36\times\dfrac{1}{2}\underset{\sim}{i} + 4\sin\left(\dfrac{\pi}{2}\right)\underset{\sim}{j} + h\cos\left(\dfrac{\pi}{2}\right)\underset{\sim}{k}$

$= 18\underset{\sim}{i} + 4\underset{\sim}{j}$ A1

**iii.** $\dot{\underset{\sim}{r}}(t) = 36\underset{\sim}{i} + 4\pi\cos(\pi t)\underset{\sim}{j} - h\pi\sin(\pi t)\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(0) = 36\underset{\sim}{i} + 4\pi\underset{\sim}{j}$ A1

initial speed $\left|\dot{\underset{\sim}{r}}(0)\right| = \sqrt{36^2 + 16\pi^2} = 38.13$ m/s

$38.13\text{m/s} = \dfrac{38.13\times 60\times 60}{1000} = 137$ km/hr A1

**iv.** when the ball touches the net $\underset{\sim}{r}(t).\underset{\sim}{i} = 36t = 11.89$ so that $t = \dfrac{11.89}{36} = 0.3303$ seconds

$\underset{\sim}{r}(0.3303).\underset{\sim}{k} = h\cos(0.3303\pi) = 1.07$ $h = \dfrac{1.07}{\cos(0.3303\pi)}$

so $h = 2.105$ metres A1

| | |
|---|---|
| $\text{deSolve}\left(v'=\frac{-(v^2+196)}{20} \text{ and } v(0)=4,t,v\right)$ | $\frac{\tan^{-1}\left(\frac{v}{14}\right)}{14}-\frac{\tan^{-1}\left(\frac{2}{7}\right)}{14}=\frac{-t}{20}$ |
| $\text{solve}\left(\frac{\tan^{-1}\left(\frac{v}{14}\right)}{14}-\frac{\tan^{-1}\left(\frac{2}{7}\right)}{14}=\frac{-t}{20},t\right)$ | $t=\frac{-10\cdot\left(\tan^{-1}\left(\frac{v}{14}\right)-\tan^{-1}\left(\frac{2}{7}\right)\right)}{7}$ |
| $\text{solve}\left(t=\frac{-10\cdot\left(\tan^{-1}\left(\frac{v}{14}\right)-\tan^{-1}\left(\frac{2}{7}\right)\right)}{7},v\right)$ | $v=-14\cdot\tan\left(\frac{7\cdot t}{10}-\tan^{-1}\left(\frac{2}{7}\right)\right)$ and $7\cdot t-10\cdot\tan^{-1}\left(\frac{2}{7}\right)\geq-5\cdot\pi$ and $7\cdot t-10\cdot\tan^{-1}\left(\frac{2}{7}\right)\leq5\cdot\pi$ |
| Define $v(t)=-14\cdot\tan\left(\frac{7\cdot t}{10}-\tan^{-1}\left(\frac{2}{7}\right)\right)$ | *Done* |
| $\text{solve}(v(t)=0,t)\mid 0<t<1$ | $t=\frac{10\cdot\tan^{-1}\left(\frac{2}{7}\right)}{7}$ |
| $\text{solve}(v(t)=0,t)\mid 0<t<1$ | $t=0.397571$ |
| $tf:=0.39757094143588$ | 0.397571 |
| $\int_0^{tf} v(t)\,dt$ | 0.784716 |
| $\int_4^0 \frac{-20\cdot u}{196+u^2}\,du$ | 0.784716 |

| | |
|---|---|
| Define $r(t)=[36\cdot t \quad 4\cdot\sin(\pi\cdot t) \quad h\cdot\cos(\pi\cdot t)]$ | *Done* |
| $\text{solve}(h\cdot\cos(\pi\cdot t)=0,t)\mid 0<t<1 \text{ and } h\neq0$ | $t=\frac{1}{2}$ and $h\neq0$ |
| $r\left(\frac{1}{2}\right)$ | $[18 \quad 4 \quad 0]$ |
| $\frac{d}{dt}(r(t))$ | $[36 \quad 4\cdot\pi\cdot\cos(\pi\cdot t) \quad -h\cdot\pi\cdot\sin(\pi\cdot t)]$ |
| $\frac{d}{dt}(r(t))\mid t=0$ | $[36 \quad 4\cdot\pi \quad 0]$ |
| $\frac{\text{norm}([36 \quad 4\cdot\pi \quad 0])\cdot60\cdot60}{1000.}$ | 137.269 |
| $\text{solve}(36\cdot t=11.89,t)$ | $t=0.330278$ |
| $r(0.33027777777778)$ | $[11.89 \quad 3.44474 \quad 0.50829\cdot h]$ |
| $\text{solve}(0.50829\cdot h=1.07,h)$ | $h=2.1051$ |

**END OF SECTION 2 SUGGESTED ANSWERS**