E

Free Exam
for 2006-15 VCE study design

Engage
Education
Foundation

# *Units 3 and 4 Specialist Maths: Exam 2*

*Practice Exam Question and Answer Booklet*

Duration: 15 minutes reading time, 2 hours writing time

Structure of book:

| Section | Number of questions | Number of questions to be answered | Number of marks |
|---|---|---|---|
| A | 22 | 22 | 22 |
| B | 2 | 2 | 58 |
| | | Total | 80 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers and rulers, one bound reference, one approved CAS calculator (memory does not need to be cleared) and, if desired, one scientific calculator.
- Students are not permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied:
- This question and answer booklet of 19 pages.

Instructions:
- You must complete all questions of the examination.
- Write all your answers in the spaces provided in this booklet.

# *Section A – Multiple-choice questions*

## *Instructions*

Answer all questions by circling your choice.
Choose the response that is correct or that best answers the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

## *Questions*

### Question 1

The Cartesian form of the curve described by the parametric equation

$$\begin{cases} x = 5 + 3(\cos(2\text{t}))^2 \\ y = 3\cos(4t) \end{cases}$$

is:

A. $y = 2x - 13,\ x \in [-3, 3]$
B. $y = x^2 - 13x,\ x \in [-3, 3]$
C. $y^2 = 13 - \frac{x^2}{2},\ x \in [5, 8]$
D. $\frac{(x-5)^2}{9} + \frac{y^2}{9} = 1,\ x \in [2, 8]$
E. $y = 2x - 13,\ x \in [5, 8]$

### Question 2

Consider the relation:

$$\frac{(x-3)^2}{4} - \frac{(y+2)^2}{9} = 1$$

Which is correct?

A. The relation has asymptotes given by $y = \pm\frac{9}{4}(x-3) - 2$
B. The relation is an ellipse
C. The relation has gradient $\frac{dy}{dx} = \frac{3(x-3)}{2(y+2)}$
D. The relation has parametric equation $\begin{cases} x = 3 + 2\csc(t) \\ y = 3\cot(t) - 2 \end{cases}$
E. None of the above

### Question 3

The domain of $2\cos^{-1}(3x-5) + \frac{\pi}{4}$ is:

A. $x \in \left[\frac{4}{3}, 2\right]$
B. $x \in \left(\frac{\pi}{4}, \frac{9\pi}{4}\right)$
C. $x \in (\frac{4}{3}, 2)$
D. $x \in \left[\frac{\pi}{4}, \frac{9\pi}{4}\right]$
E. $x \in [-1, 1]$

Question 4

The graph $y = kx + \frac{1}{ax^2+bx+c}$ has asymptotes $x = \frac{-5}{2}$, $x = 4$ and $y = -x$, and a stationary point at $x = -2.777$. The values of $(a, b, c, k)$ are closest to:

A. (-5, 4, 1, -1)
B. (3, 2, 1, -1)
C. (2, -3, -20, -1)
D. (1, 1, 1, 1)
E. (-1, 2, -3, -20)

Question 5

Which of the following defines the annulus in the complex plane lying between the relations $(\boldsymbol{x-2})^2 + (\boldsymbol{y+3})^2 = \boldsymbol{4}$ and $(\boldsymbol{x-2})^2 + (\boldsymbol{y+3})^2 = \boldsymbol{9}$?

A. $\{z\colon\ 4 \leq |z-(2-3i)| \leq 9\}$
B. $\{z\colon\ 2 \leq |z-(2-3i)| \leq 3\}$
C. $\{z\colon\ 2 \leq |z+(2-3i)| \leq 3\}$
D. $\{z\colon\ 4 \leq |z-(2i-3)| \leq 9\}$
E. $\{z\colon\ 2 \leq |z-(2i-3)| \leq 3\}$

Question 6

If $z = 5\text{cis}\left(\frac{7\pi}{9}\right)$, then what is the principle argument of $z^3$?

A. $\frac{\pi}{3}$
B. $\frac{\pi}{6}$
C. $\left(\frac{7\pi}{9}\right)^3$
D. $\frac{7\pi}{3}$
E. 0

Question 7

If $z = \text{cis}(\theta)$, then $(\sin 2\theta)^3$ is equal to:

A. $\frac{-(z^4-1)}{2z^2}i$
B. $\left(\frac{z^4-1}{2z^2}\right)^3$
C. $1-(\cos(2\theta))^3$
D. $\frac{(z^4-1)^3}{8z^6}i$
E. $\frac{1-z^4}{2z^2}i$

### Question 8
All solutions to $z^3 = -4\sqrt{2} + 4\sqrt{2}i, z \in \mathbb{C}$ are:

A. $z = \sqrt{2} + \sqrt{2}i$
B. $z = \frac{\sqrt{6}}{2} - \frac{\sqrt{6}}{2}i$ or $z = -\frac{\sqrt{6}}{2} + \frac{\sqrt{6}}{2}i$ or $z = \sqrt{2} + \sqrt{2}i$
C. $z = 2cis\left(\frac{11\pi}{12}\right)$ or $z = 2cis\left(\frac{\pi}{4}\right)$ or $z = 2cis\left(-\frac{5\pi}{12}\right)$
D. $z = 256\sqrt{2} + 256\sqrt{2}i$
E. $z = \sqrt{2} + \sqrt{2}i$ or $z = -\sqrt{2} - \sqrt{2}i$ or $z = -2$

### Question 9
The definite integral $\int_0^1 \frac{1}{x((\ln(x))^2+2)} dx$ can be simplified to $\int_a^b \frac{1}{u^2+2} du$ where:

A. $u = (\ln(x))^2, a = -\infty, b = 0$
B. $u = \ln(x), a = \infty, b = 0$
C. $u = \ln(x), a = -\infty, b = 0$
D. $u = (\ln(x))^2, a = \infty, b = 1$
E. $u = \ln(x), a = \infty, b = 1$

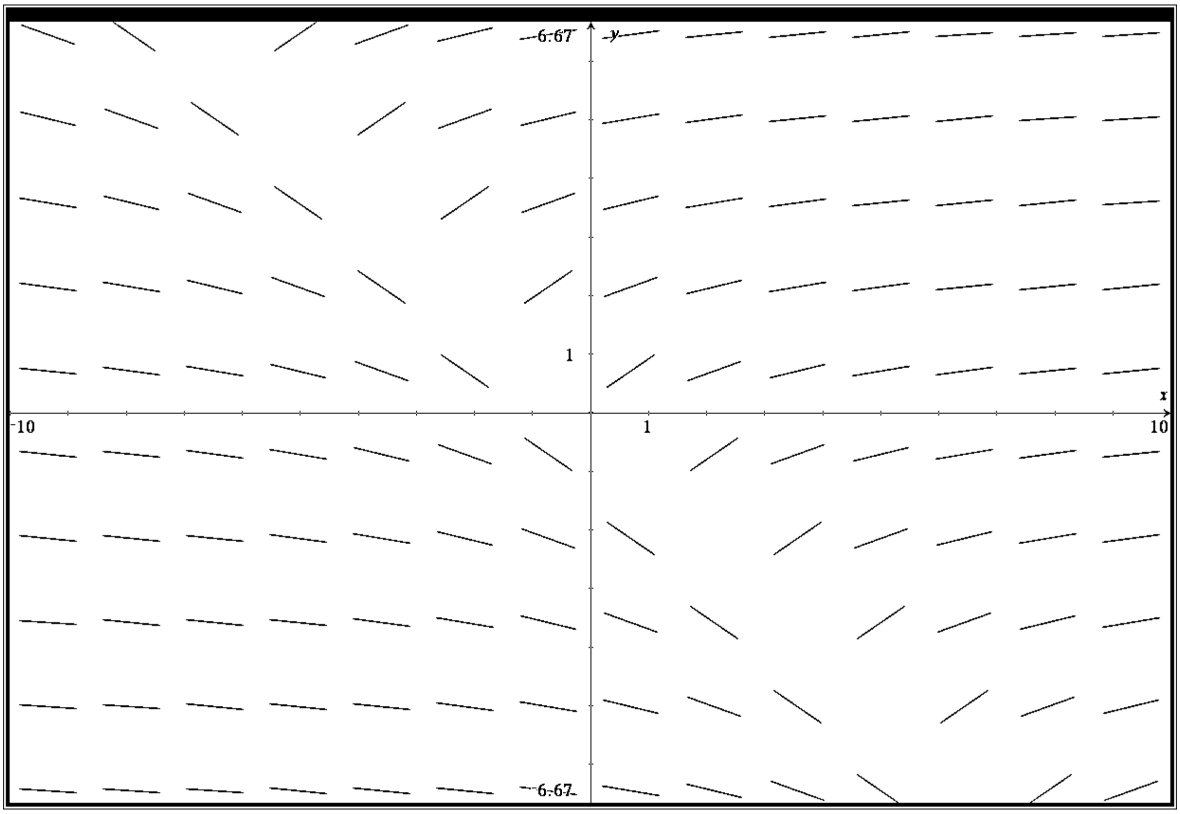
### Question 10
The region bounded by $x = 0, x = y^2(y-2)$, $y = 0$ and $y = 2$ is rotated around the y-axis. The volume of the solid of revolution is:

A. $\frac{4\pi}{3}$
B. $\frac{105\pi}{128}$
C. $\pi \int_0^2 y^2(y-2)\, dy$
D. $\frac{128}{105}$
E. $\pi \int_0^2 \left(x^2(x-2)\right)^2 dx$

### Question 11
If $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$, $x_0 = 0$ and $y_0 = 1$, and Euler's method with a step size of 0.1 is used, what would $y_3$ be?

A. 1.0099
B. 1.0290
C. 1.0558
D. 1.0890
E. 1.1269

## Question 12




The slope field above is given by:

A. $\frac{dx}{dy} = x + y$
B. $\frac{dy}{dx} = x + y$
C. $\frac{dy}{dx} = \frac{y}{x}$
D. $\frac{dy}{dx} = \frac{x}{y}$
E. $\frac{dy}{dx} = x^2$

## Question 13

A tank initially containing 100 L of pure water has a 0.5 kg/L solution of salt poured in at 15 L/minute. As the same time the concentration of salt in the tank is kept even by stirring, and water (solution) from the tank is drained at 5 L/minute. An equation linking $\frac{dQ}{dt}$, the rate of change in amount of salt with respect to time, the amount of salt $Q$ kg and time $t$ minutes is given by:

A. $\frac{dQ}{dt} = 15 - \frac{Q}{20+2t}$
B. $\frac{dQ}{dt} + 7.5 = \frac{Q}{20+2t}$
C. $\frac{dQ}{dt} = \frac{7.5-Q}{20+2t}$
D. $\frac{dQ}{dt} = \frac{Q}{20+2t}$
E. $\frac{dQ}{dt} = 7.5 - \frac{Q}{20+2t}$

Question 14

$$\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + m\boldsymbol{k}$$
$$\boldsymbol{b} = 2\boldsymbol{i} + m\boldsymbol{j} + \boldsymbol{k}$$
$$\boldsymbol{c} = m\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$$

The value(s) of $m$ such that the vectors above are linearly dependent is/are:

A. $m = -3$ or $m = 1$ or $m = 2$
B. $m = -2$ or $m = 1$ or $m = 2$
C. $m = -3$
D. $m = 3$ or $m = -1$ or $m = 2$
E. $m = 3$ or $m = -1$ or $m = -2$

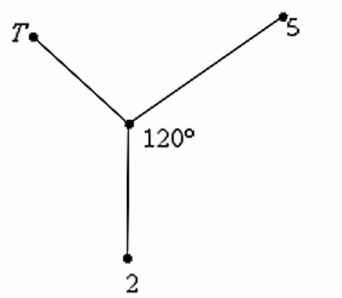
Question 15
If $\overrightarrow{AB} = (1, 2, 2)$ and $B$ is at $(3, 4, 3)$, then how far is $A$ from the origin?

A. 3
B. $4\sqrt{3}$
C. $\sqrt{7}$
D. 6
E. $\sqrt{77}$

Question 16
The Cartesian equation of the region described by $\{z: |z - (1 + 2i)| = |z - (5 - 2i)|, z \in \mathbb{C}\}$ is:

A. $3x + 2y = 1$
B. $(x - 1)^2 + (y - 2)^2 = 29$
C. $x^2 + y^2 = 1$
D. $y = 2x - 3$
E. $y = x - 3$

Question 17
A particle is acted on by three forces as shown in the diagram. The value of $T$ for the particle to be in equilibrium is:




A. $\sqrt{39}$
B. $\sqrt{19}$
C. $\sqrt{29}$
D. $\sqrt{49}$
E. 5

Question 18
If the net force on a particle of mass 2kg is given by $F = 4v^2 + 2v$ and $v = 1$ when the particle is at the origin, then the particle's velocity in terms of its displacement is given by:

A. $v = \frac{3e^{x}-1}{2}$
B. $v = \frac{3e^{2x}-1}{2}$
C. $v = e^{2x}$
D. $v = \frac{3e^{2x}+1}{2}$
E. $v = \frac{3e^{x}+1}{2}$

Question 19
A student throws a cricket ball straight up at 5m/s. How long does it take for the ball to return to its original height?

A. $\frac{5}{g}$ s
B. $\frac{10}{g}$ s
C. $\frac{15}{g}$ s
D. $\frac{20}{g}$ s
E. $15g$ s

Question 20
The normal reaction force acting on a mass $m$ on a platform accelerating downwards at 3m/s$^2$ is:

A. $mg + 3$ N
B. $m(g - 3)$ N
C. $mg - 3$ N
D. $m(g + 3)$ N
E. $3m$ N

Question 21
A 5kg particle is slowed at a constant rate from 8m/s to 2m/s in 3 seconds. $|F_{net}(t)| =$

A. $10\sin(t)$ N
B. $t^2 - 10$ N
C. $10t$ N
D. $10$ N
E. $\frac{20t}{3}$ N

Question 22
A particle of mass m is subject to two retarding forces:
$F_1 \propto x$
$F_2 \propto v^2$
An equation of motion for the particle (where $k$ and $b$ are the positive constants of proportionality for $F_1$ and $F_2$ respectively)

A. $\frac{dv}{dx} = \frac{-(kx+bv^2)}{mv}$
B. $\frac{dv}{dx} = kx + bv^2$
C. $mv\frac{dv}{dx} = kx + bv^2$
D. $\frac{dx}{dv} = \frac{-(kx+bv^2)}{m}$
E. $\frac{dv}{dx} = m(kx + bv^2)$

## *Section B – Analysis*

### *Instructions*

Answer all questions in the spaces provided.
In all questions where a numerical answer is required an exact value must be given unless otherwise specified.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.

### *Questions*

**Question 1**

Consider the relation $x^2 + xy + y^2 - 6 = 0$.

a. i. Find an expression for the derivative $\frac{dy}{dx}$.

2 marks

ii. Hence find the values of $x$ and $y$ where the relation satisfies $\frac{dy}{dx} = 0$ and $\frac{dx}{dy} = 0$.

3 marks

b. By first substituting

$$x = \frac{1}{\sqrt{2}}(u - v)$$
$$y = \frac{1}{\sqrt{2}}(u + v)$$

identify the form (shape) of the relation.

3 marks

c. i. Find $u$ and $v$ in terms of $x$ and $y$ (which can be treated as unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$), and hence sketch $u$ and $v$ axes onto the axes on the next page. Include a scale indicating positive/negative directions.

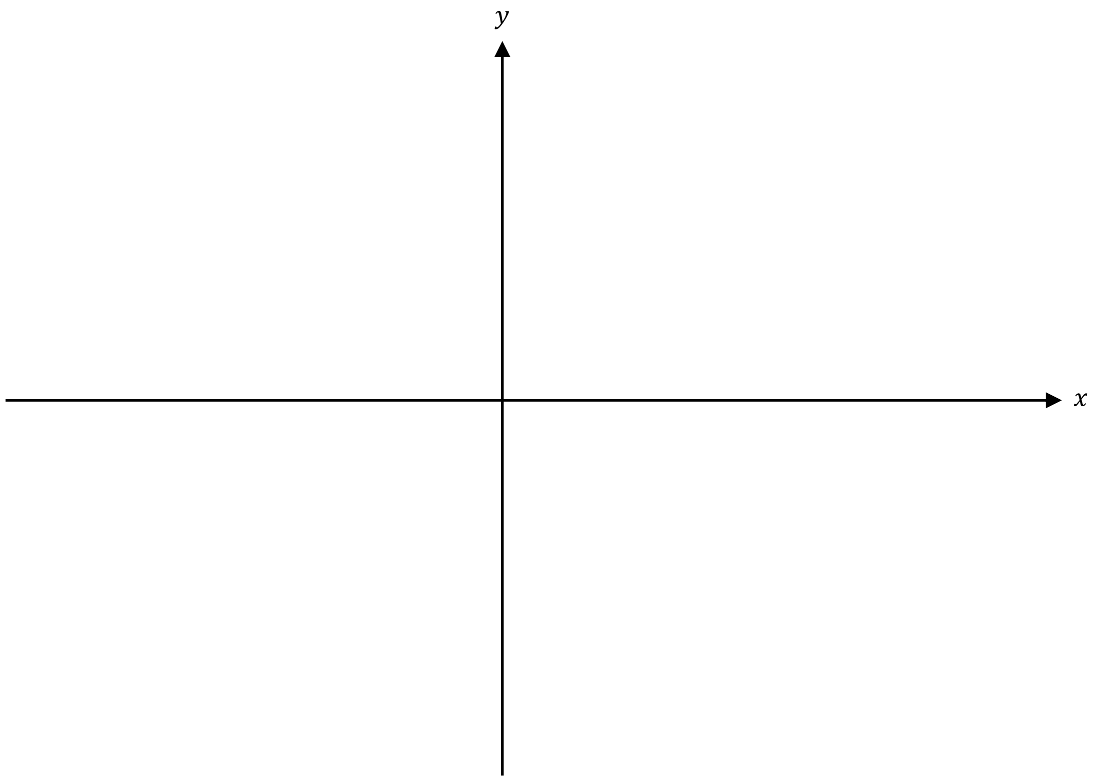
4 marks

ii. Hence, given your answer to part b, sketch

$$x^2 + xy + y^2 - 6 = 0$$

on the axes below. Include $x$ and $y$ intercepts as coordinates, and $u$ and $v$ intercepts in the form $u = \#$ or $v = \#$.

5 marks



d. Find the area bounded by the relation:

$$x^2 + xy + y^2 - 6 = 0$$

2 marks

e. Consider the related functions

$$x^2 + xy + y^2 - c = 0, \quad c > 0$$

i. Write down the relation in terms of $u$ and $v$, given the substitution in part b.

1 mark

ii. The region bounded by the relations can be rotated around the $u$ or $v$ axes to form two different solids of revolution. Let $V_u$ be the volume generated by rotating around the u axis; let $V_v$ be the volume generated by rotated around the v axis. By first expressing $V_u$ and $V_v$ as two different definite integrals, find $k$ such that $V_u = k \cdot V_v$.

4 marks

Total: 24 marks

Question 2

Consider three vectors:

$$\boldsymbol{a} = -3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$$
$$\boldsymbol{b} = -3\boldsymbol{i}$$
$$\boldsymbol{c} = 3\boldsymbol{i} + w\boldsymbol{j} + 4\boldsymbol{k}$$

a. i. Find a value of $w$ that will make the set of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ linearly dependent.

3 marks

ii. Hence write $\boldsymbol{c}$ as a linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$.
(i.e. find $k$ and $h$ such that $\boldsymbol{c} = k\boldsymbol{a} + h\boldsymbol{b}$)

2 marks

b. For this part, use your value for $\boldsymbol{w}$ found in part a.

i. Find the vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

2 marks

ii. Hence find the vector resolute of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$.

2 marks

Consider now that vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ represent forces (in newtons) acting on a particle of mass $m$ (kg) initially at rest at the origin.

c. Do not use the value of $w$ you found in part a for this part of the question.

i. Find the range of angles at which force $\boldsymbol{c}$ can possibly act relative to force $\boldsymbol{a}$. (The particular angle depends on the value of $w$.) Express your answer in the form $\theta \in (s, t]$, with $s$ and $t$ expressed in degrees correct to two decimal places.

4 marks

ii. Find the value of $w$ such that the force $\boldsymbol{c}$ acts perpendicular to force $\boldsymbol{a}$.

2 marks

The particle is also subject to a retarding force (a force that acts in the opposite direction to the particle's motion) with magnitude proportional to the particle's speed.

Let the particle's velocity vector be given by $\dot{\boldsymbol{r}}(t) = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}$.

For the remainder of this question, use the value of $w$ found in part c ii.

d. i. Find an expression for the unit velocity vector in terms of $\dot{x},\ \dot{y},$ and $\dot{z}$.

2 marks

i. Hence find an expression for the retarding force, $\boldsymbol{F}$, using $k$ to represent the positive constant of proportionality, in terms of $\dot{x},\ \dot{y}$ and $\dot{z}$.

3 marks

e. Find an expression for the net force on the particle in terms of $k$, $\dot{x},\ \dot{y}$ and $\dot{z}$.

1 mark

f. i. Find the velocity vector in terms of the time $t$, $k$ and the particle's mass ($m$).

4 marks

ii. Find the terminal speed of the particle in terms of $k$.

3 marks

g. Find the Cartesian equation that describes the position of the particle, expressing the particle's $z$ coordinate in terms of its $y$ coordinate. (Ignore the particle's motion in the $x$ direction.)

3 marks

h. If $k = 2$ and $m = 5$, find the time taken (from $t = 0$) for the particle to travel a total distance of 20 metres, correct to two decimal places.

3 marks

**Total: 34 marks**

End of Booklet

Looking for solutions? Visit www.engageeducation.org.au/practice-examswww.engageeducation.org.au/practice-exams

To enrol in one of our Specialist Mathematics lectures head to: www.engageeducation.org.au/lectures

# *Formula sheet*

## Mensuration

| | |
|---|---|
| area of a trapezium | $\frac{1}{2}(a+b)h$ |
| curved surface area of a cylinder | $2\pi rh$ |
| volume of a cylinder | $\pi r^2 h$ |
| volume of a cone | $\frac{1}{3}\pi r^2 h$ |
| volume of a pyramid | $\frac{1}{3}Ah$ |
| volume of a sphere | $\frac{4}{3}\pi r^3$ |
| area of a triangle | $\frac{1}{2}bc\sin A$ |
| sine rule | $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ |
| cosine rule | $c^2 = a^2 + b^2 - 2ab\cos C$ |

## Coordinate geometry

ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

## Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$

$1 + \tan^2(x) = \sec^2(x)$

$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$

| function | $\sin^{-1}$ | $\cos^{-1}$ | $\tan^{-1}$ |
|---|---|---|---|
| domain | $[-1, 1]$ | $[-1, 1]$ | $\mathbb{R}$ |
| range | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | $[0, \pi]$ | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |

## Algebra (complex numbers)

$z = x + yi = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$

$z^n = r^n\,\mathrm{cis}(n\theta)$ (de Moivre's theorem)

$|z| = \sqrt{x^2 + y^2} = r$

$-\pi < \mathrm{Arg}\,z \le \pi$

$z_1 z_2 = r_1 r_2\,\mathrm{cis}(\theta_1 + \theta_2)$

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\mathrm{cis}(\theta_1 - \theta_2)$

## Calculus

$\dfrac{d}{dx}(x^n) = nx^{n-1}$

$\displaystyle\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \ne -1$

$\dfrac{d}{dx}(e^{ax}) = ae^{ax}$

$\displaystyle\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

$\displaystyle\int \frac{1}{x} dx = \log_e|x| + c$

$\dfrac{d}{dx}(\sin(ax)) = a\cos(ax)$

$\displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$

$\dfrac{d}{dx}(\cos(ax)) = -a\sin(ax)$

$\displaystyle\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

$\dfrac{d}{dx}(\tan(ax)) = \dfrac{a}{\cos^2(ax)} = a\sec^2(ax)$

$\displaystyle\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$

$\dfrac{d}{dx}(\sin^{-1}(x)) = \dfrac{1}{\sqrt{1-x^2}}$

$\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}(\cos^{-1}(x)) = -\dfrac{1}{\sqrt{1-x^2}}$

$\displaystyle\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c, a > 0$

$\dfrac{d}{dx}(\tan^{-1}(x)) = \dfrac{1}{1+x^2}$

$\displaystyle\int \frac{a}{a^2+x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$

| | |
|---|---|
| product rule | $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ |
| quotient rule | $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\left(v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)}{v^2}$ |
| chain rule | $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ |
| Euler's method | If $\frac{dy}{dx} = f(x), x_0 = a$ and $y_0 = a$, then $y_{n+1} = y_n + hf(x_n)$ |
| acceleration | $a = \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt} = v\dfrac{dv}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$ |
| constant (uniform) acceleration | $v = u + at,\ s = ut + \frac{1}{2}at^2,\ v^2 = u^2 + 2as,\ s = \frac{1}{2}(u+v)t$ |

## *Vectors in two and three dimensions*

$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$\boldsymbol{r} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$

$|\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$\boldsymbol{r}_1.\boldsymbol{r}_2 = r_1 r_2 \cos\theta = x_1 x_2 + y_1 y_2 + z_1 z_2$

## *Mechanics*

| | |
|---|---|
| momentum | $\boldsymbol{p} = m\boldsymbol{v}$ |
| equation of motion | $\boldsymbol{R} = m\boldsymbol{a}$ |
| friction | $F \leq \mu N$ |