A

Free Exam
for 2006-15 VCE study design

Engage
Education
Foundation

Units 3 and 4 Specialist Maths: Exam 2

Practice Exam Solutions

Stop!

Don't look at these solutions until you have attempted the exam.

Any questions?

Check the Engage website for updated solutions, then email practiceexams@ee.org.au.

Section A – Multiple-choice questions

Question 1
The correct answer is A.

Question 2
The correct answer is A.

Looking at the answers, should group $cos(x-y+z)$ as both $\cos[(x-y)+z]$ and $\cos[x-(y+z)]$

Using trigonometric identities:

$\cos[(x-y)+z] = \cos(x-y)\cos(y) - \sin(x-y)\sin(y)$, which is not a selectable answer.

$\cos[x-(y+z)] = \cos(x)\cos(y+z) + \sin(x)\sin(y+z)$, which is answer A.

Question 3
The correct answer is B.

$$\int \frac{1}{2}\sin(2x)\sqrt{1-cosx}\ dx$$

$$= \int \sin(x)\cos(x)\sqrt{1-cosx}\ dx$$

Notice how all the answers have $1-\cos x$, so let $u = 1-cosx$, $\frac{du}{dx} = \sin(x)\,, 1-u = \cos(x)$

$$= \int (1-u)\sqrt{u}\ du$$

$$= \int u^{0.5} - u^{1.5}\ du$$

$$= \frac{2}{3}u^{1.5} - \frac{2}{5}u^{2.5} + c$$

$$= \frac{2}{3}(1-cosx)^{1.5} - \frac{2}{5}(1-cosx)^{2.5} + c$$

Now c could be zero, hence B is an anti-derivative.

Question 4
The correct answer is A.

$f(x) = tan^{-1}(x)$

$f'(x) = \frac{1}{x^2 + 1}$

$f''(x) = -\frac{2x}{(x^2 + 1)^2}$

Let $f^{'}(x) = f''(x)$

$\frac{1}{x^2 + 1} = -\frac{2x}{(x^2 + 1)^2}$

$(x^2 + 1)^2 = -2x(x^2 + 1)$

$(x^2 + 1) = -2x$

$x^2 + 2x + 1 = 0$

$(x + 1)^2 = 0$

When $f^{'}(x) = f''(x)$, $x = -1$

Question 5
The correct answer is C.

Using linear approximation with step size of -1, $y(-2) = y(-1) - \frac{dy}{dx}(-1)$, $\frac{dy}{dx}(-1) \approx 0.79$

$y(-2) \approx c - 0.79$

Question 6
The correct answer is D.

Both D and E cannot be true.

Logically, 3 or more 2 dimensional vectors are dependant.

Alternatively,

$\frac{1}{7}(-\boldsymbol{a} + 2\boldsymbol{b}) = \boldsymbol{j}$ and $\frac{1}{7}(3\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{i}$

$\boldsymbol{c} = 3\boldsymbol{a} - 2\boldsymbol{b}$

Hence $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are dependent, and so D is the correct answer.

Question 7
The correct answer is D.

Graph is of the form $y = ax^3 + c$, with a and c as constants.

Question 8
The correct answer is B.

$$\frac{d\big(\boldsymbol{r}(t)\big)}{dt} = \sec^2(t)\,\boldsymbol{i} + \tan(t)\sec^2(t)\boldsymbol{j}$$

$$\boldsymbol{v}\left(\frac{3\pi}{4}\right) = 2\boldsymbol{i} - 4\boldsymbol{j}$$

Question 9
The correct answer is A.

Question 10
The correct answer is D.

$$\text{area} = \int_0^{\frac{\pi}{2}} 2\cos^{-1}(2x)\,dx = 1$$

Question 11
The correct answer is A.

$$\int \frac{e^{2x}-1}{e^{2x}+1} - \frac{e^{2x}+1}{e^{2x}-1}\,dx$$

$$= -\int \frac{4e^{2x}}{e^{4x}-1}\,dx$$

Let $u = e^{2x}$, $\frac{du}{dx} = 2e^{2x}$

$$= -\int \frac{2}{u^2-1}\,du$$

$$= \int \frac{1}{u+1}\,du - \int \frac{1}{u-1}\,du$$

$$= \log_e(u+1) - \log_e(u-1)$$

$$= \log_e\left(\frac{u+1}{u-1}\right) + c$$

$$= \log_e\left(\frac{e^{2x}+1}{e^{2x}-1}\right) + c$$

Question 12
The correct answer is C.

Let equation 1 be $\frac{x^2}{2} + y^2 = 1$

Let equation 2 be $\frac{x^2}{2} + y = c$

$equation\ 1 - equation\ 2: y^2 - y = 1 - c$

$y^2 - y + c - 1 = 0$

Using the quadratic formula:

$$y = \frac{1 \pm \sqrt{1 - 4c + 4}}{2}$$

For there to be real solutions for P,

$1 - 4c + 4 \geq 0$

$$c \leq \frac{5}{4}$$

Question 13
The correct answer is C.

Question 14
The correct answer is E.

Question 15
The correct answer is E.

Visually, $|z|$ is the distance from the origin to the point z. Using Pythagoras's rule, $\text{distance} = \sqrt{a^2 + b^2}$.

Question 16
The correct answer is E.

The xz-plane has normal vector $\hat{\boldsymbol{k}}$. The angle, α, between $\hat{\boldsymbol{k}}$ and $\boldsymbol{p}$ is given by:

$\hat{\boldsymbol{k}}.\boldsymbol{p} = |\hat{\boldsymbol{k}}||\boldsymbol{p}|\cos(\alpha)$

$$\cos(\alpha) = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

The angle, θ, between the xz-plane and the vector $\boldsymbol{p}$ equals $90 - \alpha$.

$\therefore \alpha = 90 - \theta$

$$\therefore \cos(90 - \theta) = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$\cos(90 - \theta) = \sin(\theta)$

Hence $\theta = \sin^{-1}\left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

Question 17
The correct answer is D.

$(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{c}+\boldsymbol{d})=\boldsymbol{0}$, $\boldsymbol{a}.\boldsymbol{c}+\boldsymbol{b}.\boldsymbol{c}+\boldsymbol{a}.\boldsymbol{d}+\boldsymbol{b}.\boldsymbol{d}=\boldsymbol{0}$ (1)

$(\boldsymbol{b}+\boldsymbol{c})(\boldsymbol{a}+\boldsymbol{d})=\boldsymbol{0}$, $\boldsymbol{b}.\boldsymbol{d}+\boldsymbol{b}.\boldsymbol{a}+\boldsymbol{c}.\boldsymbol{d}+\boldsymbol{c}.\boldsymbol{a}=\boldsymbol{0}$ (2)

(1) - (2): $\boldsymbol{b}.\boldsymbol{c}-\boldsymbol{c}.\boldsymbol{d}-\boldsymbol{b}.\boldsymbol{a}+\boldsymbol{a}.\boldsymbol{d}=\boldsymbol{0}$

$(\boldsymbol{c}-\boldsymbol{a}).(\boldsymbol{b}-\boldsymbol{d})=\boldsymbol{0}$

Since $\boldsymbol{c}-\boldsymbol{a}\neq 0$ and $\boldsymbol{b}-\boldsymbol{d}\neq 0$

$\boldsymbol{c}-\boldsymbol{a}$ and $\boldsymbol{b}-\boldsymbol{d}$ are perpendicular.

Question 18
The correct answer is A.

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx}=-2(x-3)^3$$

$$\frac{1}{2}v^2=-\frac{(x-3)^4}{2}+c$$

$$v^2=-(x-3)^4+2c$$

At $x=3+\sqrt{2}, v=0$, thus $c=2$.

$$v^2=4-(x-3)^4$$

Maximum velocity occurs when $(x-3)^4=0, v=2$

Minimum displacement from O when $v=0, x=3$. To see this more clearly, rearrange the expression to make x the subject:

$$(x-3)^4=4-v^2$$

$$x=3+\sqrt[4]{4-v^2}$$

Question 19
The correct answer is B.

$$Velocity=\frac{momentum}{mass}$$

Hence written in $(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$ form, the change in velocity is from $(15,-5,5)$ to $(5,0,-5)$, where the magnitude of each of these vectors is the speed.

$$V_1=\sqrt{15^2+(-5)^2+5\text{^}2}=\sqrt{275}\approx 16.6$$

$$V_2=\sqrt{5^2+0^2+(-5)\text{^}2}=\sqrt{50}\approx 7.1$$

V_2-V_1 is closest to -10.

Question 20
The correct answer is E.

$1.3 = \sqrt{1.2^2 + 0.5^2}$, hence it is a right angle triangle.

Thus $\frac{Ts}{Tc} = \frac{1.2}{0.5} \approx 0.42$

Question 21
The correct answer is B.

$z + \bar{z} = a + bi + (a - bi) = 2a$

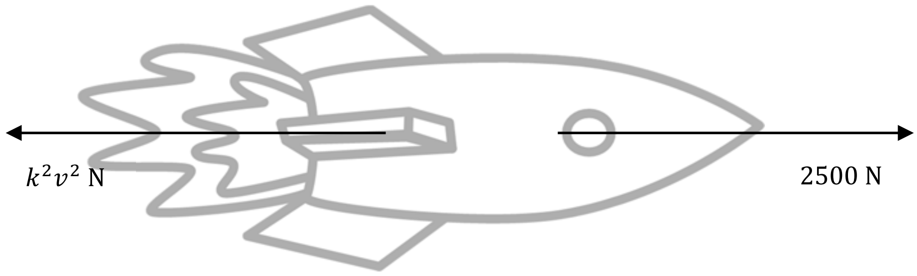
Question 22
The correct answer is D.

Let $z = a\,\mathrm{cis}(b)$. Then $iz = \mathrm{cis}\left(\frac{\pi}{2}\right) \times a\,\mathrm{cis}(b) = a\,\mathrm{cis}\left(b + \frac{\pi}{2}\right)$, which is (geometrically speaking) z rotated 90° around the origin.

Section B – Short-answer questions

Marks allocated are indicated by a number in square brackets, for example, [1] indicates that the line is worth one mark.

Question 1a



[1 mark for both arrows in correct direction and with correct values]

Question 1b i

$a = \frac{F}{m} = \frac{1}{1000}(2500 - k^2v^2)$ [1]

Question 1b ii

$a = \frac{dv}{dt} = \frac{1}{1000}(2500 - k^2v^2)$

$\frac{dt}{dv} = \frac{1000}{2500 - k^2v^2}$

$t = \int \frac{1000}{2500 - k^2v^2} dv + c$ [1]

Use partial fractions:

$\frac{1000}{2500 - k^2v^2} = \frac{A}{50 - kv} + \frac{B}{50 + kv}$

⋮

$A = B = 10$ [1]

$\therefore t = \int \frac{10}{50 - kv} dv + \int \frac{10}{50 + kv} dv + c$

$t = -\frac{10}{k}\log_e|50 - kv| + \frac{10}{k}\log_e|50 + kv| + c$

$t = \frac{10}{k}\log_e\left|\frac{50 + kv}{50 - kv}\right| + c$ [1]

When $t = 0$, $v = 0$: $0 = \frac{10}{k}\log_e|1| + c$

$\therefore c = 0$ [1]

$t = \frac{10}{k}\log_e\left|\frac{50 + kv}{50 - kv}\right|$

$e^{\frac{tk}{10}} = \frac{50 + kv}{50 - kv}$ (can remove the absolute value since $e^a > 0$ for all $a \in \mathbb{R}$)

$50e^{\frac{tk}{10}} - kve^{\frac{tk}{10}} = 50 + kv$

$50\left(e^{\frac{tk}{10}} - 1\right) = kv\left(1 + e^{\frac{tk}{10}}\right)$

$v = \frac{50\left(e^{\frac{tk}{10}} - 1\right)}{k\left(e^{\frac{tk}{10}} + 1\right)} = \frac{50(\alpha - 1)}{k(\alpha + 1)}$ [1]

Question 1b iii

When $t = \frac{10}{k}$, $v = 50$, $\alpha = e^{\frac{10k}{10k}} = e$

$\therefore 50 = \frac{50(e - 1)}{k(e + 1)}$

$k = \frac{e - 1}{e + 1}$ [1]

Question 1c

$t = \frac{10}{k} = \frac{10(e + 1)}{e - 1} \approx 22$ s [1]

Question 1d

$v_m = \frac{50(e+1)}{e-1}$ [1]

Question 2a

$\text{area} = \int_0^5 \left(\frac{1}{10}x^2 + 1\right) dx + \int_5^6 3.5dx$ [1]

$$= \left[\frac{1}{30}x^3 + x\right]_0^5 + [3.5x]_5^6$$

$$= \frac{1}{30} \times 125 + 5 + 3.5 \times 6 - 3.5 \times 5$$

$= \frac{38}{5}$ [1]

Question 2b

$x^2 = 10(y-1)$ [1]

$V = \pi \int_1^{3.5} x^2 dy$ [1]

$$= 10\pi \int_1^{3.5} (y-1)dy$$

$$= 10\pi \left[\frac{y^2}{2} - y\right]_1^{3.5}$$

$$= 10\pi \left(\frac{3.5^2}{2} - 3.5 - \frac{1}{2} + 1\right)$$

$= \frac{250\pi}{8} = \frac{125\pi}{4}\ \text{m}^2$ [1]

Question 2c

$$\frac{dV}{dt} = \frac{4}{\pi}\ \text{m}^3/\text{minute}$$

∴ it will take $\frac{125\pi}{4} \times \frac{4}{\pi} = 125$ minutes to fill [1]

During this time, the water level rises from 0 to 2.5 m.

Average rate at which water rises $= \frac{2.5\text{ m}}{125\text{ min}} = \frac{2500\text{ mm}}{125\times 60\text{ seconds}} = \frac{1}{3}\ \text{mm/s}$ [1]

Question 2d i

$V(h) = 10\pi \int_1^{1+h} (y+1)dy, 0 \le h \le 2.5$ [1]

Question 2d ii

$$V(h) = 10\pi\left[\frac{y^2}{2} - y\right]_1^{1+h}$$

$$= 10\pi\left(\frac{1+2h+h^2}{2} - h - \frac{1}{2}\right)$$

$$= 10\pi\left(\frac{(2h+h^2)}{2} - \frac{2h}{2}\right)$$

$= 10\pi h^2$ [1]

$\therefore \frac{dV}{dh} = 20\pi h$ [1]

$\frac{dh}{dt} = \frac{dh}{dV}\frac{dV}{dt}$ [1]

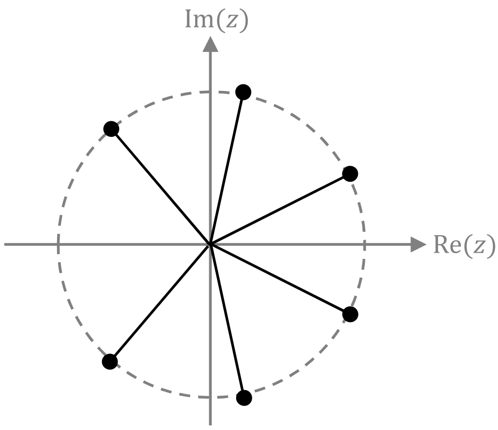
$= \frac{1}{20\pi h} \times \frac{\pi}{4} = \frac{1}{80h}$ [1]

Question 3a

Use long division or CAS calculator to find: $P(z) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$ [1]

Then read off values: $A = C = E = G = 1, B = D = F = -1$ [1]

Question 3b



Note that $z = -1$ due to the $z + 1$ term in $P(z)$.

[2 for roots]

Question 3c

$$\frac{z^7+1}{z+1} = 0$$

$z^7 = -1 = \text{cis}(-\pi)$ [1]

$z = \text{cis}\left(\frac{-\pi+2n\pi}{7}\right), n = -2, -1, 0, 1, 2, 3$ (note that $n = -3$ is excluded due to the $z + 1$ term in $P(z)$) [1]

$z = \text{cis}\left(-\frac{5\pi}{7}\right), \text{cis}\left(-\frac{3\pi}{7}\right), \text{cis}\left(-\frac{\pi}{7}\right), \text{cis}\left(\frac{\pi}{7}\right), \text{cis}\left(\frac{3\pi}{7}\right), \text{cis}\left(\frac{5\pi}{7}\right)$ [1]

Question 4a
$f = 0$ since the object is at rest [1]

Question 4b
Using resolution of forces:

Perpendicular to the plank: $N - mg\cos(30°) = 0$, so $N = 21.22$ N [1]

Parallel to the plank: $f - mg\sin(30°) = 0$, so $f = 12.25$ N (which is less than the maximum possible friction force, $\mu N = 14.85$ N) [1]

Question 4c
Again, using resolution of forces:

Perpendicular to the plank: $N = mg\cos(40°)$ [1]

Net force is parallel to the plank:

$ma = f - mg\sin(40°)$

$ma = \mu N - mg\sin(40°)$

$ma = 0.7 \times 2.5 \times 9.8 \times \cos(40°) - 2.5 \times 9.8 \times \sin(40°)$

$a = -\frac{2.61}{2.5} = -1.04\ \mathrm{m/s^2}$ (ie. acceleration down the plank) [1 mark, must include direction]

Question 4d
Use constant acceleration equation $v^2 = u^2 + 2as$ with $u = 0$, $a = 1.04\ \mathrm{m/s^2}$, $d = 3$ m and v unknown:

$v = \sqrt{2 \times 1.04 \times 3} = 2.50\ \mathrm{m/s^2}$ [1]

$p = mv = 2.5 \times 2.50 = 6.25\ \mathrm{kg\ m/s}$ [1]

Question 4e
Use resolution of forces:

Perpendicular to plank:

$\text{normal reaction force} - \text{gravity component} + \text{pulling force component} = 0$

$N - mg\cos(40°) + 20\sin(30°) = 0$ (30° is the angle between the plank and the pulling force)

$N = 2.5 \times 9.8 \times \cos(40°) - 20\sin(30°) = 8.77$ N [1]

Parallel to the plank, ignoring friction:

$\text{pulling force component} - \text{gravity component} = 20\cos(30°) - mg\sin(40°)$

$= 1.57$ N up the plane [1]

Therefore, friction will act to oppose the motion of the box up the plane (ie. it will act down the plane)

$f_{max} = \mu N = 0.7 \times 8.77$ N [1]

Since the maximum frictional force is greater than the resolved force, the net force is 0. [1]

Question 4f
It will remain stationary. [1]

Question 5a
AC = $\boldsymbol{c} - \boldsymbol{a}$, BC = $\boldsymbol{c} - \boldsymbol{b}$, BA = $\boldsymbol{a} - \boldsymbol{b}$ [2]

Question 5b
OM = $\frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$, ON = $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c})$, OP = $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ [2]

Question 5c i
We know that OM ⊥ BC and ON ⊥ AC. Hence $OM.BC = 0$ and $ON.AC = 0$. [1]

$\frac{1}{2}(\boldsymbol{b} + \boldsymbol{c}).(\boldsymbol{c} - \boldsymbol{b}) = 0$

$\boldsymbol{b}.\boldsymbol{c} - \boldsymbol{b}.\boldsymbol{b} + \boldsymbol{c}.\boldsymbol{c} - \boldsymbol{c}.\boldsymbol{b} = 0$

$|\boldsymbol{c}|^2 = |\boldsymbol{b}|^2$

$\therefore |\boldsymbol{c}| = |\boldsymbol{b}|$ since $|\boldsymbol{c}| > 0$ and $|\boldsymbol{b}| > 0$. [0.5]

$\frac{1}{2}(\boldsymbol{a} + \boldsymbol{c}).(\boldsymbol{c} - \boldsymbol{a}) = 0$

$\boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{a} + \boldsymbol{c}.\boldsymbol{c} - \boldsymbol{c}.\boldsymbol{a} = 0$

$|\boldsymbol{c}|^2 = |\boldsymbol{a}|^2$

$\therefore |\boldsymbol{c}| = |\boldsymbol{a}|$ since $|\boldsymbol{c}| > 0$ and $|\boldsymbol{a}| > 0$. [0.5]

$\therefore |\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$ [1]

Question 5c ii
OP.BA = $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b}).(\boldsymbol{a} - \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{a}.\boldsymbol{a} - \boldsymbol{a}.\boldsymbol{b} + \boldsymbol{a}.\boldsymbol{b} - \boldsymbol{b}.\boldsymbol{b}) = \frac{1}{2}(|\boldsymbol{a}|^2 - |\boldsymbol{b}|^2) = \frac{1}{2}(|\boldsymbol{a}|^2 - |\boldsymbol{a}|^2) = 0$

∴ OP ⊥ BA since OP and BA ≠ $\boldsymbol{0}$. [1]

Question 5d
$|AC|^2 = (\boldsymbol{c} - \boldsymbol{a}).(\boldsymbol{c} - \boldsymbol{a}) = \boldsymbol{c}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{c} + \boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{c}|^2 + |\boldsymbol{a}|^2 - 2|\boldsymbol{a}||\boldsymbol{c}|\cos\alpha = d^2 + d^2 - 2d^2\cos\alpha$

$= 2d^2(1 - \cos\alpha)$ [1]

Similarly, $|BC|^2 = 2d^2(1 - \cos\beta)$ [1] and $|BA|^2 = 2d^2(1 - \cos\gamma)$ [1]

Hence,

$|AC|^2 + |BC|^2 + |BA|^2 = 2d^2(1 - \cos\alpha + 1 - \cos\beta + 1 - \cos\gamma)$

$= 2d^2(3 - (\cos\alpha + \cos\beta + \cos\gamma)$ [1]

Question 6a
$x_0 = 1, y_0 = 1$

$x_1 = 1.1, y_1 = y(1.1) = y_0 + hf'(x_0) = 1 + 0.1\left(\frac{2+1}{2}\right) = 1 + 0.1 \times 1.5 = 1.15$ [1]

$x_2 = 1.2, y_2 = y(1.2) = 1.15 + 0.1\left(\frac{2.2+1}{2.1}\right) = 1.15 + 0.1 \times \frac{3.2}{2.1} = 1.302$ [1]

Question 6b

$\frac{dy}{dx} = \frac{2x+1}{x+1} = 2 - \frac{1}{x+1}$ using partial fractions or any other suitable method

$$y = \int 2dx - \int \frac{1}{x+1} dx + c$$

$y = 2x - \log_e|x+1| + c$ [1]

When $x = 1, y = 1$:

$1 = 2 - \log_e 2 + c$

$c = \log_e 2 - 1$

$\therefore y = 2x - \log_e|x+1| + \log_e 2 - 1$

$$y = 2x - 1 + \log_e \left|\frac{2}{x+1}\right|$$

$y(1.2) = 2.4 - 1 + \log_e\left(\frac{2}{2.2}\right) = 1.4 + \log_e\left(\frac{1}{1.1}\right)$ [1]