The Mathematical Association of Victoria

Trial Exam 2014

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 25 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **name** in the space provided above on this page and on the answer sheet for multiple-choice questions.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

If $\operatorname{cosec}(\phi) = \frac{4}{3}$, then $\cot(\phi)$ could equal

A. $-\frac{3}{5}$

B. $\frac{4}{7}$

C. $-\frac{4}{7}$

D. $-\frac{\sqrt{7}}{3}$

E. $\frac{\sqrt{7}}{4}$

Question 2

Consider the function $f: D \to R,\ f(x) = \frac{x^2}{x^2 - 4}$, where D is the maximal domain of f, and let $g(x) = f(x-1)$. The maximal domain and range of g are

A. Domain $R \setminus \{-2, 2\}$, Range $R \setminus \{0\}$

B. Domain $R \setminus \{-3, 1\}$, Range $R \setminus \{0\}$

C. Domain $R \setminus \{-2, 2\}$, Range $R \setminus \{1\}$

D. Domain $R \setminus \{-1, 3\}$, Range $R \setminus \{1\}$

E. Domain $R \setminus \{-1, 3\}$, Range $R \setminus \{0\}$

Question 3

The value of $\sin\left(2\tan^{-1}\left(\frac{1}{2}\right)\right)$ is

A. $\frac{4}{5}$

B. $\frac{3}{5}$

C. $\frac{2}{\sqrt{5}}$

D. $\frac{\sqrt{5}}{5}$

E. $\frac{\sqrt{2}}{2}$

Question 4

If $u = 3 + 4i$, then $\dfrac{u}{6 - 2\overline{u}}$ is equal to

A. $\dfrac{i}{8}$

B. $-\dfrac{i}{8}$

C. $\dfrac{1 - 3i}{8}$

D. $\dfrac{3 + 4i}{8}$

E. $\dfrac{4 - 3i}{8}$

Question 5

Which one of the following is **not** a fourth root of $1 + i\sqrt{3}$?

A. $\sqrt[4]{2}\,\text{cis}\left(-\dfrac{11\pi}{12}\right)$

B. $\sqrt[4]{2}\,\text{cis}\left(-\dfrac{5\pi}{12}\right)$

C. $\sqrt[4]{2}\,\text{cis}\left(\dfrac{\pi}{12}\right)$

D. $\sqrt[4]{2}\,\text{cis}\left(\dfrac{7\pi}{12}\right)$

E. $\sqrt[4]{2}\,\text{cis}\left(\dfrac{11\pi}{12}\right)$

Question 6

Let $M = \{z : |z - 1| = |z + i|\}$ and $P = \{z : |z + i| = |z - 4 + i|\}$ be subsets of the complex plane. $M \cap P$ is

A. $\{z : z = 1 - i\}$

B. $\{z : z = -1 + i\}$

C. $\{z : z = -2 + 2i\}$

D. $\{z : z = 2 - 2i\}$

E. $\{z : -4\,\text{Re}(z) + \text{Im}(z) = 0\}$

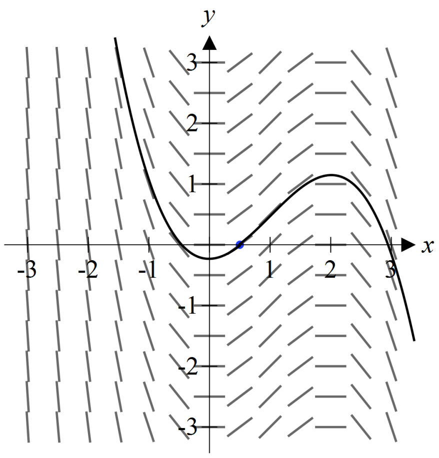
Question 7

Let p be a polynomial function with integer coefficients. If -5, $\sqrt{3}$ and $1 - 2i$ are roots of p, then the **minimum** degree of the polynomial is

A. 2
B. 3
C. 4
D. 5
E. 6

Question 8

The direction (slope) field of a particular differential equation is shown below.



The differential equation could be

A. $\dfrac{dy}{dx} = -x^2$

B. $\dfrac{dy}{dx} = 2x - x^2$

C. $\dfrac{dy}{dx} = x^2 - 2x$

D. $\dfrac{dy}{dx} = x^2 - x^3$

E. $\dfrac{dy}{dx} = -x\left(x+2\right)^2$

Question 9

Using a suitable substitution, $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\tan(x)\log_e\left(\sec(x)\right)\right)dx$ can be expressed completely in terms of *u* as

A. $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} (u)\,du$

B. $\displaystyle\int_{-\log\left(\frac{\sqrt{3}}{2}\right)}^{\log_e(2)} (u)\,du$

C. $\displaystyle-\int_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}} (u)\,du$

D. $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\log_e(u)\right)du$

E. $\displaystyle\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} \left(\log_e(u)\right)du$

Question 10

Euler's method is used to solve the differential equation $\frac{dy}{dx} = \log_e\left(\frac{1}{\sqrt{x}}\right)$, with step size of $\frac{1}{10}$ and initial values $x = 1$ and $y = 2$. The value of y when $x = \frac{6}{5}$ is given by

A. $\frac{1}{4}\log_e\left(\frac{6}{5}\right) + 2$

B. $\frac{1}{4}\log_e\left(\frac{10}{11}\right) + 2$

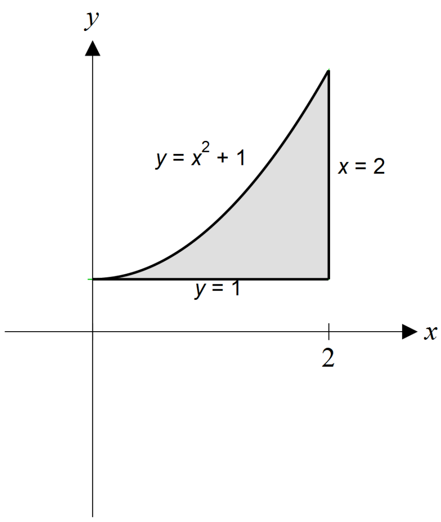
C. $2 - \frac{1}{20}\log_e\left(\frac{5}{6}\right)$

D. $2 + \frac{1}{20}\log_e\left(\frac{11}{10}\right)$

E. $2 + \frac{1}{20}\log_e\left(\frac{10}{11}\right)$

Question 11

The shaded region shown below, which is enclosed by the graph of $y = x^2 + 1$ and the lines $y = 1$ and $x = 2$, is rotated about the ***x*-axis** to form a solid of revolution.



The volume of the solid is given by

A. $\pi\int_1^5 x^4\,dx$

B. $\pi\int_0^2 \left(x^4 + 2x^2\right)dx$

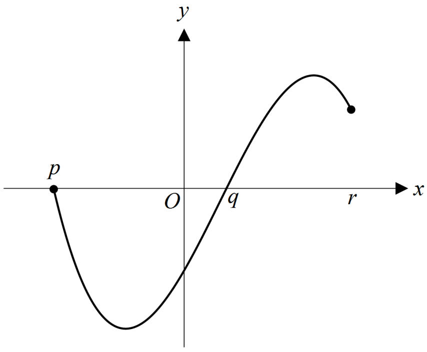
C. $\pi\int_0^2 \left(x^4 + 2x^2 + 1\right)dx$

D. $\pi\int_0^2 \left(x^2 + 1\right)dx$

E. $\pi\int_1^5 \left(x^2 + 1\right)dx$

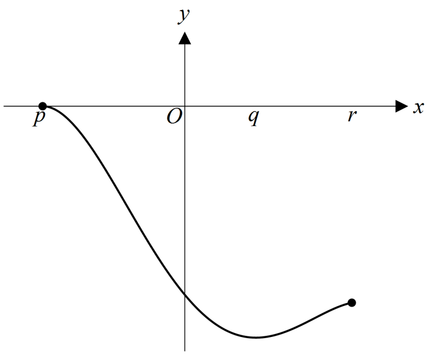
Question 12

The graph of a function *f* is shown below.

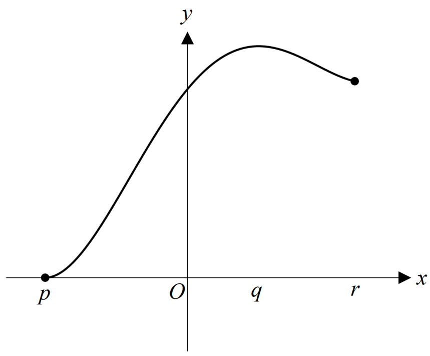


If $g(x)=\int_{p}^{x} f(t)\,dt$, then the graph of *g* could be

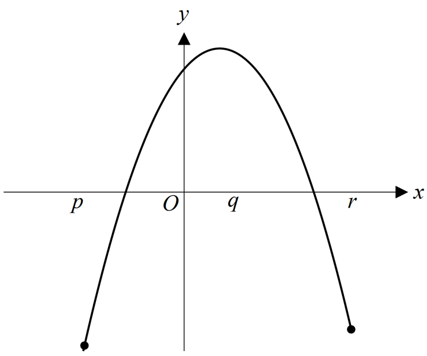
A.



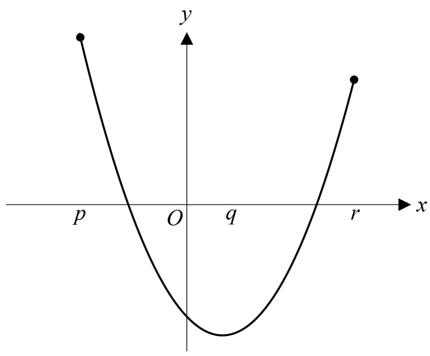
B.



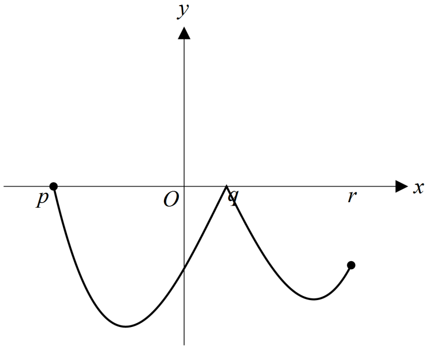
C.



D.



E.



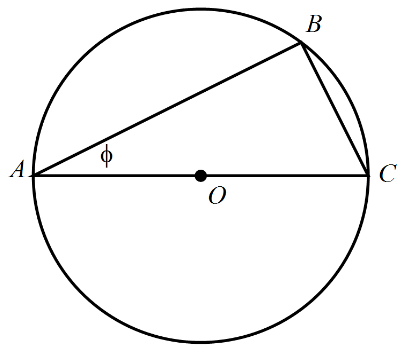
Question 13
The gradient of the **normal** to the graph of $x^2 - y^2 + xy - 1 = 0$, at the points where $y = 0$, is equal to

A. 1
B. 2
C. $\frac{1}{2}$
D. -1
E. $-\frac{1}{2}$

Question 14
The angle between the vectors $\underset{\sim}{p} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{q} = -2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ is

A. 120°
B. 90°
C. 60°
D. 45°
E. 30°

Question 15
In the diagram below, the vertices of triangle *ABC* are on the circumference of the circle and the centre of the circle, *O*, is on side *AC*.



Which one of the following is **false**?

A. $\overrightarrow{AC} \cdot \overrightarrow{BC} = \left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|\sin(\phi)$

B. $\overrightarrow{AC} \cdot \overrightarrow{AB} = \left|\overrightarrow{AC}\right|\left|\overrightarrow{AB}\right|\cos(\phi)$

C. $\left(\overrightarrow{AC} - \overrightarrow{AB}\right) \cdot \overrightarrow{AB} = \left|\overrightarrow{BC}\right|^2$

D. $\left(\overrightarrow{AC} - \overrightarrow{BC}\right) \cdot \overrightarrow{AB} = \left|\overrightarrow{AB}\right|^2$

E. $\left|\overrightarrow{CA} - \overrightarrow{BA}\right|^2 = \left|\overrightarrow{CA}\right|^2 - \left|\overrightarrow{AB}\right|^2$

Question 16

The position vector of a particle moving along a curve is $\underset{\sim}{r} = \cos(2t)\underset{\sim}{i} - 2\sin(t)\underset{\sim}{j}$.

The cartesian equation of the curve is

A. $x^2 - 2y + 2 = 0$

B. $y^2 + 2x - 2 = 0$

C. $x^2 - y + 1 = 0$

D. $y^2 + x - 1 = 0$

E. $y^2 + 2x - 1 = 0$

Question 17

At all points of a particular curve, $\frac{d^2y}{dx^2} = -12x$. The point P with coordinates $(-1, 4)$ is on the curve and the gradient of the tangent to the curve at P is -1. The equation of the curve is

A. $y = -2x^3 + 5x + 7$

B. $y = 2x^3 - 5x^2 + 7$

C. $y = -2x^3 + 7x + 5$

D. $y = 2x^3 - 7x^2 + 5$

E. $y = -12x^3 - x + 4$

Question 18

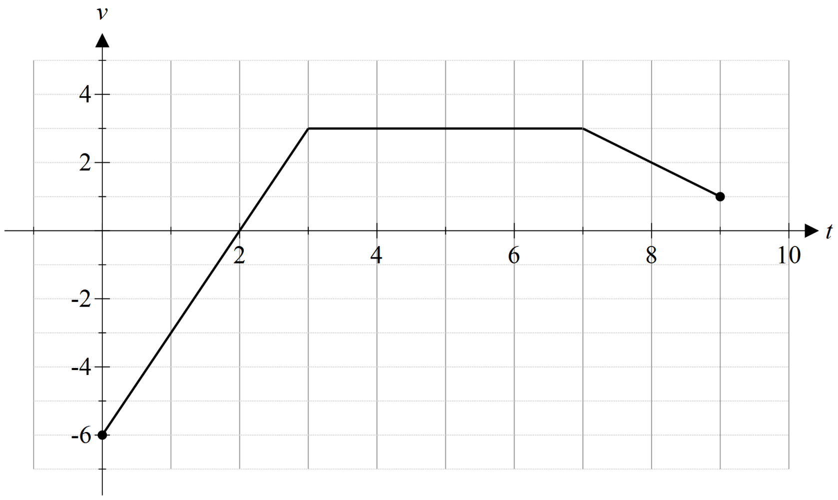
A golf ball is hit from an origin, O, on horizontal ground. The position vector of the ball is given by $\underset{\sim}{r} = 35t\underset{\sim}{i} + (20t - 5t^2)\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the forward and upward directions, respectively.

The ball strikes the ground at a point P. The length of the line segment OP is

A. 20
B. 35
C. 80
D. 140
E. 175

Question 19

The velocity-time graph of a particle travelling **from the origin** along the *x*-axis on the cartesian plane is shown.



The coordinates of the final position of the particle are

A. $(9,1)$

B. $(9,0)$

C. $\left(\frac{23}{2},0\right)$

D. $\left(\frac{25}{2},1\right)$

E. $\left(\frac{29}{2},0\right)$

Question 20

A particle moves in a straight line such that its acceleration, *a* m/s, is given by $a = \frac{9}{x^2}$, where *x* metres is its displacement from a fixed origin. If $v = -2$ m/s at $x = 9$, where *v* is the velocity of the particle, then the relationship between *v* and *x* is

A. $v = 3 - \frac{9}{x}$

B. $v = \frac{9}{x} + 1$

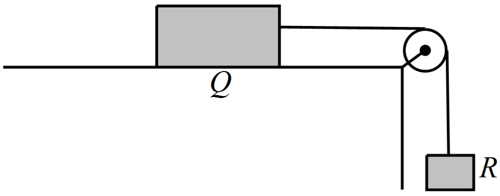
C. $v = \sqrt{6 - \frac{18}{x}}$

D. $v = -\sqrt{\frac{6x - 18}{x}}$

E. $v = -\sqrt{\frac{9 + 4x}{x}}$

Question 21

A body Q of mass $5m$ is on a rough horizontal plane. The coefficient of friction between body Q and the plane is $\frac{1}{10}$. Body Q is connected to body R, of mass m, by means of an inextensible string, over a smooth pulley. R is suspended over the edge of the plane, as shown below.



When the system is released from rest, the acceleration of the system is

A. $6mg$

B. $6g$

C. $\frac{g}{6}$

D. $\frac{mg}{12}$

E. $\frac{g}{12}$

Question 22

A particle of mass m is projected vertically upwards from the ground with an initial speed of $\sqrt{\frac{g}{k}}$, where k is a positive real constant. When the speed of the particle is v, air resistance exerts a force on the particle of magnitude mkv^2, so that the equation of motion is $-mg - mkv^2 = ma$, where a is the acceleration of the particle. The maximum height reached by the particle is given by

A. $\frac{1}{2g}\sqrt{\frac{g}{k}}$

B. $\frac{\log_e(2)}{2k}$

C. $\frac{m}{2g}\sqrt{\frac{g}{k}}$

D. $\frac{m\log_e(2)}{2k}$

E. $\frac{\tan^{-1}\left(\sqrt{\frac{k}{g}}\right)}{\sqrt{gk}}$

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
Unless otherwise specified an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1 (13 marks)

Consider the complex number $w = 4 - 4i$.

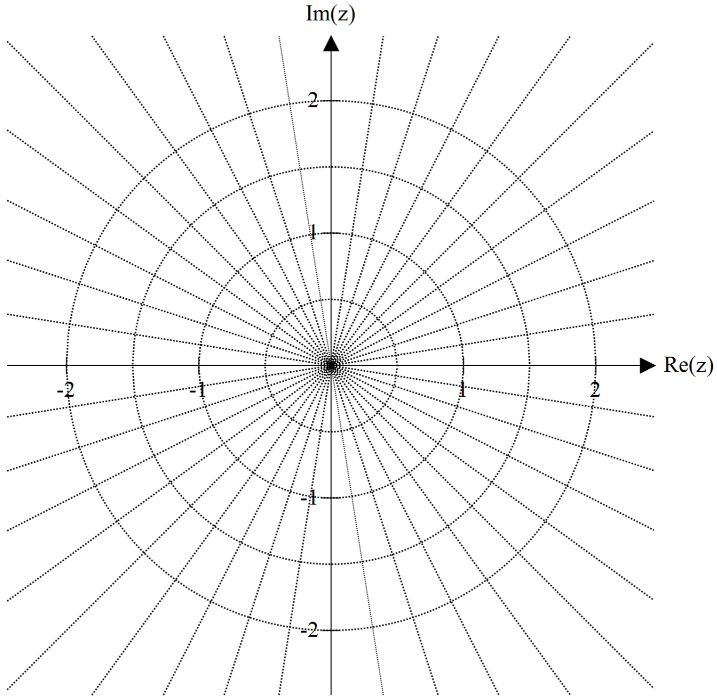
a. i. Show that $|w| = 4\sqrt{2}$. 1 mark

ii. Show that $\operatorname{Arg}(w) = -\dfrac{\pi}{4}$. 1 mark

b. Find $\{z : z^5 = w,\ z \in C\}$. Express each answer in the form $a\operatorname{cis}(\theta)$, where a and θ are real numbers with $-\pi < \theta \le \pi$. 3 marks

c. i. On the diagram of the complex plane below, sketch $\left\{z : |z| = \sqrt{2}\right\}$. 1 mark

ii. Plot all solutions to the equation $z^5 = w$ as points on the complex plane below. 1 mark



Consider a different complex number $u = \sqrt{3} - i$.

d. Find the least positive integer k for which $u^k \in R^+$. 3 marks

e. If u is a root of the equation $z^9 + 16(1+i)z^3 + c + id = 0$, find the values of the real constants c and d. 3 marks

End of Question 1

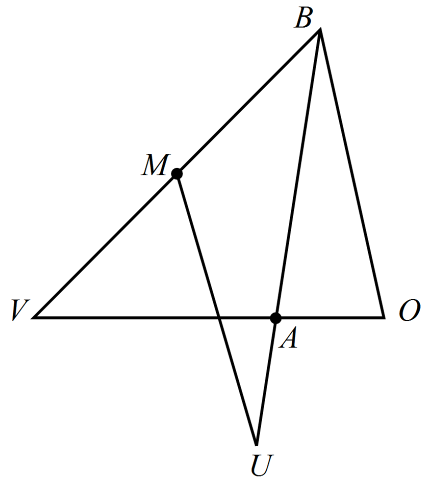
Question 2 (12 marks)

Consider the vectors $\underset{\sim}{m} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{n} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$.

a. If ϕ is the angle that $\underset{\sim}{m}$ makes with the y-axis, find the value of $\cos(\phi)$. 1 mark

b. Let $\hat{\underset{\sim}{r}} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, where x, y and z are real numbers, be a unit vector perpendicular to both $\underset{\sim}{m}$ and $\underset{\sim}{n}$. Find the possible values of x, y and z. 3 marks

In the diagram shown below, A is a point on side OV of triangle OVB, such that $\left|\overrightarrow{OV}\right| = 3\left|\overrightarrow{OA}\right|$. M is the midpoint of side VB.



Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

c. i. Express $\overrightarrow{BA}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

1 mark

ii. Express $\overrightarrow{BM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

1 mark

d. The points B, A and U are collinear. If $\overrightarrow{BU} = p\overrightarrow{BA}$, where p is a scalar, express $\overrightarrow{MU}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and p.

2 marks

e. If the line segment MU bisects the line segment OV, use a vector method to show that MU is parallel to BO.

2 marks

f. Given that MU is parallel to BO, find the value of p.

2 marks

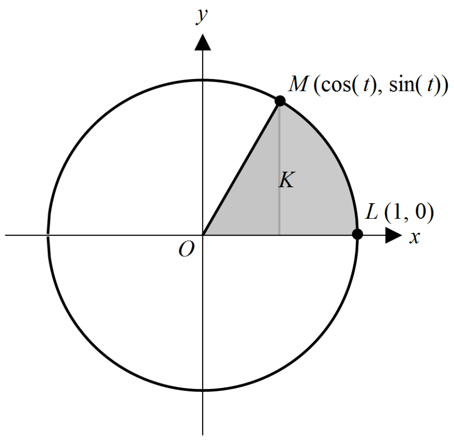
End of Question 2

Question 3 (12 marks)

The position vector of a point M is $\underset{\sim}{r} = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}$, where t is a real variable such that $t \in [0, 2\pi]$.

a. Show that the locus of M is a unit circle. 1 mark

Let K be the area of the shaded region shown below.

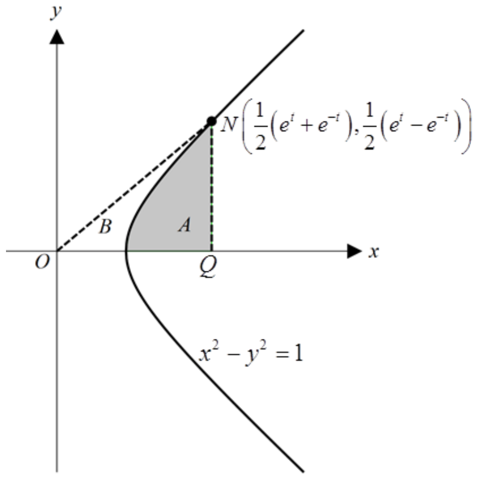


b. Find an expression for K in terms of t. 2 marks

A point N has position vector $\underset{\sim}{r} = \frac{1}{2}\left(e^{t} + e^{-t}\right)\underset{\sim}{i} + \frac{1}{2}\left(e^{t} - e^{-t}\right)\underset{\sim}{j}$, where t is a non-negative real variable.

c. Show that the locus of N has cartesian equation $x^2 - y^2 = 1$. 2 marks

Let *A* be the area enclosed by the graph of $x^2 - y^2 = 1$, the *x*-axis and the line $x = \frac{1}{2}\left(e^t + e^{-t}\right)$. Let *B* be the area enclosed by the graph of $x^2 - y^2 = 1$, the *x*-axis and the line segment *ON*, as shown below.



d. It can be shown that for $t \geq 0$, $A = \dfrac{e^{-2t}\left(e^{4t} - 4te^{2t} - 1\right)}{8}$.

Express *B* in terms of *t*, in simplest form. 3 marks

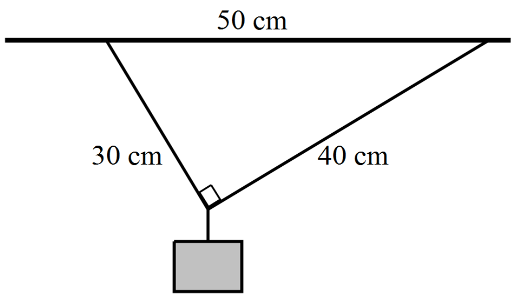
e. Determine the cartesian coordinates of N for which the rate of change in the x-coordinate with respect to t is equal to $\frac{3}{4}$. 2 marks

f. When $t = a$ the gradient of the normal to the hyperbola with equation $x^2 - y^2 = 1$ is $-\frac{3}{5}$. Find the value of a. 2 marks

End of Question 3

Question 4 (10 marks)

A block, A, is suspended by two light inextensible strings of lengths 30 cm and 40 cm, attached to two points on a horizontal beam, 50 cm apart. The tension in the shorter string is $4g$ newtons.

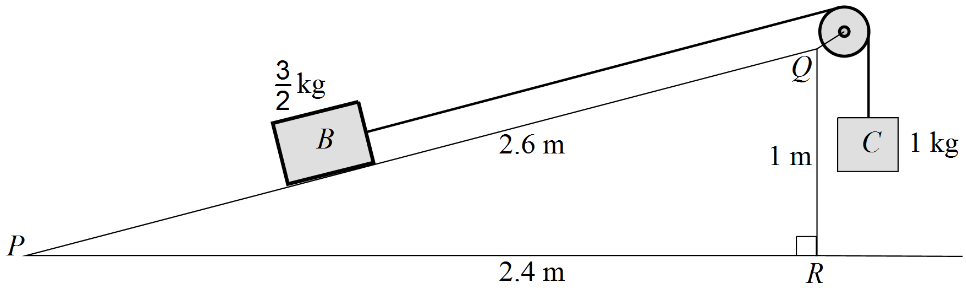


a. Determine the mass of block A. 2 marks

b. Determine the tension in the longer string. 1 mark

A different block, *B*, is placed on a smooth inclined plane. The cross section of the plane forms a right-angled triangle with height, *QR*, of 1 m and hypotenuse, *PQ*, of length 2.6 m.

The block is connected by a light inextensible string which passes over a frictionless pulley, to a particle, *C*, of mass of 1 kg which is hanging freely, as shown below.



Particle *C* is 1 m above the ground when the system is released from rest.

c. Show that when the system is released, the magnitude of the acceleration of block *B* up the plane is $\frac{11g}{65}$ ms^{-2}.

2 marks

d. When particle *C* has travelled vertically a distance of 1 metre, it hits the ground and the string becomes slack.

i. Find the speed of block *B* at this instant when *C* hits the ground. Express the answer in ms^{-1}, correct to two decimal places.

2 marks

ii. Find the magnitude of the acceleration of the block *B* immediately after *C* hits the ground. Express the answer in ms^{-2}, correct to two decimal places. 1 mark

iii. Find the total distance travelled by block *B* **up the plane** before it stops and starts sliding down. Express the answer in metres, correct to two decimal places. 2 marks

End of Question 4

Question 5 (11 marks)

Lin, a skydiver, drops vertically from a hot-air balloon that is at rest, so that her velocity immediately before dropping is zero. Lin falls freely from the hot-air balloon for 10 seconds before her parachute is opened. Let v_0 be Lin's speed immediately before the parachute is opened.

a. Assuming that air resistance is negligible during free fall, find the value of v_0 in ms^{-1}, correct to the nearest integer.

1 mark

Lin is using a new type of parachute that introduces a resistive force as soon as it is opened. As a result, Lin's speed, v ms^{-1}, t seconds after her parachute is opened, is given by

$v(t)=(v_0-4)e^{-\frac{g}{4}t}+4$. Lin hits the ground 6 seconds after her parachute is opened.

b. What was Lin's speed immediately before she hits the ground? Express the answer in ms^{-1}, correct to one decimal place.

1 mark

c. What was Lin's altitude above the ground when the parachute was opened? Give the answer in metres, correct to one decimal place.

2 marks

d. How high above the ground was the hot-air balloon when Lin dropped from it? Express the answer in metres, correct to one decimal place.

1 mark

A parachute made by a different company is designed so that when a skydiver of mass m kilograms opens the parachute while in free-fall, the parachute immediately introduces a drag force proportional to the velocity, v, of the skydiver. The equation of motion is given by $m\frac{dv}{dt} = mg - kv$, where $t \geq 0$ is the time from when the parachute is opened, and k is a positive real constant.

e. Show that $v = \frac{mg}{k} + \left(u - \frac{mg}{k}\right) \times e^{-\frac{k}{m}t}$, where u is the speed of the skydiver immediately before the parachute is opened. 3 marks

f. Given that the parachute is designed so the magnitude of the terminal velocity of a 90 kg skydiver is $4.2\ \text{ms}^{-1}$, find the value of k, in kg s^{-1}, correct to the nearest integer. 1 mark

END OF QUESTION AND ANSWER BOOKLET