Year 2014

VCE

Specialist Mathematics

Trial Examination 1

Solutions

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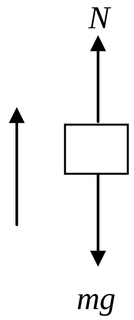
Question 1



$ma = N - mg$

$m = 60$ kg, $N = 788$ newtons M1

$60a = 788 - 60 \times 9.8$

$60a = 788 - 588 = 200$

$a = \dfrac{10}{3}$ m/s^2 A1

$s = ut + \dfrac{1}{2}at^2$

$s = ?$, $u = 0$, $a = \dfrac{10}{3}$, $t = 3$

$s = 0 + \dfrac{1}{2} \times \dfrac{10}{3} \times 9$

$s = 15$ metres A1

Question 2

$y = \arccos\left(\dfrac{3x}{4}\right) = \cos^{-1}\left(\dfrac{3x}{4}\right)$

$\dfrac{dy}{dx} = \dfrac{-3}{\sqrt{16-9x^2}} = -3\left(16-9x^2\right)^{-\frac{1}{2}}$ A1

$\dfrac{d^2y}{dx^2} = -3 \times \dfrac{-1}{2} \times -18x\left(16-9x^2\right)^{-\frac{3}{2}} = \dfrac{-27x}{\sqrt{\left(16-9x^2\right)^3}}$ A1

$\dfrac{d^2y}{dx^2} = ax\left(\dfrac{dy}{dx}\right)^3$ substituting

$\dfrac{-27x}{\sqrt{\left(16-9x^2\right)^3}} = ax\left(\dfrac{-3}{\sqrt{16-9x^2}}\right)^3 = \dfrac{-27ax}{\sqrt{\left(16-9x^2\right)^3}}$

$\Rightarrow \quad a = 1$ A1

Question 3

$xe^{2y} - y = c$ using implicit differentiation and the product rule

$$e^{2y} + 2xe^{2y}\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$e^{2y} = \left(1 - 2xe^{2y}\right)\frac{dy}{dx}$$ M1

$$\frac{dy}{dx} = \frac{e^{2y}}{1 - 2xe^{2y}}$$

$m_N = 3 \Rightarrow m_T = -\frac{1}{3} = \frac{dy}{dx}$ when it crosses x-axis $y = 0$ A1

$-\frac{1}{3} = \frac{1}{1-2x} \Rightarrow 1 - 2x = -3 \quad 2x = 4 \Rightarrow x = 2 \quad P(2,0)$

$c = xe^{2y} - y = 2e^0 - 0$

$c = 2$ A1

Question 4

a. $z = (1-i)^3\left(-\sqrt{3} + i\right)^4$

$$z = \left[\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right]^3\left[2\,\text{cis}\left(\frac{5\pi}{6}\right)\right]^4$$ M1

$$\arg(z) = -3\times\frac{\pi}{4} + 4\times\frac{5\pi}{6} = -\frac{3\pi}{4} + \frac{10\pi}{3} = \frac{31\pi}{12}$$

$$\text{Arg}(z) = \frac{31\pi}{12} - 2\pi$$

$$\text{Arg}(z) = \frac{7\pi}{12} \qquad k = \frac{7}{12}$$ A1

b. $z^2 = -8i = 8\,\text{cis}\left(-\frac{\pi}{2} + 2k\pi\right)$

$$z = \sqrt{8}\,\text{cis}\left(-\frac{\pi}{4} + k\pi\right)$$

$$k = 0 \quad z = \sqrt{8}\,\text{cis}\left(-\frac{\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 2 - 2i$$ A1

$$k = 1 \quad z = \sqrt{8}\,\text{cis}\left(\frac{3\pi}{4}\right) = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -2 + 2i$$ A1

Question 5 $v(t)=\frac{dx}{dt}=\frac{30t}{\sqrt{25+2t^2}}$

a. distance travelled in 10 seconds $D=\int_0^{10}\frac{30t}{\sqrt{25+2t^2}}dt$ A1

let $u=25+2t^2$ $\frac{du}{dt}=4t$

terminals when $t=0$ $u=25$ and when $t=10$ $u=225$

$D=\frac{30}{4}\int_{25}^{225}u^{-\frac{1}{2}}du$ M1

$D=\frac{15}{2}\left[2u^{\frac{1}{2}}\right]_{25}^{225}=15\left[\sqrt{225}-\sqrt{25}\right]=15(15-5)$

$D=150$ metres A1

b. $\lim\limits_{t\to\infty}v(t)$

$=\lim\limits_{t\to\infty}\left(\frac{30t}{\sqrt{25+2t^2}}\right)=\lim\limits_{t\to\infty}\left(\frac{30}{\sqrt{\frac{25}{t^2}+2}}\right)=\frac{30}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$

$=15\sqrt{2}$ ms^{-1} A1

Question 6

$V=\pi\int_a^b y^2dx$ $y=4\cos\left(\frac{x}{3}\right)$ $y=0\Rightarrow\frac{x}{3}=\frac{\pi}{2}$ $x=\frac{3\pi}{2}$

$V=\pi\int_0^{\frac{3\pi}{2}}16\cos^2\left(\frac{x}{3}\right)dx$ A1

$V=8\pi\int_0^{\frac{3\pi}{2}}\left(1+\cos\left(\frac{2x}{3}\right)\right)dx$

$V=8\pi\left[x+\frac{3}{2}\sin\left(\frac{2x}{3}\right)\right]_0^{\frac{3\pi}{2}}$ M1

$V=8\pi\left[\left(\frac{3\pi}{2}+\frac{3}{2}\sin(\pi)\right)-\left(0+\frac{3}{2}\sin(0)\right)\right]$

$V=12\pi^2$ units^3 A1

Question 7

$y = \dfrac{x^4 - 81}{3x^3} = \dfrac{x}{3} - \dfrac{27}{x^3} = \dfrac{x}{3} - 27x^{-3}$

crosses x-axis when $y = 0 \Rightarrow x^4 - 81 = 0 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow x = \pm 3$

$(3,0) \ (-3,0)$ A1

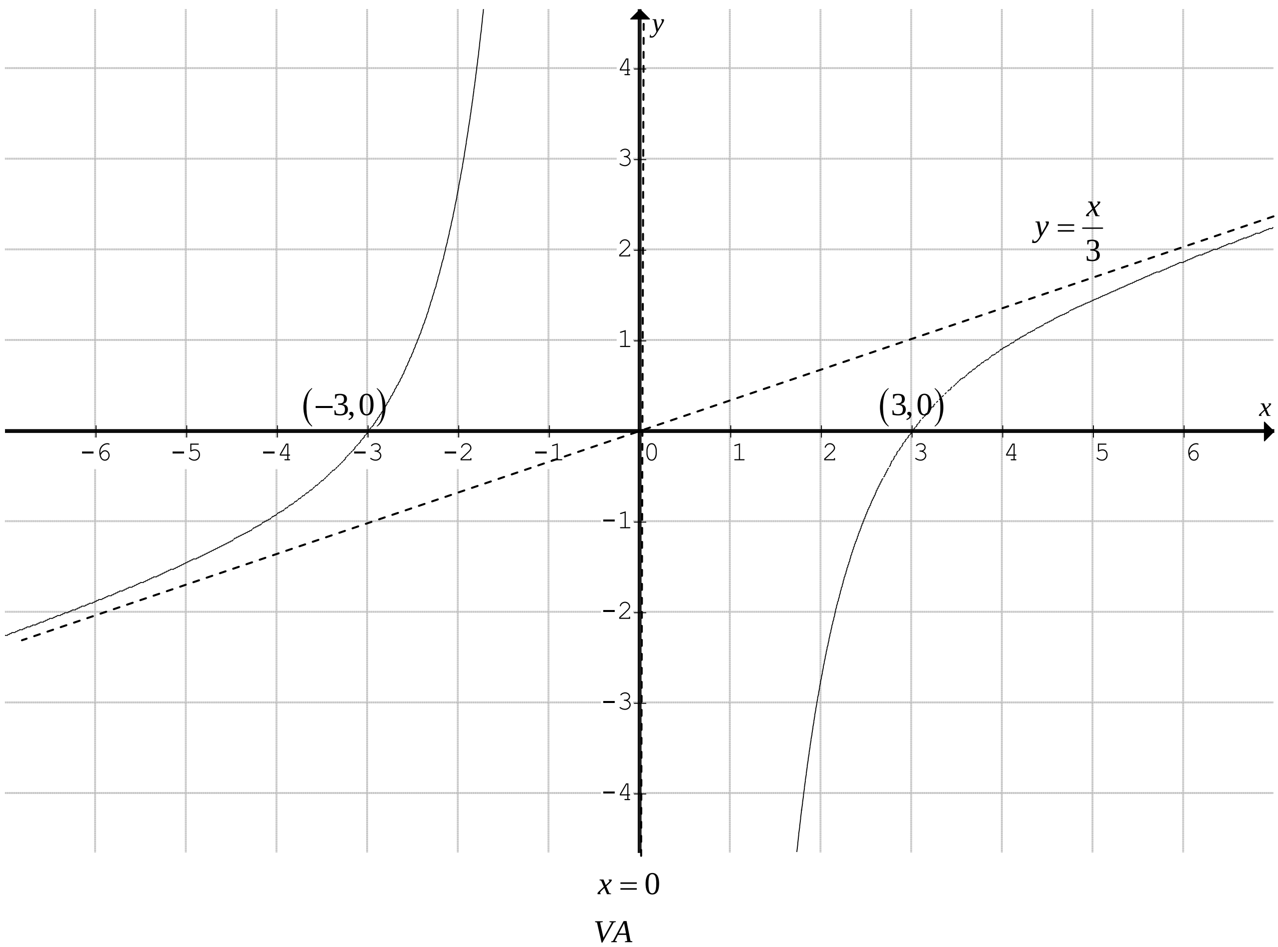
does not cross the y-axis

$x = 0$ is a vertical asymptote and $y = \dfrac{x}{3}$ is oblique asymptote A1

for turning points $\dfrac{dy}{dx} = \dfrac{1}{3} + 81x^{-4} = \dfrac{1}{3} + \dfrac{81}{x^4} = 0 \ \Rightarrow x^4 = -27$

this has no real solutions, there are no turning points A1

correct graph, shape asymptotes G1



Question 8

a. $\underset{\sim}{a}=2\underset{\sim}{i}-\underset{\sim}{j}-2\underset{\sim}{k}$, $\underset{\sim}{b}=5\underset{\sim}{i}+4\underset{\sim}{j}+3\underset{\sim}{k}$ and $\underset{\sim}{c}=4\underset{\sim}{i}+11\underset{\sim}{j}+z\underset{\sim}{k}$.

$\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent $\Rightarrow$ $\underset{\sim}{c}=m\underset{\sim}{a}+n\underset{\sim}{b}$, $m,n\in R\setminus\{0\}$

$\underset{\sim}{i}:\ \Rightarrow(1)\ \ 4=2m+5n$

$\underset{\sim}{j}:\ \Rightarrow(2)\ \ 11=-m+4n$ M1

$\underset{\sim}{k}:\ \Rightarrow(3)\ \ z=-2m+3n$

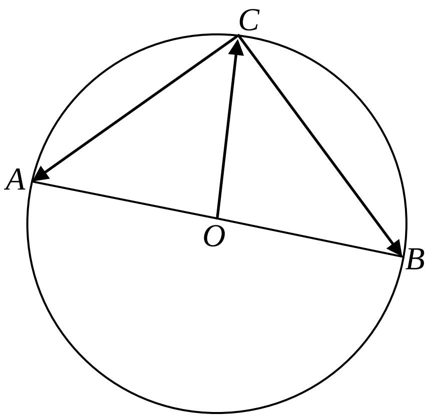
$(1)\ \ 4=2m+5n$

$2\times(2)\ \ 22=-2m+8n$ adding $26=13n\Rightarrow n=2$

and $2m=4-5n=4-10=-6\ \Rightarrow m=-3$

into $(3)\ \ z=12$ A1

b.



Let $\overrightarrow{OC}=\underset{\sim}{c}$ and $\overrightarrow{OB}=\underset{\sim}{b}$

$\overrightarrow{OA}=-\underset{\sim}{b}$, since O is the midpoint of AB

$\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=-\underset{\sim}{b}-\underset{\sim}{c}=-(\underset{\sim}{b}+\underset{\sim}{c})$ M1

$\overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}=\underset{\sim}{b}-\underset{\sim}{c}$

$\overrightarrow{CA}.\overrightarrow{CB}=-(\underset{\sim}{b}+\underset{\sim}{c}).(\underset{\sim}{b}-\underset{\sim}{c})=-(\underset{\sim}{b}.\underset{\sim}{b}+\underset{\sim}{c}.\underset{\sim}{b}-\underset{\sim}{b}.\underset{\sim}{c}-\underset{\sim}{c}.\underset{\sim}{c})$ A1

but $\underset{\sim}{c}.\underset{\sim}{b}=\underset{\sim}{b}.\underset{\sim}{c}$ and $\underset{\sim}{b}.\underset{\sim}{b}=|\underset{\sim}{b}|^2$ and $\underset{\sim}{c}.\underset{\sim}{c}=|\underset{\sim}{c}|^2$

$\overrightarrow{CA}.\overrightarrow{CB}=|\underset{\sim}{c}|^2-|\underset{\sim}{b}|^2$

however $|\underset{\sim}{b}|=|\underset{\sim}{c}|$ since both are radii of the circle

so that $\overrightarrow{CA}.\overrightarrow{CB}=0\ \Rightarrow \overrightarrow{CA}$ is perpendicular to $\overrightarrow{CB}$ A1

Question 9

a. $\underset{\sim}{r}(t)=(3-2\cos(2t))\underset{\sim}{i}+(4-3\sin(2t))\underset{\sim}{j}$ for $t\geq 0$

$x=3-2\cos(2t) \quad y=4-3\sin(2t)$

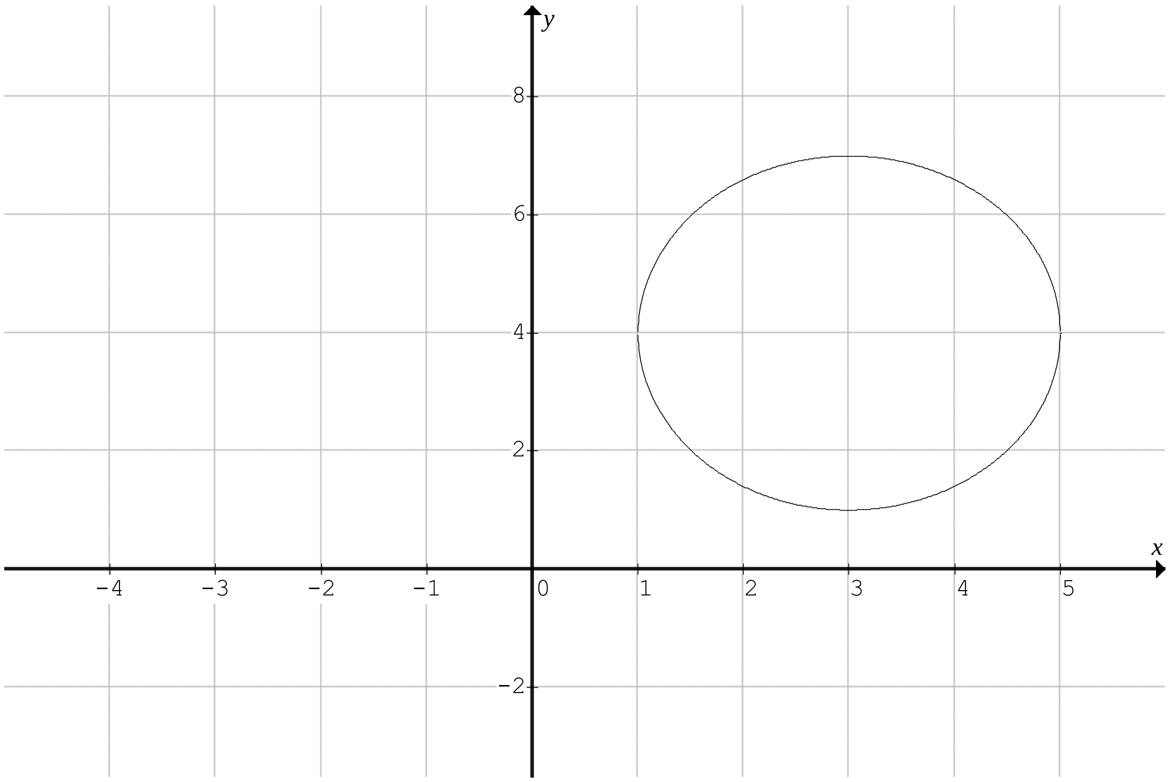
$\cos(2t)=\frac{3-x}{2} \quad , \quad \sin(2t)=\frac{4-y}{3}$

$\sin^2(2t)+\cos^2(2t)=1$

$\frac{(x-3)^2}{4}+\frac{(y-4)^2}{9}=1$ ellipse centre $(3,4)$

domain $[1,5]$ range $[1,7]$

graph correct G1



b. $\underset{\sim}{r}(t)=(3-2\cos(2t))\underset{\sim}{i}+(4-3\sin(2t))\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t)=4\sin(2t)\underset{\sim}{i}-6\cos(2t)\underset{\sim}{j}$ A1

$\left|\dot{\underset{\sim}{r}}(t)\right|=\sqrt{16\sin^2(2t)+36\cos^2(2t)}$

$=\sqrt{16(1-\cos^2(2t))+36\cos^2(2t)}$ M1

$=\sqrt{20\cos^2(2t)+16}$

$\left|\dot{\underset{\sim}{r}}(t)\right|_{max}=\sqrt{36}=6$

when $\cos(2t)=1 \Rightarrow 2t=2k\pi \quad t=k\pi$ A1

$\left|\dot{\underset{\sim}{r}}(t)\right|_{min}=\sqrt{16}=4$

when $\cos(2t)=0 \Rightarrow 2t=(2k+1)\frac{\pi}{2} \quad t=(2k+1)\frac{\pi}{4}, k\in Z$ A1

Question 10

$\int \frac{5x+3}{4x^2+81}dx$ separate out into two integrals

$=5\int \frac{x}{4x^2+81}dx+3\int \frac{1}{4x^2+81}dx$

let $u=4x^2+81$ let $v=2x$

$\frac{du}{dx}=8x$ $\frac{dv}{dx}=2$ M1

$=\frac{5}{8}\int \frac{1}{u}du+\frac{3}{2}\int \frac{1}{81+v^2}dv$

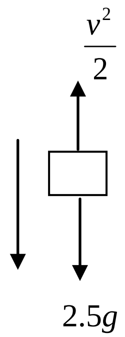
$=\frac{5}{8}\log_e(|u|)+\frac{3}{2}\times\frac{1}{9}\tan^{-1}\left(\frac{v}{9}\right)+c$

$=\frac{5}{8}\log_e(4x^2+9)+\frac{3}{2}\times\frac{1}{9}\tan^{-1}\left(\frac{v}{9}\right)+c$ since $4x^2+9>0$

$=\frac{5}{8}\log_e(4x^2+81)+\frac{1}{6}\tan^{-1}\left(\frac{2x}{9}\right)+c$ A1

Question 11

a.



$$2.5a = 2.5 \times 9.8 - \frac{v^2}{2}$$

$$\frac{5a}{2} = \frac{5 \times 9.8}{2} - \frac{v^2}{2}$$

$$5a = 49 - v^2$$

$$a = \frac{dv}{dt} = \frac{49 - v^2}{5}$$ M1

b.

$$\frac{dt}{dv} = \frac{5}{49 - v^2}$$ inverting both sides

$$t = \int \frac{5}{49 - v^2}\,dv$$ by partial fractions

$$\frac{5}{49 - v^2} = \frac{A}{7+v} + \frac{B}{7-v} = \frac{A(7-v) + B(7+v)}{(7+v)(7-v)} = \frac{7(A+B) + v(B-A)}{49 - v^2}$$ M1

$$7(A+B) = 5$$

$$B - A = 0 \quad \Rightarrow A = B = \frac{5}{14}$$

$$t = \frac{5}{14}\int \left(\frac{1}{7+v} + \frac{1}{7-v}\right) dv$$

$$t = \frac{5}{14}\left[\log_e(|7+v|) - \log_e(|7-v|)\right] + c$$ A1

when $t = 0 \quad v = 0 \quad \Rightarrow c = 0$

$$t = \frac{5}{14}\log_e\left(\frac{|7+v|}{|7-v|}\right)$$ A1

END OF SUGGESTED SOLUTIONS