2014 VCAA Specialist Mathematics Exam 1 Solutions

Q1a $\hat{a} = \frac{\tilde{a}}{|\tilde{a}|} = \frac{\sqrt{3}\,\tilde{i} - \tilde{j} - \sqrt{2}\,\tilde{k}}{\sqrt{3+1+2}} = \frac{\sqrt{3}}{\sqrt{6}}\tilde{i} - \frac{1}{\sqrt{6}}\tilde{j} - \frac{\sqrt{2}}{\sqrt{6}}\tilde{k}$

$= \frac{1}{\sqrt{2}}\tilde{i} - \frac{1}{\sqrt{6}}\tilde{j} - \frac{1}{\sqrt{3}}\tilde{k}$

Q1b $\cos\theta = \hat{a}\,.\,\tilde{i} = \frac{1}{\sqrt{2}}$, .: $\theta = \frac{\pi}{4}$

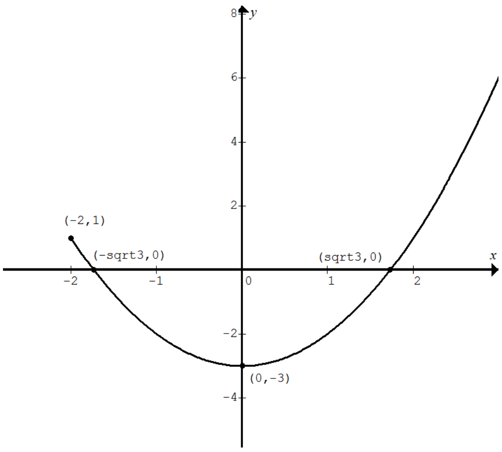
Q1c $\tilde{b}\,.\tilde{a} = 0$, $6 - m + 5\sqrt{2} = 0$, $m = 6 + 5\sqrt{2}$

Q2a $x = t - 2$ where $t \ge 0$

$y = t^2 - 4t + 1 = t^2 - 4t + 4 - 3 = (t-2)^2 - 3$

.: $y = x^2 - 3$

Q2b



Q2c $\tilde{v}(t) = \tilde{i} + (2t-4)\tilde{j}$, $\tilde{v}(1) = \tilde{i} - 2\tilde{j}$

Speed $= |\tilde{v}(1)| = \sqrt{1+4} = \sqrt{5}$

Q3a $z - i$ is a factor of $f(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$ which has real coefficients, .: $z + i$ is also a factor.

.: $f(z) = (z-i)(z+i)(z^2 + bz + c)$

.: A quadratic factor of $f(z)$ is $(z-i)(z+i) = z^2 + 1$.

Q3b $f(z) = (z^2+1)(z^2+bz+c) = z^4 + bz^3 + (c+1)z^2 + bz + c$

.: $b = -4$ and $c = 6$

Let $z^2 + bz + c = 0$

.: $z = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{4 \pm \sqrt{16-24}}{2} = 2 \pm \sqrt{2}\,i$

The solutions are: $\pm i\,, 2 \pm \sqrt{2}\,i$

Q4 $y = -3e^{3x}e^{y} = -3e^{3x+y}$

.: $\frac{dy}{dx} = -3e^{3x+y}\left(3 + \frac{dy}{dx}\right)$

$\frac{dy}{dx} = -9e^{3x+y} - 3e^{3x+y}\frac{dy}{dx}$, $\left(1 + 3e^{3x+y}\right)\frac{dy}{dx} = -9e^{3x+y}$

.: $\frac{dy}{dx} = \frac{-9e^{3x+y}}{1 + 3e^{3x+y}}$.

At $(1, -3)$, $\frac{dy}{dx} = -\frac{9}{4}$

.: gradient of the normal $= \frac{4}{9}$

Q5a $f(x) = a\sin(6x) = 2a\sin(3x)\cos(3x) = 96\sin(3x)\cos(3x)$

.: $a = 48$

Q5b and Q5c $u = \cos(6x)$, $\frac{du}{dx} = -6\sin(6x)$

When $x = \frac{\pi}{36}$, $u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$; when $x = \frac{\pi}{12}$, $u = 0$

$\int_{\frac{\pi}{36}}^{\frac{\pi}{12}} 96\cos(3x)\sin(3x)\cos^2(6x)dx = \int_{\frac{\pi}{36}}^{\frac{\pi}{12}} 48\sin(6x)\cos^2(6x)dx$

$= \int_{\frac{\pi}{36}}^{\frac{\pi}{12}} 48u^2\left(-\frac{1}{6}\frac{du}{dx}\right)dx$

$= \int_{\frac{\sqrt{3}}{2}}^{0} -8u^2 du = \int_0^{\frac{\sqrt{3}}{2}} 8u^2 du$

$= \left[\frac{8u^3}{3}\right]_0^{\frac{\sqrt{3}}{2}} = \frac{8}{3}\times\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{8}{3}\times\frac{3\sqrt{3}}{8} = \sqrt{3}$

Q6a $\frac{a}{a-4} = \frac{(a-4)+4}{a-4} = 1 + \frac{4}{a-4}$

Q6b $V = \int_3^4 \pi y^2 dx = \pi\int_3^4 \left(\frac{x}{\sqrt{x^2-4}}\right)^2 dx = \pi\int_3^4 \frac{x^2}{x^2-4}dx$

$= \pi\int_3^4 \frac{x^2}{x^2-4}dx = \pi\int_3^4\left(1 + \frac{4}{x^2-4}\right)dx = \pi\int_3^4\left(1 + \frac{4}{(x-2)(x+2)}\right)dx$

$= \pi\int_3^4\left(1 + \frac{1}{x-2} - \frac{1}{x+2}\right)dx$

$= \pi\left[x + \log_e(x-2) - \log_e(x+2)\right]_3^4$

$= \pi\left[x + \log_e\left(\frac{x-2}{x+2}\right)\right]_3^4 = \pi\left[\left(4 + \log_e\frac{2}{6}\right) - \left(3 + \log_e\frac{1}{5}\right)\right]$

$= \pi\left(1 + \log_e\frac{5}{3}\right)$

Q7a The range of $f(x) = 3x \arctan(2x)$ is $[0, \infty)$.

Q7b $f'(x) = 3\arctan(2x) + 3x\left(\frac{2}{1+(2x)^2}\right)$

$= 3\arctan(2x) + \frac{6x}{1+4x^2}$

Q7c From part b, $\arctan(2x) = \frac{1}{3}f'(x) - \frac{2x}{1+4x^2}$

Area $= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \arctan(2x)dx = \frac{1}{3}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} f'(x)dx - \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2x}{1+4x^2}dx$

$\frac{1}{3}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} f'(x)dx = \frac{1}{3}\left[3x\arctan(2x)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$

$= \frac{1}{3}\left(\frac{3\sqrt{3}}{2}\arctan\sqrt{3} - \frac{3}{2}\arctan 1\right) = \left(\frac{\sqrt{3}}{6} - \frac{1}{8}\right)\pi$

$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2x}{1+4x^2}dx = \frac{1}{4}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{u}\frac{du}{dx}dx = \frac{1}{4}\int_2^4 \frac{1}{u}du$

$= \frac{1}{4}\left[\log_e u\right]_2^4 = \frac{1}{4}\log_e 2$

$u = 1+4x^2, \frac{du}{dx} = 8x$

$x = \frac{1}{2}, u = 2$

$x = \frac{\sqrt{3}}{2}, u = 4$

.: Area $= \left(\frac{\sqrt{3}}{6} - \frac{1}{8}\right)\pi - \frac{1}{4}\log_e 2$

Q8ai Horizontally: $T_2 = T_1 \sin\theta$

Vertically: $T_1 \cos\theta = 5g$, .: $T_1 = \frac{5g}{\cos\theta}$

Q8aii $T_2 = T_1 \sin\theta = \frac{5g\sin\theta}{\cos\theta} = 5g\tan\theta$

Q8b For $0 < \theta < \frac{\pi}{2}$, $0 < \sin\theta < 1$ and $\cos\theta > 0$

.: $\frac{0}{\cos\theta} < \frac{\sin\theta}{\cos\theta} < \frac{1}{\cos\theta}$, .: $0 < \tan\theta < \sec\theta$

Q8c Neither string will break:

$T_1 = \frac{5g}{\cos\theta} \le 98$ and $T_2 = 5g\tan\theta \le 98$

.: $\sec\theta \le 2$ and $\tan\theta \le 2$

.: $\sec\theta \le 2$, .: $\cos\theta \ge \frac{1}{2}$, .: $0 < \theta \le \frac{\pi}{3}$

Maximum value of θ is $\frac{\pi}{3}$.

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